Operator Valued Weights in von Neumann Algebras, II

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Let $M$ be a von Neumann algebra, and $N$ a sub von Neumann algebra of $M$. We prove that if $\omega$ and $\psi$ are n.f.s. weights on $N$ and $M$ respectively, such that $\omega$ extends $\psi$, then there is a unique n.f.s. operator valued weight $T$ from $M$ to $N$, such that $\psi = \omega \circ T$. Moreover we generalize the notion of modular automorphism groups associated with conditional expectations to operator valued weights.

Introduction

In [9] we defined operator valued weights from a von Neumann algebra $M$ to a sub von Neumann algebra $N$. We refer the reader to [9] for definitions and notation. The sections of this paper are denoted sections 5 and 6, and all references to sections 1–4 should be understood as references to [9, sects. 1–4].

In the following we list the main results of this paper. In [9, sect. 4] we proved, that when $T$ is a n.f.s. operator valued weight from $M$ to $N$, and $\phi$ is a n.f.s. weight on $N$, then $\phi^T$ extends $\phi$. The first theorem here proves the converse, namely:

**Theorem (5.1).** Let $M$, $N$ be von Neumann algebras $N \subseteq M$, and let $\omega$, $\psi$ be n.f.s. weights on $N$ and $M$ respectively. If $\phi^T(x) = \phi^T(x)$ for any $x \in N$, then there exists a unique n.f.s. operator valued weight $T$ such that $\psi = \omega \circ T$.

The proof of theorem 5.1 relies on a reduction of the problem to the case, where $\phi$, $\psi$ are traces on semifinite algebras by using the crossed products $R(M, \phi^*)$ and $R(N, \psi^*)$, which are semifinite (cf. [15]). From Theorem 5.1 we obtain a simple criterion for semifiniteness of the centralizer of a weight, namely:

**Theorem (5.7).** The centralizer $M_\phi$ of a n.f.s. weight $\phi$ on a von Neumann algebra $M$ is semifinite if and only if there exists a $\phi^*$-invariant n.f.s. operator valued weight from $M$ to $M_\phi$.
It should be noted, that there exists a n.f.s. weight $\phi$ on the hyperfinite type II$_\infty$ factor, such that $M_\phi$ is of type III (cf. [8]).

The following result is perhaps surprising, since it is in marked contrast to what is valid for conditional expectations. Recall that $P(M, N)$ is the set of all n.f.s. operator valued weights from $M$ to $N$, $N \subseteq M$.

**Theorem (5.9).** $P(M, N) \neq \emptyset \iff P(N', M') \neq \emptyset$.

$M'$ and $N'$ are of course the commutant of $M$ and $N$. As a corollary we obtain that there is always a n.f.s. operator valued weight from the algebra $B(H)$ of all bounded operators on a Hilbert space $H$ to a von Neumann algebra $N$ on $H$. Note that there is a normal conditional expectation from $B(H)$ to $N$ if and only if $N$ is a direct sum of type I factors (cf. [11]). In the last section (Section 6) we generalize some recent results due to Combes and Delaroche on conditional expectations (cf. [2]). The modular automorphism group $\sigma^T$, $T \in P(M, N)$ on $N^c = N' \cap M$ and the cocycle Radon Nikodym derivatives $(DT_2 : DT_1)$, $T_1, T_2 \in P(M, N)$ can be defined exactly as in [2].

**Theorem (6.5).** Let $T_0$ be a fixed element in $P(M, N)$. The map $T \to (DT : DT_0)$ is a bijection of $P(M, N)$ onto the set of $\sigma^T$-cocycles in the relative commutant $N^c$.

Not all results from [2] can be generalized to operator valued weights. For instance the restriction map $T \to T | N^c$ is a bijection of $P(M, N)$ onto $P(N^c, Z(N))$ if and only if there is a faithful family of conditional expectations from $M$ to $N$ (cf. Theorem 6.6). In general, however, the restriction of $T$ to $N^c$ need not to be semifinite. Using Connes' and Takesaki's relative commutant Theorem [5, Chap. II, Theorem 5.1] we show:

**Corollary (6.11).** If $M = R(N, \theta)$ is the continuous decomposition of a properly infinite von Neumann algebra due to Takesaki [15], then $P(M, N) \neq \emptyset$, but there is no normal conditional expectation from $M$ to $N$ (we identify $N$ with its natural injection in the crossed product).

