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Finitely Generated Ideals in $A^\infty(D)$

PAOLO DE BARTOLOMEIS

Istituto di Matematica Applicata, Università di Firenze, Florence, Italy

AND

GIUSEPPE TOMASSINI

Department of Mathematics, Scuola Normale Superiore Pisa, Pisa, Italy

INTRODUCTION

Let $D \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain with C^∞-smooth boundary bD and let V be an analytic subvariety of a neighborhood of $\bar{D}$. Let $\mathscr{I}_V$ denote the sheaf of ideals of V. From the general theory of Oka–Cartan–Serre it follows that if $f_1,\ldots,f_k \in \Gamma(\bar{D}, \mathscr{I}_V)$ generate $\mathscr{I}_{V,z}$ at every point $z \in D$, then they generate $\Gamma(D, \mathscr{I}_V)$ over $\mathscr{O}(D)$ $(=\Gamma(D, \mathscr{O}))$.

Let A be a subalgebra (or more generally a vector subspace) of $\mathscr{O}(D)$ containing $f_1,\ldots,f_k$. It is a natural question to ask whether $f_1,\ldots,f_k$ generate $\Gamma(D, \mathscr{I}_V) \cap A$ over A. There are several results in this direction in the special case V is a point and A is the algebra of holomorphic functions satisfying some regularity condition at the boundary [3, 6, 7, 10, 14].

The situation becomes much more complicated when V has positive dimension, as the following elementary example shows: the holomorphic function $f(z_1, z_2) = z_2(1 - z_1)^{-1/4}$ is continuous up to the boundary on the unit ball B in $\mathbb{C}^2$ but its factor in z_2 is not even bounded (and it can be easily proved that the ideal of the complex line $z_2 = 0$ is not finitely generated over $A^0(B)$, the algebra $\mathscr{O}(B) \cap C^0(\bar{B})$).

The aim of the present paper is to prove that similar phenomena disappear in a high regularity situation, i.e., for the algebra $A^\infty(D) = \mathscr{O}(D) \cap C^\infty(\bar{D})$. Namely, we prove that if D is strictly pseudoconvex, bD and V are "regularly separated" (see Section 1) and V is smooth near bD then $f_1,\ldots,f_k$ generate $I^\infty(V) = \Gamma(D, \mathscr{I}_V) \cap C^\infty(D)$ over $A^\infty(D)$. Some of these results were announced in [5].

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1. The Local and Semi-Local Cases

1. From now on D will denote a bounded domain in $\mathbb{C}^n$ with C^∞-smooth boundary bD, D' a neighborhood of $\bar{D}$ and $V' \subset D'$ a closed complex analytic subset, such that $\operatorname{Sing} V' \cap bD = \varnothing$.

Let $\mathscr{E}_{V'}$ be the ideal of germs of holomorphic functions vanishing on V'.

A *complete defining system* for V' is a finite set of holomorphic functions on $D', f_1, ..., f_k$ such that for every $z \in D$ the germs $f_{1,z}, ..., f_{k,z}$ generate $\mathscr{E}_{V',z}$. It is well known that such a system always exists if D' is a Stein neighborhood of $\bar{D}$.

Let $V = V' \cap D \neq \varnothing$. For every relative open subset U of $\bar{D}$ we set

$$\mathscr{O}_{\bar{D}}^{(\infty)}(U) = \{f \in \mathscr{O}(\mathring{U}) : f \in C^\infty(\mathring{U} \cup (bD \cap U))\},$$

$$\mathscr{O}_{\bar{V}}^{(\infty)}(U \cap \bar{V}) = \{f \in \mathscr{O}(\mathring{U} \cap V) : f = \tilde{f}|_{\mathring{U} \cap V}, \tilde{f} \in C^\infty(U \cup (bD \cap U))\};$$

