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ON GENERALISED MINIMAL DOMINATION PARAMETERS FOR PATHS

B. BOLLOBÁS
Trinity College, Cambridge, UK

E.J. COCKAYNE
University of Victoria, B.C., Canada

C. M. MYNHARDT
University of South Africa, Pretoria, South Africa

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A subset X of vertices of a graph is a *k-minimal P-set* if X has property P, but the removal of any l vertices from X, where $l \leq k$, followed by the addition of any $(l-1)$ vertices destroys the property P. We note that 1-minimality is the usual minimality concept. In this paper we determine $\Gamma_k(P_n)$, the largest cardinality of a k-minimal dominating set of the n-vertex path P_n. We also prove for any n-vertex graph G, $\Gamma_2(G)\gamma(\bar{G}) \leq n$ and finally a 'Gallai-type' theorem for k-minimal parameters is established.

1. Introduction

The concept of minimality may be generalised as follows. Let S be a set and P a property enjoyed by some of the subsets of S. A subset of S with (without) property P is called a P-set ($\bar{P}$-set). A subset X of S is called a *k-minimal P-set* if X has the property P, but for all l satisfying $1 \leq l \leq k$, all l-subsets U of X and all $(l-1)$-subsets R of S, $(X-U)\cup R$ is a $\bar{P}$-set. We note that 1-minimality is the usual concept of minimality. In this paper the set S will be the vertex set V of a graph and a subset $X \subseteq V$ has property P if and only if X is dominating. This specialization of the above defines a *k-minimal dominating set of a graph*. We define $\Gamma_k(G)$ to be the largest cardinality of a k-minimal dominating set of G. Let $\gamma(G)$ ($\Gamma(G)$) be the smallest (largest) cardinality of a minimal dominating set of G. The following inequalities are obvious for any graph G:

$$\gamma(G) \leq \cdots \leq \Gamma_k(G) \leq \cdots \leq \Gamma_2(G) \leq \Gamma_1(G) = \Gamma(G).$$

In this work, we first strengthen the theorem of Jaeger and Payan [6] which asserts that the product of domination numbers of a graph and its complement is at most the number of vertices in the graph. The examination of their proof, in fact, motivated the new extended definitions of minimality given here. The principal result of the paper is the exact determination of $\Gamma_k(P_n)$. This calculation is surprisingly complex although the evaluations of $\gamma(P_n)$ and $\Gamma(P_n)$ are trivial.

It is clear that many other parameters of graphs and more general structures, which are defined in terms of minimality may also be similarly generalized and that the maximality concept may also be extended. In [2], Cockayne, MacGillivray and Mynhardt compute $\beta_k(P_n)$ and $\beta_k(C_n)$ where $\beta_k(G)$ is the smallest cardinality of a k-maximal independent set of vertices of G. The final result of this paper is a generalization of a theorem of Gallai [3] concerning certain k-maximal and k-minimal parameters.

For an excellent bibliography of the study of domination in graphs, the reader is referred to [5]. Extensions of Gallai's Theorem are given in [1].

2. The results

2.1. *The Jaeger-Payan generalisation*

In [6] Jaeger and Payan proved the following Nordhaus–Goddum type result for the domination number.

Theorem 1 (Jaegar and Payan). *For any n vertex graph G, $\gamma(G)\gamma(\bar{G}) \leqslant n$.*

We show here that their proof may be adapted to prove the stronger result:

Theorem 2. *For any n-vertex graph G, $\Gamma_2(G)\gamma(\bar{G}) \leqslant n$.*

Proof. The result is trivial for $\Gamma_2(G) = 1$, hence we assume $\Gamma_2(G) = t \geqslant 2$. Let $X = \{x_1, \ldots, x_t\}$ be a largest 2-minimal dominating set. Since X is dominating, there exists a partition of $V(G)$ into classes $V_1, \ldots, V_t$, such that for each $i = 1, \ldots, t$, $x_i \in V_i$ and x_i is adjacent to all other vertices in V_i. Let $\mathcal{P}$ be such a partition such that the number of vertices which are adjacent to all other vertices in their class, is maximum. We show that each class of $\mathcal{P}$ is a dominating set of $\bar{G}$.