### 5. Conditions for Existence of Operator Valued Weights

**Theorem 5.1.** Let $M$ and $N$ be von Neumann algebras, $N \subseteq M$, and let $\phi$ and $\psi$ be n.f.s. weights on $M$ and $N$ respectively, such that $\phi(x) = \psi(x) \forall x \in N$. Then there exists a unique n.f.s. operator valued weight $S$ from $M$ to $N$ such that $\phi = \psi \circ S$.

For the proof we need some results about crossed products from [15]: Let $M$ be a von Neumann algebra, and $\phi$ a n.f.s. weight on $M$. The crossed
product $R(M, \sigma^\phi)$ is generated by $M$ (or more precisely a subalgebra isomorphic to $M$) and a one parameter group of unitaries $\lambda(t), t \in \mathbb{R}$ such that

$$\sigma^\phi(t) = \lambda(t)x \lambda(t)^*, \quad x \in M.$$

There is a dual action $s \mapsto \theta_s^\phi$ of $\mathbb{R}$ on $R(M, \sigma^\phi)$. The automorphisms $\theta_s^\phi$ can be characterized by their action on the generators:

$$\theta_s^\phi(x) = x, \quad x \in M,$$

$$\theta_s^\phi(\lambda(t)) = e^{-ist}\lambda(t), \quad t \in \mathbb{R}.$$

In particular $M \subseteq \{y \in R(M, \sigma^\phi) \mid \theta_s^\phi y = y \ \forall s \in \mathbb{R}\}$. In fact $M$ is exactly the fixed point algebra under $\theta_s^\phi$.

**Lemma 5.2.** In the above situation, put

$$T_x = \int_{-\infty}^{\infty} \theta_s^\phi x \ ds, \quad x \in R(M, \sigma^\phi)_+. $$

Then:

1. $T$ is a n.f.s. operator valued weight from $R(N, \sigma^\phi)$ to $M$.
2. There is a (unique) n.f.s. trace $\tau$ on $R(M, \sigma^\phi)$ such that $(D\phi \circ T : D\tau)_t = \lambda(t), \quad t \in \mathbb{R}$.
3. The trace $\tau$ satisfies $\tau \circ \theta_s^\phi = e^{-s\tau}, \quad s \in \mathbb{R}$.

*Proof.* (1) The above integral defines an element in the extended positive part of $R(M, \sigma^\phi)$ by

$$\langle \phi, T_x \rangle = \int_{-\infty}^{\infty} \phi \circ \theta_s^\phi(x) \ ds, \quad \phi \in R(M, \sigma^\phi)_+. $$

We extend $\theta_s^\phi$ to an automorphism group on $R(M, \sigma^\phi)_+$ as in the proof of proposition 4.9. Since

$$\langle \phi, \theta_s^\phi(T_x) \rangle = \langle \phi \circ \theta_s^\phi, T_x \rangle = \int_{-\infty}^{\infty} \phi \circ \theta_s^\phi \circ \theta_t^\phi(x) \ dt = \langle \phi, T_x \rangle$$

it follows that

$$\theta_s^\phi(T_x) = T_x, \quad s \in \mathbb{R}.$$

Let $T_x = \int_{0}^{\infty} \lambda d\lambda_1 + \infty p$ be the spectral resolution of $T_x$, then $\theta_s^\phi(e_\lambda) = e_\lambda$ and $\theta_s^\phi(p) = p$.

Hence $e_\lambda, \ p \in M$ (fixed point algebra under $\theta_s^\phi$), which proves that $T_x$ can be regarded as an element of $\hat{M}_+$. It is easy to check that $T$ is a normal, faithful operator valued weight from $R(M, \sigma^\phi)$ to $M$. To prove that $T$ is semifinite, we consider $T(\lambda(f) \mu(f))$, where $f$ is a continuous function with compact support,
and \( \lambda(f) = \int_{-\infty}^{\infty} f(t) \lambda(t) \, dt \). Put \( \phi_n(s) = \exp(s^2/2n^2) \), \( n \in \mathbb{N} \), then \( \phi_n(s) \not\to 1 \) for \( n \to \infty \) uniformly on compact sets.