$U \mapsto \mathscr{O}_{\bar{D}}^{(\infty)}(U)$ and $U \cap \bar{V} \mapsto \mathscr{O}_{\bar{V}}^{(\infty)}(U \cap \bar{V})$ define two sheaves on $\bar{D}$ and $\bar{V}$ respectively, $\mathscr{O}_{\bar{D}}^{(\infty)}$ and $\mathscr{O}_{\bar{V}}^{(\infty)}$ such that

$$\mathscr{O}_{\bar{D}}^{(\infty)}|_D = \mathscr{O}_D \qquad \text{and} \qquad \mathscr{O}_{\bar{V}}^{(\infty)}|_V = \mathscr{O}_V.$$

Let $\mathscr{E}_{\bar{V}}^{(\infty)} \subset \mathscr{O}_{\bar{D}}^{(\infty)}$ be the ideal of germs vanishing on $\bar{V}$. In particular we have

$$A^\infty(D) = \Gamma(\bar{D}, \mathscr{O}_{\bar{D}}^{(\infty)}), \qquad A^\infty(V) = \Gamma(\bar{V}, \mathscr{O}_{\bar{V}}^{(\infty)}), \qquad I^\infty(V) = \Gamma(\bar{V}, \mathscr{E}_{\bar{V}}^{(\infty)}).$$

We say that V' and bD are *regularly separated at* $p \in bD \cap V'$ if there exist $N \in \mathbb{N}$, $c > 0$ and a neighborhood U of p such that

$$\operatorname{dist}(z, U \cap bD \cap V')^N \leqslant C \operatorname{dist}(z, U \cap V') \qquad (*)$$

for every $z \in U \cap bD$.

We say V' and bD are *regularly separated* if in the previous definition U can be chosen in such a way as to be a neighborhood of bD.

Note that if bD is real analytic $(*)$ is the well known Łojasiewicz's inequality [12]. It is worth noting also that $(*)$ does not follow from strict pseudoconvexity of D.

Suppose now V' is smooth near bD and let d be its complex dimension. We say that V' is *transversal* to bD at $z \in bD$ if $V' \cap bD$ is smooth of real dimension $2d - 1$ and

$$T_z^{\mathbb{C}}(V' \cap bD) = T_z^{\mathbb{C}}(V') \cap T_z^{\mathbb{C}}(bD)$$

(where $T_z^{\mathbb{C}}(\cdot)$ is the complex tangent space at z).

It is a very simple matter to prove that if V' and bD are transversal at z then they are regularly separated at z.

2. We shall consider first the case where V' is a linear subspace: our first statement is the following:

PROPOSITION 1. *Let D be pseudoconvex and let $V' = \{z_{k+1} = \cdots = z_n = 0\}$. Assume $V = V' \cap D \neq \emptyset$ and V' and bD are regularly separated. Then for every $f \in A^\infty(D)$ such that $f|_V = 0$ there exist $\lambda_{k+1}, \ldots, \lambda_n \in A^\infty(D)$ such that*

$$f = \sum_{j=k+1}^{n} \lambda_j z_j.$$

We prove first the following result in codimension 1:

LEMMA 2. *Let $D \subset \mathbb{C}^n$ be a bounded domain with C^∞-smooth boundary and let V' be the hyperplane $z_n = 0$. Assume $V = V' \cap D \neq \emptyset$ and V' and bD are regularly separated. Then*:

(i) *every $f \in A^\infty(D)$ such that $f|_V = 0$ can be written as $f = gz_n$, where $g \in A^\infty(D)$,*

(ii) *if D is pseudoconvex the restriction homomorphism $A^\infty(D) \to A^\infty(V)$ is onto.*

Proof. We shall divide the proof into several steps.

(a) If $f \in C^\infty(\bar{D})$ satisfies $f|_V = 0$ then $f = gz_n + h\bar{z}_n$, where $g, h \in C^\infty(\bar{D})$.