For suppose V_1 (say) does not dominate $\bar{G}$. Then there exists a vertex x which is in V_2 (say) and is adjacent in G to all vertices of V_1. Vertex x is not adjacent to all vertices of V_2, for otherwise $(X - \{x_1, x_2\}) \cup \{x\}$ dominates G, contrary to 2-minimality. Therefore $x \neq x_2$ and we now consider the partition $\mathcal{P}' = V_1', \ldots, V_t'$ where $V_1' = V_1 \cup \{x\}$, $V_2' = V_2 - \{x\}$, $V_j' = V_j$ for $2 < j \leqslant t$. We have $x_i \in V_i'$ for $i = 1, \ldots, t$ and $\mathcal{P}'$ has at least one extra vertex (the vertex x) which is adjacent to all other members of its class. This contradicts the maximum property of $\mathcal{P}$. Thus each V_i dominates $\bar{G}$ and

$$n = \sum_{i=1}^{t} |V_i| \geqslant \sum_{i=1}^{t} \gamma(\bar{G}) = \Gamma_2(G)\gamma(\bar{G}). \qquad \square$$

The *domatic number* $d(G)$ of G is the largest order of a partition of $V(G)$ into dominating sets of G. The following deduction from the proof of Theorem 2 is obvious.

Corollary 1. *For any graph G, $\Gamma_2(G) \leqslant d(\bar{G})$.*

2.2. *The calculation of $\Gamma_k(P_n)$*

Define an l-subset Q of a dominating set X of a graph G to be *stable* (*unstable*) if and only if there does not exist (there exists) an $(l-1)$-subset R of $V - X$ such that $(X - Q) \cup R$ is dominating. Notice that a dominating set X is k-minimal if for each $1 \leqslant l \leqslant k$, all l-subsets of X are stable. Let P_n have the (left to right) vertex sequence $v_1, \ldots, v_n$. The simple proof of the following proposition is omitted.

Proposition 1. *If X is a $(k-1)$-minimal but not k-minimal dominating set of P_n, then any unstable k-set in X consists of consecutive vertices of X, i.e. if $X = \{v_{m_1}, v_{m_2}, \ldots, v_{m_t}\}$ where $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_t$, then any unstable k-set in X may be written as $\{v_{m_{i+1}}, v_{m_{i+2}}, \ldots, v_{m_{i+k}}\}$ for some i.*

In what follows $P_n\langle i, m\rangle$ where $1 \leqslant i \leqslant m \leqslant n$, will denote the subgraph of P_n induced by the vertex subset $\{v_i, v_{i+1}, \ldots, v_m\}$. Our first Lemmas 1(a) and (b) give necessary conditions for consecutive vertices of P_n to appear in a k-minimal dominating set.

Lemma 1(a). *For any $k, n \geqslant 1$, if X is a k-minimal dominating set of P_n such that $\{v_{i-1}, v_i\} \subseteq X$ for some i, then exactly one of the following holds:*

(i) *$n < i + 3k$, in which case $n = i + 3r + 1$, where $0 \leqslant r \leqslant k-1$ and*

$$V(P_n\langle i, n\rangle) \cap X = \{v_{i+3} \mid j = 0, 1, \ldots, r\};$$

(ii) *$n \geqslant i + 3k$ and*

$$V(P_n\langle i, i+3k\rangle) \cap X = \{v_{i+3j} \mid j = 0, \ldots, k\}.$$

Proof. By induction on k. Let $k = 1$ and let X be a 1-minimal (i.e. minimal) dominating set of P_n with $\{v_{i-1}, v_i\} \subseteq X$ for some i. Suppose $n < i + 3$. If $n = i$, then X is not minimal, hence $n \geqslant i + 1$. If $n = i + 2$ then $v_{i+2} = v_n$ is dominated, therefore v_{i+1} or v_{i+2} is in X. In either case X is not minimal so the case $n = i + 2$ cannot occur. Therefore $n = i + 1$ and (i) holds. If $n \geqslant i + 3$, it is easily seen that v_{i+1} or v_{i+2} in X contradicts minimality and hence v_{i+3} must be in X so that v_{i+2} is dominated. Hence the assertion (ii) holds.