Put
\[
g_n(t) = \int_{-\infty}^{\infty} \exp(s^2/2n^2) e^{-ist} \, ds = (2\pi)^{1/2} n \exp(\pi t^2/2).
\]

Note that \( g_n(t) \geq 0 \) and \( \int_{-\infty}^{\infty} g_n(t) \, dt = 2\pi \). Since by Lebesgue's monotone convergence theorem:
\[
T(\lambda(f)^* \lambda(f)) = \sup_{n \in \mathbb{N}} \int_{-\infty}^{\infty} \theta_s \phi(\lambda(f)^* f) \phi_n(s) \, ds
= \sup_{n \in \mathbb{N}} \int_{-\infty}^{\infty} \lambda(f)^* f(t) e^{-ist} \phi_n(s) \, ds
= \sup_{n \in \mathbb{N}} \lambda((f)^* f) g_n
\]
and \( \| \lambda((f)^* f) g_n \| \leq \| f^* f \|_\infty \cdot \| g_n \|_1 \leq 2\pi \| f \|_2^2 \), it follows that
\[
\| T(\lambda(f)^* \lambda(f)) \| \leq 2\pi \| f \|_2^2.
\]
In particular \( \lambda(f) \in n_T \). Since \( n_T \) is a right module over \( M \) we have
\[
\lambda(f) a \in n_T \quad f, a \in M.
\]
which proves that \( n_T \) is \( \sigma \)-weakly dense in \( R(M, \sigma^\theta) \), i.e. \( T \) is semifinite.

(2) Let \( x \in R(M, \sigma^\theta)_+ \). Then for any \( t \in \mathbb{R} \):
\[
T(\lambda(t) x \lambda(t)^*) = \int_{-\infty}^{\infty} \theta_s \phi(\lambda(t) x \lambda(t)^*) \, ds = \int_{-\infty}^{\infty} \theta_s \phi(x) \, ds = Tx.
\]
Therefore
\[
\phi \circ T(\lambda(t) x \lambda(t)^*) = \phi \circ T(x) \quad x \in R(M, \sigma^\theta)_+.
\]

Hence by [3, lemma 1.2.3(c)] we have \( \sigma_t^\phi \sigma_T(\lambda(s)) = \lambda(s) \), \( s, t \in \mathbb{R} \). Together with Theorem 4.7(1) we get
\[
\sigma_t^\phi \sigma_T(x) = \sigma_t^\phi(x) = \lambda(t) x \lambda(t)^* \quad x \in M, \quad t \in \mathbb{R},
\]
\[
\sigma_t^\phi \sigma_T(\lambda(s)) = \lambda(s) = \lambda(t) \lambda(s) \lambda(t)^* \quad s, t \in \mathbb{R}.
\]
Since \( M \) and \( \lambda(s) \) generate \( R(M, \sigma^\theta) \) it follows that
\[
\sigma_t^\phi \sigma_T(x) = \lambda(t) x \lambda(t)^* \quad x \in R(M, \sigma^\theta).
\]
Hence by [12, Theorem 7.4] there is a n.f.s. trace \( \tau \) on \( R(M, \sigma^\theta) \) such that
\( \phi \circ T = \tau(h^\ast) \) where \( h \) is the positive selfadjoint operator for which \( h^\ast t = \lambda(t) \).

From [3, lemma 1.2.3(b)] it follows that:

\[
(D\phi \circ T : DT)_t = \lambda(t), \quad t \in \mathbb{R}.
\]

(3) Since \( \theta_s^\phi(\lambda(t)) = e^{ist}\lambda(t) \) we have \( \theta_s^\phi(h) = e^{ist}h \). By the definition of \( T \) the weight \( \phi \circ T \) is \( \theta_s^\phi \)-invariant. Hence for \( x \in R(M, \sigma^\phi)_+ \):

\[
\tau \circ \theta_s^\phi(x) = \phi \circ T(h^{-1} \cdot \theta_s^\phi(x)) = \phi \circ T(\theta_s^\phi(h^{-1}) \cdot x) = e^{-s}\phi \circ T(h^{-1} \cdot x) = e^{-s}\tau(x).
\]

**Remark.** In [15] Takesaki proved that \( R(M, \sigma^\phi) \) is semifinite. Note that lemma 5.2 gives a proof of this result without using Hilbert algebra technique. In [10] we will prove that \( \phi \circ T \) is proportional to the dual weight \( \tilde{\phi} \) constructed in [15].