Using the fact that V' and bD are regularly separated, via Whitney's extension theorem we can find a ball B containing $\bar{D}$ and $F \in C^\infty(\bar{B})$ such that $F|_{B \cap V'} = 0$, $F_{\bar{D}} = f$. Fix $z \in B$ and set $\varphi(t) = F(z_1, \ldots, z_{n-1}, tz_n)$. We have

$$F(z_1, \ldots, z_n) = \varphi(1) = \int_0^1 \frac{d\varphi}{dt}\, dt = z_n \int_0^1 \frac{\partial F}{\partial z_n}(z_1, \ldots, z_{n-1}, tz_n)\, dt$$

$$+ \bar{z}_n \int_0^1 \frac{\partial F}{\partial \bar{z}_n}(z_1, \ldots, z_{n-1}, tz_n)\, dt.$$

(b) If $f \in C^\infty(\bar{D})$ is $\bar{n}$-*flat* on V (i.e., $(\partial^m f/\partial \bar{z}_n^m)|_V = 0\ \forall\, m \geqslant 0$) then for any $k \in \mathbb{N}$ there exist $g, h \in C^\infty(\bar{D})$ such that $f = gz_n + h\bar{z}_n^k$.

By induction on k. Assume it is true for k; we have

$$\frac{\partial^k f}{\partial \bar{z}_n^k} = \frac{\partial^k g}{\partial \bar{z}_n^k} z_n + k!\, h \sum_{j=1}^{k} \binom{k}{j} \frac{k!}{j!} \frac{\partial^j h}{\partial \bar{z}_n^j} \bar{z}_n^j;$$

from the $\bar{n}$-flatness it follows that $h|_V = 0$ and so by step (a) $h = z_n u + \bar{z}_n v$, where $u, v \in A^\infty(\bar{D})$ and $f = z_n(g + \bar{z}_n^k u) + \bar{z}_n^{k+1} v$.

(c) If $f \in C^\infty(\bar{D})$ is $\bar{n}$-flat on V then $z_n^{-1} f \in C^\infty(\bar{D})$.

For any $k \in \mathbb{N}$ we have $z_n^{-1}f = g + \bar{z}_n^{k+2} z_n^{-1} h \in C^k(\bar{D})$. Now if $f \in A^{\infty}(\bar{D})$ and $f|_V = 0$ then f is $\bar{n}$-flat on V and so setting $g = fz_n^{-1}$ we have $f = gz_n$, where $g \in A^{\infty}(D)$: this proves (i).

Now let $\pi: \mathbb{C}^n \to V'$ be the natural projection; for every $f \in A^{\infty}(V)$ let $\tilde{f}: V' \to \mathbb{C}$ be a C^{∞}-smooth extension of f and let $\tilde{F} = \tilde{f} \circ \pi|_{\bar{D}}$. We have $\tilde{F} \in C^{\infty}(\bar{D})$ and $\bar{\partial}\tilde{F} = \sum_{j=1}^n b_j \, d\bar{z}_j$, where $b_1, ..., b_n$ are $\bar{n}$-flat on V. Then $\alpha = z_n^{-1} \bar{\partial}\tilde{F}$ is C^{∞}-smooth on $\bar{D}$ so that we can find $u \in C^{\infty}(\bar{D})$ such that $\bar{\partial}u = \alpha$ [11]. Then $F = \tilde{F} - z_n u$ belongs to $A^{\infty}(D)$ and it is the required extension of f (cf. also [8]).

We are now in a position to prove Proposition 1.

Proof of Proposition 1. Let V_p' be defined by $z_{k+1+p} = \cdots = z_n = 0$, $1 \leqslant p \leqslant n-k-1$, and set $D_p = D \cap V_p'$.

We may assume (possibly after a linear transformation of coordinates) that $V_p' \cap bD$ is C^{∞}-smooth and V_p' and $V_{p+1}' \cap bD$ are regularly separated.

Let $f \in A^{\infty}(D)$ be such that $f|_V = 0$. In view of Lemma 2 we have $f|_{D_1} = z_{k+1} g_{k+1}$, where g_{k+1} is the restriction to D_1 of a function in $A^{\infty}(D_2)$.