Now suppose that the result is true for $k-1$ where $k \geqslant 2$ and let X be a k-minimal dominating set of P_n with $\{v_{i-1}, v_i\} \subseteq X$.

Case 1. Let $n < i + 3(k-1)$. Since X is $(k-1)$-minimal, by the induction hypothesis, condition (i) holds.

Case 2. Let $i + 3(k-1) \leqslant n$. By the induction hypothesis,

$$V(P_n\langle i, i+3(k-1)\rangle) \cap X = \{v_{i+3j} \mid j = 0, \ldots, k-1\}. \tag{1}$$

If $n = i + 3(k-1)$, then

$$X' = (X - \{v_{i+3j} \mid j = 0, \ldots, k-1\}) \cup \{v_{i+3j+2} \mid j = 0, \ldots, k-2\}$$

dominates P_n, which contradicts the k-minimality of X. If $n = i + 3(k-1) + 2$, then since v_n is dominated, v_n or v_{n-1} is in X. Again X' dominates and we conclude that either $n = i + 3(k-1) + 1$ in which case (i) is satisfied, or $n \geq i + 3k$. In this case, it follows from (1) and the 1-minimality of X, that neither $v_{i+3(k-1)+1}$ nor $V_{i+3(k-1)+2}$ is in X. Therefore $v_{i+3k} \in X$ to dominate its predecessor and (ii) holds. This completes the proof of Lemma 1(a). $\square$

Lemma 1(a) has a 'dual' form concerning the vertices to the left of a consecutive pair v_{i-1}, v_i in a k-minimal dominating set. The proof is similar to that of Lemma 1(a) and is omitted.

Lemma 1(b). *For any $k, n \geq 1$, if X is a k-minimal dominating set of P_n such that $\{v_{i-1}, v_i\} \subseteq X$ for some i, then exactly one of the following holds:*

(i) *$i - 1 - 3k < 1$ (i.e. $i < 2 + 3k$), in which case $1 = i - 1 - 3r - 1$ (i.e. $i = 3r$) for some r, $1 \leq r \leq k$, and*

$$V(P_n\langle 1, i-1\rangle) \cap X = \{v_{i-1-3j} \mid 0 \leq j \leq r-1\};$$

(ii) *$i - 1 - 3k \geq 1$ (i.e. $i \geq 3k + 2$) and*

$$V(P_n\langle i-1-3k, i-1\rangle) \cap X = \{v_{i-1-3j} \mid 0 \leq j \leq k\}.$$

By a *Γ_k-set* we mean a k-minimal dominating set of largest cardinality.

Theorem 3. *For any $k, n \geq 1$, P_n has a Γ_k-set which is independent.*

Proof. Let n and k be such that the statement is false. Since the result follows easily if $k = 1$, it is clear that $k \geq 2$. Let X be a Γ_k-set in P_n such that $P_n\langle X\rangle$ has as few edges as possible and let i be the largest integer such that $v_{i-1}, v_i \in X$. By the choice of X and Lemma 1, respectively, neither v_{i+1} nor v_{i+2} is an end vertex of P_n and $\{v_{i+1}, v_{i+2}\} \cap X = \emptyset$; hence $n \geq i + 3$ and $v_{i+3} \in X$. Consider the set $X' = (X - \{v_i\}) \cup \{v_{i+1}\}$ which clearly dominates P_n. By the choice of X, X' is not k-minimal. Hence there is a smallest integer $l \leq k$ for which there exists an l-set $Q \subseteq X'$ such that $(X' - Q) \cup R$ dominates P_n for some $(l-1)$-set $R \subseteq V(P_n)$.