**Proof of Theorem 5.1.** Assume that \( M, N, \phi \) and \( \psi \) satisfy the conditions of the Theorem.

Since \( \sigma_1^\phi \subseteq \sigma_1^\psi \) we can regard \( R(N, \sigma^\psi) \) as a subalgebra of \( R(M, \sigma^\phi) \) namely the von Neumann algebra generated by \( N \) and the one parameter group \( \lambda(t) \in R(M, \sigma^\phi) \). Let \( \theta_s^\phi \) be the dual action on \( R(M, \sigma^\phi) \). Then

\[
\theta_s^\phi(x) = x \quad \forall x \in N,
\]

\[
\theta_s^\phi(\lambda(t)) = e^{-ist}\lambda(t), \quad t \in \mathbb{R}.
\]

Hence the dual action \( \theta_s^\phi \) on \( R(N, \sigma^\psi) \) is just the restriction of \( \theta_s^\phi \).

Put

\[
T_1x = \int_{-\infty}^{\infty} \theta_s^\phi(x) \, ds, \quad x \in R(M, \sigma^\phi)_+;
\]

and

\[
T_2x = \int_{-\infty}^{\infty} \theta_s^\psi(x) \, ds, \quad x \in R(N, \sigma^\psi)_+.
\]

Then \( T_2 \subseteq T_1 \). According to lemma 5.2 there exist n.f.s. traces \( \tau_1 \) and \( \tau_2 \) on \( R(M, \sigma^\phi) \) and \( R(N, \sigma^\psi) \) respectively such that

\[
(D\phi \circ T_1 : D\tau_1)_t = \lambda(t) = (D\phi \circ T_2 : D\tau_2)_t, \quad t \in \mathbb{R}.
\]

By Theorem 2.7 there is a unique n.f.s. operator valued weight \( \tilde{S} \) from \( R(M, \sigma^\phi) \) to \( R(N, \sigma^\psi) \) such that \( \tau_1 = \tau_2 \circ \tilde{S} \). Put \( \tilde{S}_s = \theta_s^\phi \circ \tilde{S} \circ \theta_s^\phi, \quad s \in \mathbb{R} \) then for \( a \in R(N, \sigma^\psi) \) and \( x \in R(M, \sigma^\phi)_+ \):

\[
R_s(a^*xa) = \theta_s^\psi \circ \tilde{S}(\theta_s^\phi(a)^*\theta_s^\phi(x) \theta_s^\phi(a)) = \theta_s^\phi(\theta_s^\psi(a)^*\tilde{S}(\theta_s^\phi(x)) \theta_s^\phi(a)) = a^*R_s(x)a.
\]
Hence for $s \in \mathbb{R}$, $R_s$ is an operator valued weight from $R(M, \sigma^\phi)$ to $R(N, \sigma^\psi)$. Since $\tau_1 \circ \theta^\phi_s = e^{-s} \tau_1$, and $\tau_2 \circ \theta^\phi_s = e^{-s} \tau_2$ it follows that $\tau_2 \circ R_s = \tau_1$. Thus $R_s = \mathcal{S}$ for any $s \in \mathbb{R}$, or equivalently

$$\mathcal{S} \circ \theta^\phi_s(x) = \theta^\phi_s \circ \mathcal{S}(x), \ x \in R(M, \sigma^\phi)_+.$$

Extending $T_1$, $T_2$ and $\mathcal{S}$ as in remark 2.4 we get for any $x \in R(M, \sigma^\phi)_+$:

$$\mathcal{S} \circ T_1(x) = \int_{-\infty}^{x} \mathcal{S} \circ \theta^\phi_s(x) \, ds = \int_{-\infty}^{x} \theta^\phi_s \circ \mathcal{S}(x) \, ds = T_2 \circ \mathcal{S}(x).$$

Hence we have the following commutative diagram

$$
\begin{array}{ccc}
R(M, \sigma^\phi)_+ & \xrightarrow{\mathcal{S}} & R(N, \sigma^\psi)_+ \\
T_1 \downarrow & & T_2 \downarrow \\
\hat{M}_+ & \xrightarrow{\mathcal{S}} & \hat{N}_+.
\end{array}
$$

According to proposition 2.5, $T_1$ maps $R(M, \sigma^\phi)_+$ onto $\hat{M}_+$. Hence $\mathcal{S}(\hat{M}_+) \subseteq \hat{N}_+$. Let $S$ be the restriction of $\mathcal{S}$ to $M_+$.