It follows that $f - z_{k+1} g_{k+1}$ vanishes on D_1 so that on D_2 we have $f - g_{k+1} z_{k+1} = g_{k+2} z_{k+2}$, etc.

With the same notations, as a consequence of Proposition 1, we get the following:

COROLLARY 3. *Assume D is pseudoconvex, V' and bD are regularly separated and* Sing $V' \cap bD = \varnothing$. *Let $f_1, ..., f_k$ be a complete defining system for V'. Then the sheaf homomorphisms*

(i) $(\mathcal{O}_{\bar{D}}^{(\infty)})^{\oplus k} \to \mathcal{E}_V^{(\infty)}$ *(given by* $(\lambda_1, ..., \lambda_k) \mapsto \sum_{j=1}^k \lambda_j f_j$*),*

(ii) $\mathcal{O}_{\bar{D}}^{(\infty)} \to \mathcal{O}_V^{(\infty)}$ *(given by "restriction to V") are onto.*

2. THE GENERAL CASE

1. In view of Corollary 3 the problem we are dealing with is locally solvable.

In order to get the global result, in the case V' is a *global complete intersection* (i.e., $k = n\text{-dim}_{\mathbb{C}} V'$) cohomological techniques can be employed proving that the sheaf $\mathcal{R}$ of $\mathcal{O}_{\bar{D}}^{(\infty)}$-relations between $f_1, ..., f_k$ actually satisfies $H^1(\bar{D}, \mathcal{R}) = 0$ [1].

In the general case here we use a construction which reduces the problem to the linear case, via an extension theorem.

THEOREM 4. *Assume D is strongly pseudoconvex and let V_1', V_2' be analytic subvarieties of D' such that if we set $V_3' = V_1' \cap V_2'$ we have:*

(i) Sing $V_j' \cap bD = \varnothing$, $j = 1, 2, 3$,

(ii) *V_j' and bD are regularly separated*, $j = 1, 2, 3$.

Assume also that V_1' and V_2' intersect transversally along bD and let $Z' = V_1' \cup V_2'$, $V_j = V_j' \cap D$, $j = 1, 2, 3$. Then the restriction homomorphism $A^\infty(D) \to A^\infty(Z)$ is onto, where $Z = Z' \cap D$.

Proof. (1) First of all we prove that the restriction homomorphism $\mathcal{O}_{\bar{D}}^{(\infty)} \to \mathcal{O}_{\bar{Z}}^{(\infty)}$ is onto.

For this let $p \in bD$ and U be a small neighborhood of p. Let $f \in A^\infty(U \cap Z)$: in view of Corollary 3(ii) we may assume $f|_{V_3 \cap U} = 0$. Assume for the moment we may choose complex coordinates $z_1, \ldots, z_n$ in such a way that

$$V_1' \cap U = \{z \in U: z_1 = \cdots = z_k = 0\},$$

$$V_2' \cap U = \{z \in U: z_s = \cdots = z_m = 0, s \leqslant k + 1\}.$$

Let $f_i = f|_{V_i \cap U}$; from Corollary 3(i) we get

$$f_1 = \sum_{j=k+1}^{m} h_j z_j, \qquad h_j \in A^\infty(V_1 \cap U),\ k + 1 \leqslant j \leqslant m,$$

and in view of part (ii) we can extend h_j as $H_j \in A^\infty(U \cap D)$. If we set $F_1 = \sum_{j=k+1}^m H_j z_j$ we have $F_1 \in A^\infty(U \cap D)$ and $F_1|_{V_2 \cap D} = 0$.