Suppose $v_{i+1} \in Q$. By the choice of l, $v_{i+1} \notin R$. But then $(X - ((Q - \{v_{i+1}\}) \cup \{v_i\})) \cup R = (X' - Q) \cup R$ which dominates P_n, contradicting the k-minimality of X. Hence $v_{i+1} \notin Q$. By the choice of l and by Proposition 1, either $Q \subseteq \{v_1, v_2, \ldots, v_{i-1}\}$ or $Q \subseteq \{v_{i+3}, v_{i+4}, \ldots, v_n\}$. If $Q \subseteq \{v_1, v_2, \ldots, v_{i-1}\}$, then since $(X' - Q) \cup R$ dominates P_n, it is clear that $(X - Q) \cup R$ dominates P_n.

Hence $Q \subseteq \{v_{i+3}, v_{i+4}, \ldots, v_n\}$. If $v_{i+3} \notin Q$ then $v_{i+3} \in (X' - Q) \cup R$ in which case $(X - Q) \cup R$ dominates P_n.

Hence, again by Proposition 1 and Lemma 1, $Q = \{v_{i+3j} \mid j = 1, 2, \ldots, l\}$. By the choice of i, $v_{i+3l+1} \notin X'$. Hence at least $3l - 3 + 1$ vertices of P are not dominated by $X' - Q$ and since $\gamma(P_{3l-3+1}) = \lceil (3(l-1)+1)/3 \rceil$, no $(l-1)$-set

$R \subseteq V(P_n)$ exists such that $(X' - Q) \cup R$ dominates P_n. This contradiction completes the proof. □

We now state and prove the principal result of the paper.

Theorem 4. *For all $k \geqslant 1$ and $n \geqslant 2$,*

$$\Gamma_k(P_n) = \begin{cases} \left\lceil \dfrac{(k+1)n}{3k+1} \right\rceil & \text{if } n \equiv 3l + 1 \pmod{3k+1} \\ & \text{for some } l \in \{0, 1, \ldots, k-1\}, \\ \left\lfloor \dfrac{(k+1(n+1)}{3k+1} \right\rfloor & \text{otherwise.} \end{cases}$$

Proof of lower bound. Write n as $n = h + r$, where $h = m(3k+1)$ for some integer m, and $0 < r \leqslant 3k$. Let

$$X = \{v_i \in V(P_n) \mid i \equiv 1 \pmod{3k+1} \text{ or } i \equiv 3l \pmod{3k+1},\ 1 \leqslant l \leqslant k\}.$$

Clearly, X dominates P_n unless $r = 3l + 2$ for some l, $1 \leqslant l \leqslant k - 1$. Define

$$Y = \begin{cases} (X - \{v_{h+3j} \mid 1 \leqslant j \leqslant l\}) \cup \{v_{h+3j+1} \mid 1 \leqslant j \leqslant l\} \\ \quad \text{if } r = 3l + 2 \text{ for some } l,\ 1 \leqslant l \leqslant k-1, \\ X \quad \text{otherwise.} \end{cases}$$

Then Y dominates P_n for all n. Moreover, we show that Y is k-minimal unless $r = 3l$ for some l, $1 \leqslant l \leqslant k - 1$.

In order to see this, suppose firstly that $r = 3l$ for some l, $1 \leqslant l \leqslant k - 1$. Let $Q = \{v_{h+1}\} \cup \{v_{h+3j} \mid 1 \leqslant j \leqslant l\}$. Then $|Q| = l + 1 \leqslant k$ and exactly $3l$ vertices of P_n, the vertices of $P_n\langle h+1, n\rangle$, are not dominated by $Y - Q$. But $\gamma(P_{3l}) = \lceil 3l/3 \rceil = l$ and hence there exists an l-set R such that $(Y - Q) \cup R$ dominates P_n. In particular, $R = \{v_{h+3j-1} \mid 1 \leqslant j \leqslant l\}$.