It is easy to check that $S$ is a normal operator valued weight from $M_+$ to $N_+$. Since $\tau_1 = \tau_2 \circ \mathcal{S}$ and since $\mathcal{S}$ preserves cocycle Radon Nikodym derivatives we get using the formula

$$(D\phi \circ T_1 : D\tau_1) = (D\psi \circ T_2 : D\tau_2)$$

that

$$\phi \circ T_1 = (\psi \circ T_2) \circ \mathcal{S}.$$

Thus for any $x \in R(M, \sigma^\phi)_+$:

$$(\psi \circ S) \circ T_1(x) = \psi \circ T_2 \circ \mathcal{S}(x) = \phi \circ T_1(x).$$

Since $T_1$ maps $R(M, \sigma^\phi)_+$ onto $\hat{M}_+$ it follows that

$$\psi \circ S(y) = \phi(y) \ \forall y \in M_+.$$  

By lemma 2.6 $S$ is semifinite and faithful. The uniqueness of $S$ follows from lemma 4.8.

By Theorem 4.7(1) and Theorem 5.1 we get:

**Corollary 5.3.** Let $M, N$ be von Neumann algebras, $N \subseteq M$. Then $P(M, N)$ is non empty if and only if there exist n.f.s. weights $\phi, \psi$ on $M$ and $N$ respectively such that

$$\sigma^\phi_t(x) = \sigma^\psi_t(x) \ \forall x \in N.$$
The following is a converse of theorem 4.7:

**Corollary 5.4.** Let $M, N$ be von Neumann algebras, $N \subseteq M$, and let $\phi \to \bar{\phi}$ be a map of $P(N)$ into $P(M)$ satisfying

1. $\omega_{\phi}(x) = \omega_{\bar{\phi}}(x)$, $x \in N, \phi \in P(N)$,
2. $(D\bar{\phi} : D\bar{T})_{\phi} = (D\phi : D\phi)_{\phi}, \phi, \psi \in P(N)$.

Then there is a unique operator valued weight $T \in P(M, N)$ such that $\bar{\phi} = \phi \circ T$ for any $\phi \in P(N)$.

**Proof.** We choose $\phi_0 \in P(N)$. By Theorem 5.1 there is a unique $T \in P(M, N)$ such that $\phi_0 \circ T = \bar{\phi}_0$. For any $\phi \in P(N)$ we get by Theorem 4.7(2) that

$$(D\phi \circ T : D\phi_0 \circ T) = (D\phi : D\phi_0) = (D\bar{\phi} : D\phi_0 \circ T).$$

Hence $\phi \circ T = \bar{\phi}_0$ (cf. [3, Theorem 1.2.4]).

**Theorem 5.5.** Let $M_1, M_2$ be von Neumann algebras and let $N_{1}, N_2$ be sub von Neumann algebras of $M_1$ and $M_2$ respectively. Let $T_i \in P(M_i, N_i)$ $i = 1, 2$. There is a unique operator valued weight $T \in P(M_1 \otimes M_2, N_1 \otimes N_2)$ such that

$$(\phi_1 \circ T_1) \otimes (\phi_2 \circ T_2) = (\phi_1 \otimes T_1) \otimes (\phi_2 \circ T_2)$$

for any pair $(\phi_1, \phi_2)$ of n.f.s. weights on $N_1$ and $N_2$ respectively.

**Proof.** Let $\psi_1$ and $\psi_2$ be fixed n.f.s. weights on $N_1$ and $N_2$ then for $x \in N_1$ and $y \in N_2$

$$\sigma_{\phi_1 T_1}(x \otimes y) = \sigma_{\psi_1 T_1}(x) \otimes \sigma_{\psi_2 T_2}(y)$$

Hence by theorem 5.1 there is a unique operator valued weight $T$ from $M_1 \otimes M_2$ to $N_1 \otimes N_2$ such that

$$(\psi_1 \circ T_1) \otimes (\psi_2 \circ T_2) = (\psi_1 \otimes T_1) \otimes (\psi_2 \circ T_2).$$

Since the map $\omega \to \omega \circ T$ preserves cocycle Radon Nikodym derivatives (Theorem 4.7(2)) it follows from the formula in [2, lemma 1.5]:

$$(D\phi_1 \circ T_1 : D\phi_2 \circ T_2) = (D\phi_1 : D\psi_1) \otimes (D\phi_2 : D\psi_2),$$

for $\phi_i \in P(N_i), i = 1, 2$, that

$$(\phi_1 \circ T_1) \otimes (\phi_2 \circ T_2) = (\phi_1 \circ T_1) \otimes (\phi_2 \circ T_2)$$

for any $\phi_1 \in P(N_1)$ and $\phi_2 \in P(N_2)$. 
**Definition 5.6.** The operator valued weight \( T \) in proposition 5.4 is called the tensor product of \( T_1 \) and \( T_2 \) and is denoted by \( T_1 \otimes T_2 \).