In the same way we can construct $F_2 \in A^\infty(U \cap D)$ extending f_2 and vanishing on $V_1 \cap U$. The function $F = F_1 + F_2$ satisfies $F|_{Z \cap U} = f$. This shows that the sequence of sheaves

$$0 \to \mathscr{E}_{\bar{Z}}^{(\infty)} \to \mathcal{O}_{\bar{D}}^{(\infty)} \to \mathcal{O}_{\bar{Z}}^{(\infty)} \to 0$$

is exact and so in order to conclude the proof we have to show that the group $H^1(\bar{D}, \mathscr{E}_{\bar{Z}}^{(\infty)})$ is actually zero. This amounts to proving the following claim: let F be a $(0, 1)$-form, C^∞-smooth up to bD, $\bar{\partial}$-closed and such that $F|_Z = 0$; then there exists $u \in C^\infty(\bar{D})$ such that $\bar{\partial} u = F$ and $u|_Z = 0$.

(2) The proof of this claim is based on a construction on the theme of the "bumps lemma" of Andreotti and Grauert [2, p. 237] (cf. also [4] and [9]).

2. Consider a finite covering $\{B_j\}$, $1 \leqslant j \leqslant q$, of bD where $B_j = B(\zeta_j, \rho)$ is the open ball of radius ρ centered at $\zeta_j \in bD$, in such a way that $\{B(\zeta_j, \rho/2)\}$, $1 \leqslant j \leqslant q$, is also a covering and $Z \cap B_j$ is holomorphically equivalent to a plane crossing, $1 \leqslant j \leqslant q$. In view of the bumps lemma we can find an increasing family of pseudoconvex domains $\{D_j\}$, $0 \leqslant j \leqslant q$, with C^∞-smooth boundary, such that $D_0 = D$, $D_{j-1} \cup \{\zeta_j\} \subset D_j \subset D_{j-1} \cup B(\zeta_j, \rho/2)$ and $\bar{D} \subset D_q \Subset D'$.

Let $\alpha_1 \in C^\infty(\bar{D})$ be such that $\bar{\partial} \alpha_1 = F$; in particular on $Z \backslash \text{Sing}\, Z$ we have

$\bar{\partial}\alpha_1 = 0$ and so in view of a result of Malgrange [13] $\alpha_1|_Z$ is holomorphic. From part (1) we deduce that there exists $g_1 \in A^{\infty}(D \cap B_1)$ such that $g_1|_{Z' \cap B_1} = \alpha_1$; it follows that $u_1 = \alpha_1 - g_1 \in C^{\infty}(\overline{D \cap B_1})$, $\bar{\partial}u_1 = F$ on $D \cap B_1$ and $u_1 = 0$ on $Z' \cap B_1$. Let $\eta_1 \in C_0^{\infty}(B_1)$ be such that $\eta_1 = 0$ on $B(\zeta_1, \rho/2)$ and set $F_1 = \bar{\partial}[(1-\eta_1)u_1]$. We have the following: F_1 is C^{∞}-smooth on $\bar{D}$, $\bar{\partial}F_1 = 0$ and $F_1 = 0$ on $Z' \cap D_1$. Moreover on $\bar{D}$ we have $F_1 = F - \bar{\partial}\beta_1$, where $\beta_1 = \eta_1 u_1 \in C^{\infty}(\bar{D})$ and $\beta_1|_Z = 0$. By iteration we get a (0, 1)-form F_q, C^{∞}-smooth on $\bar{D}_q$ such that $\bar{\partial}F_q = 0$, $F_q = 0$ on $Z' \cap D_q$ and on $\bar{D}$, $F_q = F - \bar{\partial}\beta_q$, where $\beta_q \in C^{\infty}(\bar{D})$ and $\beta_q|_Z = 0$.

Now let $\gamma \in C^{\infty}(D_q)$ be such that $\bar{\partial}\gamma = F_q$; then $\gamma|_{Z' \cap D_q}$ is holomorphic. Let G be holomorphic on D_q and such that $G = \gamma$ on $Z' \cap D_q$ and set $u = \gamma - G + \beta_q$. We have $u \in C^{\infty}(\bar{D})$, $\bar{\partial}u = F$ and u vanishes on Z.