Now suppose $r \neq 3l$ for any l, $1 \leqslant l \leqslant k - 1$. Let S be any subset of Y consisting of k consecutive vertices of Y. Note that at least $3(k-1) + 1$ consecutive vertices of P_n are not dominated by $Y - S$ and since $\gamma(P_{3(k-1)+1}) = \lceil (3(k-1)+1)/3 \rceil = k$, no set T with fewer than k vertices exists such that $(Y - S) \cup T$ dominates P_n. Hence S is stable and therefore any subset of Y consisting of fewer than k consecutive vertices of P_n is also stable. Now let s be the smallest integer such that Y is not s-stable. By Proposition 1, any unstable s-subset of Y consists of s consecutive vertices of Y. Since any set of k or fewer consecutive vertices of Y is stable, $s > k$ and Y is k-minimal as asserted.

In view of the above, define

$$Z = \begin{cases} (Y - Q) \cup R & \text{if } r = 3l \text{ for some } l,\ 1 \leqslant l \leqslant k-1, \\ Y & \text{otherwise,} \end{cases}$$

where Q and R are as above. Z is a k-minimal dominating set of P_n for all n; hence $\Gamma_k(P_n) \geqslant |Z|$. Moreover,

$$|Z| = (k+1)m + |Z \cap V(P_n\langle h+1, n\rangle)| \tag{2}$$

(where $n = h + r$ with $h = m(3k + 1)$ and $0 \leqslant r \leqslant 3k$), and

$$|Z \cap V(P_n\langle h+1, n\rangle)| = \begin{cases} l & \text{if } r = 3l \text{ for some } l,\ 1 \leqslant l \leqslant k-1, \\ k+1 & \text{if } r = 3k, \\ l+1 & \text{if } r = 3l+2 \text{ for some } l,\ 0 \leqslant l \leqslant k-1, \\ l+1 & \text{if } r = 3l+1 \text{ for some } l,\ 0 \leqslant l \leqslant k-1. \end{cases} \tag{3}$$

It is easily verified from (2) and (3), that in the first three cases of (3), $|Z| = \lfloor (k+1)(n+1)/(3k+1) \rfloor$ and in the fourth case $|Z| = \lceil (k+1)n/(3k+1) \rceil$. This completes the proof of the lower bound.

Proof of upper bound. We first prove the following lemma.

Lemma 2. *If X is an independent Γ_k set of P_n, then among any $3k+1$ consecutive vertices of P_n, at most $k+1$ are in X.*

Proof. Suppose there are $3k+1$ consecutive vertices $S = \{v_i, v_{i+1}, \ldots, v_{i+3k}\}$ of P_n such that $|S \cap X| \geqslant k+2$. Let j (l respectively) be the smallest (largest) integer such that $v_j \in S \cap X$ ($v_l \in S \cap X$) and let $S' = \{v_{j+2}, v_{j+3}, \ldots, v_{l-2}\}$. Then $|S' \cap X| = |S \cap X| - 2 \geqslant k$ while $|S'| \leqslant 3k - 3$. But $\gamma(P_{3k-3}) = \lceil (3k-3)/3 \rceil = k - 1$ and therefore there exists a set R with $|R| \leqslant k - 1$ which dominates S'. Clearly $(X - (S' \cap X)) \cup R$ dominates P_n, contradicting the k-minimality of X. Hence the lemma is proved. □

To continue with the proof of the upper bound, we again put $n = h + r$ with $h = m(3k+1)$ for some integer m, and $0 \leqslant r \leqslant 3k$. Suppose, contrary to the result,

$$\Gamma_k(P_n) > \begin{cases} \left\lceil \dfrac{(k+1)n}{3k+1} \right\rceil & \text{if } r = 3l+1 \text{ for some } l,\ 0 \leqslant l \leqslant k-1, \\ \left\lfloor \dfrac{(k+1)(n+1)}{3k+1} \right\rfloor & \text{otherwise,} \end{cases}$$

i.e.,

$$\Gamma_k(P_n) \geqslant \begin{cases} (k+1)m + l + 1 & \text{if } r = 3l \text{ for some } l,\ 1 \leqslant l \leqslant k-1, \\ (k+1)m + l + 2 & \text{if } r = 3l+1 \text{ for some } l,\ 0 \leqslant l \leqslant k-1, \\ (k+1)m + l + 2 & \text{if } r = 3l+2 \text{ for some } l,\ 0 \leqslant l \leqslant k-1, \\ (k+1)m + k + 2 & \text{if } r = 3k. \end{cases}$$