Note that when \( T_1 \) and \( T_2 \) are normal faithful conditional expectations, then our definition of \( T_1 \otimes T_2 \) coincides with the usual definition of the tensor product (cf. [2, proposition 2.1]).

**Theorem 5.7.** Let \( \phi \) be a n.f.s. weight on a von Neumann algebra \( M \). The centralizer \( M_\phi \) of \( \phi \) is semifinite if and only if there exists a \( \sigma_\phi \)-invariant n.f.s. operator valued weight from \( M \) to \( M_\phi \).

**Proof.** Assume that \( M_\phi \) is semifinite and choose a n.f.s. trace \( \tau \) on \( M_\phi \). Then by Theorem 5.1 there exists a unique \( T \in P(M, M_\phi) \) such that \( \phi = \tau \circ T \).

It is easily seen that for any \( t \in \mathbb{R} \) the map \( S_t = T \circ \sigma_t^\phi \) is also a n.f.s. operator valued weight from \( M \) to \( M_\phi \) such that \( \phi = \tau \circ S_t \). Hence by lemma 4.8 \( S_t = T \).

Therefore \( T \) is \( \sigma_\phi \)-invariant.

Conversely, assume that there is a \( \sigma_\phi \)-invariant operator valued weight \( T \in P(M, M_\phi) \), and let \( \psi \in P(M_\phi) \). Clearly the weight \( \psi \circ T \) on \( M \) is \( \sigma_\phi \)-invariant. Hence by [12, Theorem 5.12] there exists a positive selfadjoint operator \( h \) affiliated with \( M_\phi \), such that \( \psi \circ T = \phi(h) \) and thus by [12, Theorem 4.6]:

\[
\sigma_t^\phi T(x) = h^{it}xh^{-it}, \quad x \in M.
\]

In particular

\[
\sigma_t^\phi(y) = h^{it}y h^{-it}, \quad y \in M_\phi.
\]

Since \( \sigma_\phi \) is inner, we conclude that \( M_\phi \) is semifinite [12, Theorem 7.4].

**Remark.** It is known that \( M_\phi \) is semifinite for two classes of n.f.s. weights on a von Neumann algebra \( M \), namely for strictly semifinite weights (cf. [11]) and for integrable weights (cf. [5, Sect. 2]). In these two cases it is easy to give an explicit formula for a \( \sigma_\phi \)-invariant operator valued weight \( T \in P(M, M_\phi) \). If \( \phi \) is strictly semifinite, there is a faithful family of \( \sigma_\phi \)-invariant normal states on \( M \). Hence by [13] \( \sigma_t^\phi \) acts weakly almost periodic on the predual of \( M \). Let \( m \) be a left invariant mean on the weakly almost periodic functions (cf. [7, Sect. 3.1]), then it is easy to check that

\[
E_{\phi}x = \int_{-\infty}^{\infty} \sigma_t^\phi x \, dm(t) = \lim_{\mu \to +\infty} \frac{1}{2\mu} \int_{-\mu}^{\mu} \sigma_t^\phi x \, dt \quad (\sigma\text{-weakly})
\]

defines a normal, \( \sigma_\phi \)-invariant conditional expectation from \( M \) to \( M \) (see also [16, Sect. 3]). Moreover \( E_{\phi} \) is faithful, because its support projection is \( \sigma_\phi \)-invariant, and thus belongs to \( M_\phi \). If \( \phi \) is integrable, then by definition the set

\[
\text{span} \left\{ x \in M_+ \left| \int_{-\infty}^{\infty} \sigma_t^\phi x \, dt \in M_\phi \right. \right\}
\]
is $\sigma$-weakly dense in $M$. Hence

$$T_\phi x = \int_{-\infty}^{\infty} \sigma_t^\phi(x) \, dt$$

defines a n.f.s. operator valued weight from $M$ to $M_\phi$. (Same proof as in lemma 5.2(1)).