Thus in order to establish Theorem 4 we only have to prove the following:

LEMMA 5. *Let V_1, V_2 be germs of analytic submanifolds at $p \in \mathbb{C}^n$ such that*

(i) *$V_1 \cap V_2$ is smooth,*

(ii) $T_p^{\mathbb{C}}(V_1 \cap V_2) = T_p^{\mathbb{C}}(V_1) \cap T_p^{\mathbb{C}}(V_2)$.

Then we can choose complex coordinates $z_1, \ldots, z_n$ in a neighborhood U of p in such a way that

$$V_1 \cap U = \{z \in \mathbb{C}^n : z_1 = \cdots = z_k = 0\},$$
$$V_2 \cap U = \{z \in \mathbb{C}^n : z_s = \cdots = z_m = 0, s \leqslant k+1\}.$$

Proof. We shall consider first the case where $\dim_{\mathbb{C}}(T_p^{\mathbb{C}}(V_1) + T_p^{\mathbb{C}}(V_2)) = n$.

Let V_1 be defined (in a neighborhood U of p) by $z_1 = \cdots = z_{n-d} = 0$ and V_2 by $f_1 = \cdots = f_{n-k} = 0$; it follows that $V_1 \cap V_2$ has dimension $d + k - n$ and $\partial f_1 \wedge \cdots \wedge \partial f_{n-k} \wedge \partial z_1 \wedge \cdots \wedge \cdots \wedge \partial z_{n-d}(p) \neq 0$. Then there is a system of local (holomorphic) coordinates $\zeta_1, \ldots, \zeta_n$ such that $\zeta_j = z_j$ for $1 \leqslant j \leqslant k$ and $\zeta_{k+1} = f_1, \ldots, \zeta_n = f_{n-k}$.

In order to reduce the general case to the previous one we only have to check that under our assumptions $T_p^{\mathbb{C}}(V_1) + T_p^{\mathbb{C}}(V_2)$ is the Zariski tangent space $T_p^{\mathbb{C}}(V_1 \cup V_2)$ of $V_1 \cup V_2$ at p. We have $T_p^{\mathbb{C}}(V_1) + T_p^{\mathbb{C}}(V_2) \subset T_p^{\mathbb{C}}(V_1 \cup V_2)$: in order to prove the opposite inclusion we must prove the following: given a $\mathbb{C}$-linear map $L: \mathbb{C}^n \to \mathbb{C}$ vanishing on $T_p^{\mathbb{C}}(V_1) + T_p^{\mathbb{C}}(V_2)$ there is a local holomorphic function g such that $g = 0$ on $V_1 \cup V_2$ and $\partial g(p) = L$.

With the above notations we may suppose that on U

$$V_1 = \{z \in U : z_j = 0, 1 \leqslant j \leqslant k\},$$
$$V_2 = \{z \in U : f_s(z) = \cdots = f_m(z) = 0, s \leqslant k+1\}$$

and

$$V_1 \cap V_2 = \{z \in U: z_1 = \cdots = z_k = f_{k+1}(z) = \cdots = f_m(z) = 0\},$$

where

$$\partial f_s \wedge \cdots \wedge \partial f_m(p) \neq 0, \qquad \partial z_1 \wedge \cdots \wedge \partial z_k \wedge \partial f_{k+1} \wedge \cdots \wedge \partial f_m(p) \neq 0$$

and $\dim_{\mathbb{C}}(T_p^{\mathbb{C}}(V_1) + T_p^{\mathbb{C}}(V_2)) = n + s - k - 1$.

In particular

$$V_1 \cap V_2 = \{z \in V_1 : f_{k+1}(z) = \cdots = f_m(z) = 0\}.$$

Let $\tilde{g}$ be a local holomorphic function such that $\partial \tilde{g}(p) = L$ and $\tilde{g}|_{V_2} = 0$; set $\tilde{g}_1 = \tilde{g}|_{V_1}$.