By Theorem 3, P_n has an independent Γ-set X. Let $X' = X \cap V(P_n\langle 1, h\rangle)$. By Lemma 2, $|X'| \leqslant (k+1)m$, which implies

$$|X - X'| \geqslant \begin{cases} l+1 & \text{if } r = 3l \text{ for some } l,\ 1 \leqslant l \leqslant k-1, \\ l+2 & \text{if } r = 3l+1 \text{ for some } l,\ 0 \leqslant l \leqslant k-1, \\ l+2 & \text{if } r = 3l+2 \text{ for some } l,\ 0 \leqslant l \leqslant k-1, \\ k+2 & \text{if } r = 3k. \end{cases}$$

The last case, i.e. $|X - X'| \geq k + 2$ for $r = 3k$, contradicts Lemma 2 applied to $P_n\langle h, n\rangle$. Hence three cases remain.

Case 1. Let $r = 3l + 1$ for some $l, 0 \leq l \leq k - 1$ and consider the $3(k - l)$ vertices of the subgraph $P_n\langle h - 3(k - l) + 1, h\rangle$ of P_n. At least $3(k - l) - 2$ of these are not dominated by $X - V(P_n\langle h - 3(k - l) + 1, h\rangle)$ and therefore

$$|X' \cap V(P_n\langle h - 3(k - l) + 1, h\rangle)| \geq \left\lceil \frac{3(k - l) - 2}{3} \right\rceil = k - l.$$

But then

$$|X \cap V(P_n\langle h - 3(k - l) + 1, n\rangle)| \geq k - l + l + 2 = k + 2$$

while $|V(P_n\langle h - 3(k - l) + 1, n\rangle)| = 3k + 1$, contradicting Lemma 2.

Case 2. Let $r = 3l$ where $1 \leq l \leq k - 1$. If $v_h \in X'$ or $v_{h-1} \in X'$, then $X - X'$ is unstable since $\gamma(P_{3l}) = \lceil 3l/3 \rceil = l < l + 1$. Hence $\{v_{h-1}, v_h\} \cap X' = \emptyset$ and therefore $v_{h-2} \in X'$, $v_{h+1} \in X - X'$. Consider the $3(k - l) + 1$ vertices of $P_n\langle h - 3(k - l), h\rangle$. At least $3(k - l) - 1$ of these are not dominated by $X - V(P_n\langle h - 3(k - l), h\rangle)$ and therefore

$$|X' \cap V(P_n\langle h - 3(k - l), h\rangle)| \geq \left\lceil \frac{3(k - l) - 1}{3} \right\rceil = k - l.$$

If $|X'| < (k + 1)m$, then $|X - X'| \geq l + 2$ so that

$$|X \cap V(P_n\langle h - 3(k - l), n\rangle)| \geq k - l + l + 2 = k + 2$$

while $|V(P_n\langle h - 3(k - l), n\rangle)| = 3k + 1$, contradicting Lemma 2. Hence $|X'| = (k + 1)m$. Since $\{v_{h-1}, v_h\} \cap X' = \emptyset$,

$$|X' \cap V(P_n\langle 1, h - 2\rangle)| = (k + 1)m.$$

Consider the subgraphs $P_n\langle 1, 3k - 1\rangle$ and $P_n\langle 3k, h - 2\rangle$ of P_n. Clearly, $P_n\langle 3k, h - 2\rangle$ has order $(3k + 1)(m - 1)$ so that

$$|X' \cap V(P_n\langle 3k, h - 2\rangle)| \leq (k + 1)(m - 1) \text{ by Lemma 2.}$$

Therefore $|X' \cap V(P_n\langle 1, 3k - 1\rangle)| \geq k + 1$.