Then $\tilde{g}_1 = \sum_{j=k+1}^m \alpha_j f_j$, where $\alpha_j \in \mathcal{O}(V_1)$, and

$$\partial_{V_1} \tilde{g}_1(p) = \partial \tilde{g}(p) |\, T_P^{\mathbb{C}}(V_1) = L \,|\, T_P^{\mathbb{C}}(V_1) = 0$$

∂_{V_1} being the ∂-operator for V_1 and so $\alpha_j(p) = 0$, $k+1 \leqslant j \leqslant m$. Extend α_j by A_j (holomorphically), $k+1 \leqslant j \leqslant m$, and set $G = \sum_{j=k+1}^m A_j f_j$; we have $G|_{V_1} = \tilde{g}_1$, $G|_{V_2} = 0$, $\partial G(p) = 0$. The function we are looking for is now $g = \tilde{g} - G$.

3. We are now in a position to prove our main theorem.

Let $D \subset \mathbb{C}^n$ be a bounded domain with C^∞-smooth boundary bD and let V' be a complex analytic subvariety of an open neighborhood D' of D such that $V = V' \cap D \neq \varnothing$. Let $f_1, \ldots, f_k$ be a complete defining system for V'.

THEOREM 6. *Assume*

(i) *D is strongly pseudoconvex,*

(ii) Sing $V' \cap bD = \varnothing$ *and V', bD are regularly separated.*

Then every $f \in A^\infty(D)$ vanishing on V can be written as

$$f = \sum_{j=1}^k h_j f_j,$$

where $h_1, \ldots, h_k \in A^\infty(D)$.

Proof. Consider the holomorphic map $F: D' \to \mathbb{C}^k$ given by $F(z) = (f_1(z), \ldots, f_k(z))$ and let Γ be its graph. Consider in $D' \times \mathbb{C}^k$ a bounded domain B with C^∞-smooth boundary, strongly pseudoconvex and such that $B \cap (D' \times \{0\}) = D$ and Γ intersects bB transversally. From the fact

$\{f_1,\dots,f_k\}$ is a complete defining system for V' it follows that the jacobian matrix of F has rank n-$\dim_{\mathbb{C}} V'$ near bD and in particular Γ intersects D' transversally along bB. Let $Z=(D'\cup\Gamma)\cap B$ and let $f\in A^{\infty}(D)$ be such that $f=0$ on V.

Let $\tilde{f}$ be defined by: $\tilde{f}=f$ on D and $\tilde{f}=0$ on $\Gamma\cap B$; because of the transversality, $\tilde{f}$ is holomorphic on Z and $\tilde{f}\in A^{\infty}(Z)$.

By Theorem 4 applied to Z we can find $G\in A^{\infty}(B)$ such that $G|_Z=\tilde{f}$. In particular $G=0$ on $\Gamma\cap B$. Now $\Gamma\cap B$ is holomorphically equivalent to a plane section and thus, using Proposition 1 we can find $\tilde{h}_1,\dots,\tilde{h}_k\in A^{\infty}(B)$ such that $G=\sum_{j=1}^{k}\tilde{h}_j(f_j-w_j)$ ($w_1,\dots,w_k$ complex coordinates in $\mathbb{C}^k$). By restriction to $\bar{D}$ we get $f=\sum_{j=1}^{k}h_jf_j$, where $h_j=\tilde{h}_j|_{\bar{D}}$. This concludes the proof of our main theorem.

COROLLARY 7. *In the above hypothesis assume $f\in A^{\infty}(V)$ vanishes on V of order $q>0$. Then f can be written as*

$$f=\sum_{j_1+\cdots+j_k=q} h_{j_1\cdots j_k} f_1^{j_1}\cdots f_k^{j_q},$$

where $h_{j_1\cdots j_k}\in A^{\infty}(D)$, $j_1+\cdots+j_k=q$.

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