Let j be the largest integer such that $v_j \in X' \cap V(P_n\langle 1, 3k - 1\rangle) = X''$ and consider $P_n\langle 1, j - 2\rangle$. Clearly $X'' \cap V(P_n\langle 1, j - 2\rangle) = Q$ satisfies $|Q| \geq k$, while $P_n\langle 1, j - 2\rangle$ has order at most $3k - 3$. Since $\gamma(P_{3k-3}) = k - 1$, There exists a set R containing $k - 1$ vertices which dominates $P_n\langle 1, j - 2\rangle$. But then $(X - Q') \cup R$ dominates P_n, where Q' is any k-subset of Q, contradicting the k-minimality of X.

Case 3. Let $r = 3l + 2$ where $0 \leq l \leq k - 1$. If $\{v_{h-2}, v_{h-1}, v_h\} \cap X \neq \emptyset$, then at most $3l + 3$ vertices remain to be dominated by $Q = X - X'$ which has at least $l + 2$ vertices. But $\gamma(P_{3l+3}) = l + 1$; hence there exists an $(l + 1)$-element subset R of $V(P_n)$ such that $(X - Q) \cup R$ dominates P_n, contradicting the k-minimality of X if $l \leq k - 2$. Hence in this case, $\{v_{h-2}, v_{h-1}, v_h\} \cap X' = \emptyset$ implying that v_{h-1} is not dominated which is impossible. If $l = k - 1$, then by Lemma 2, $\{v_{h-1}, v_h\} \cap X' = \emptyset$ and $|X - X'| = l + 2 = k + 1$, so that $v_{h-2} \in X'$ and $|X'| = (k+1)m$. As in Case 2, a contradiction of the k-minimality of X can now be obtained. □

2.3. *A generalization of Gallai's Theorem*

The point covering number $\alpha(G)$ of a graph G (i.e. smallest number of vertices which cover all the edges) and the independence number $\beta(G)$ (i.e. largest cardinality of an independent set of vertices) are related by the well-known result of Gallai [3].

Theorem 5 (Gallai). *For any n vertex graph, $\alpha(G) + \beta(G) = n$.*

In order to generalize this result we need three definitions. Let P be a property associated with the subsets of a set S. The subset X of S is a *k-maximal P-set* if X is a P-set but the addition of any l elements to X where $l \leq k$, followed by the removal of any $l - 1$ elements, yields a $\bar{P}$-set.

Let $Y \subseteq S$ be a *Q-set* if and only if it intersects every $\bar{P}$-set (i.e. it is a transversal of the family of $\bar{P}$-sets.

Finally property P is *hereditary* if each subset of a P-set is also a P-set.

Theorem 6. *Let P be a hereditary property. Then X is a k-maximal P-set if and only if $S - X$ is a k-minimal Q-set.*

Proof. Let X be a k-maximal P-set and $Y = S - X$. Y is a transversal of the $\bar{P}$-sets for otherwise there is a $\bar{P}$-set entirely contained in X contrary to the hereditary property. Suppose Y is not a k-minimal Q-set. Then for some l-subset T of Y where $l \leq k$ and an $(l-1)$-subset U of $S - Y$, $(Y - T) \cup U$ is a Q-set.

Consider the set $(X \cup T) - U$. It is not a $\bar{P}$-set since it does not intersect the Q-set $(Y - T) \cup U$. Hence $(X \cup T) - U$ is a P-set which contradicts the k-maximality of X. Therefore Y is a k-minimal Q-set. The proof of the converse is similar and omitted. □

If $\alpha_k(S, P)$ and $\beta_k(S, P)$ denote the smallest cardinality of a k-minimal Q-set and the largest cardinality of a k-maximal P-set, we have immediately:

Corollary 2. *If P is a hereditary property on the subsets of S, then*

$$\alpha_k(S, P) + \beta_k(S, P) = |S|.$$

In the special case where $k = 1$, $S = V(G)$ and P-sets are independent sets of vertices, Corollary 2 reduces to Gallai's Theorem.

Finally we note that Hedetniemi [4] has also obtained some generalisations of Gallai's Theorem. Most of these may be deduced from Corollary 2 by taking $k = 1$ and P to be a suitable hereditary property associated with the subsets of the set S which is either the vertex set or edge set of a graph.

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