In [8] we gave an example of a n.f.s. weight $\phi$ on the hyperfinite factor of type $II_\infty$ such that $M_\phi$ is of type III. Using a lemma from [8] we can prove

**Proposition 5.8.** Let $R$ be the hyperfinite factor of type $II_\infty$. There exists a sub von Neumann algebra $N$ of $R$, such that $P(R, N)$ is empty.

**Proof.** By [8, lemma 2], there exists an abelian subalgebra $A$ of $R$, such that the relative commutant $N = A' \cap R$ is of type III. Assume that there exists an operator valued weight $T \in P(R, N)$. Let $\tau$ be the trace on $R$ and let $\phi \in P(N)$. Then $\phi \circ T = \tau(h \cdot)$ for a positive selfadjoint operator $h$ affiliated with $R$. Since $A \subset Z(N) \subset N$ we get by [12, Theorem 4.6] that

$$h^{it}ah^{-it} = \sigma_t^\phi T(a) = \sigma_t^{\phi T}(a) = a \quad \forall a \in A.$$ 

Hence $h$ is affiliated with $A' \cap R = N$. However

$$\sigma_t^{\phi T}(x) = \sigma_t^{\phi T}(x) = h^{it}xh^{-it} \quad \forall x \in N.$$ 

Thus $t \rightarrow \sigma_t^{\phi}$ is inner, which contradicts that $N$ is of type III.

**Theorem 5.9.** Let $M$ and $N$ be von Neumann algebras on a Hilbert space $H$, such that $N \subset M$. Then

$$P(M, N) \neq \Leftrightarrow P(N', M') \neq \mathcal{C}.$$ 

**Lemma 5.10.** Let $M$ be a von Neumann algebra on a Hilbert space $H$. There exists a strongly continuous one parameter group of unitary operators $(u_t)_{t \in \mathbb{R}}$ on $H$ such that

$$\sigma_t^\phi(x) = u_t x u_t^* \quad x \in M,$$

$$\sigma_t^\psi(y) = u_t^* y u_t \quad y \in M'$$

for some pair $\phi, \psi$ of n.f.s. weights on $M$ and $M'$ respectively.

**Proof.** Note first that if $(e_i)_{i \in I}$ is a partition of 1 in projections in the center $Z(M)$ of $M$, then the lemma is true for $M$ iff it is true for each of the von Neumann algebras $e_i M$ acting on the Hilbert spaces $e_i H$. If $M$ is a semifinite, $M'$ is also semifinite. Hence in this case $u_t = 1$ can be used. Thus we may assume that $M$ is of type III. We treat first the case where $M$ and $M'$ are $\sigma$-finite. By [6, Chap.
III, Sect. 8, Corollary 11 and Sect. 1 corollary de proposition 4] $M$ has a cyclic
and separating vector $\xi_0$. Put
\[ \phi(x) = (x\xi_0 | \xi_0), \quad x \in M, \]
\[ \psi(y) = (y\xi_0 | \xi_0), \quad y \in M'. \]
and let $\Delta$ be the modular operator associated with the left Hilbert algebra
$M\xi_0$ (cf. [14]).

Then $\Delta^{-1}$ is the modular operator associated with $M'\xi_0$. Hence:
\[ \sigma_1^\phi(x) = \Delta^{it}x\Delta^{-it}, \quad x \in M, \]
\[ \sigma_1^\psi(y) = \Delta^{-it}y\Delta^{it}, \quad y \in M'. \]

Therefore $u_t = \Delta^{it}$ can be used in this case. We return now to the general
case ($M$ still of type III). Using [6, Chap. III, Sect. 1, lemma 7] on both $M$
and $M'$ it follows that there exists a partition $(e_i)_{i \in I}$ of 1 into central projections,
such that $e_iM$ and $e_iM'$ both contain partitions of $e_i$ into equivalent $\sigma$-finite
projections. Thus it is no loss of generality to assume that $M$ (resp. $M'$)
contains a partition of 1 into equivalent $\sigma$-finite projections $(p_i)_{i \in I}$
(resp. $(q_i)_{i \in I}$).

Let $\alpha$ and $\beta$ be fixed elements in $I$ and $J$ respectively. Put $r = p_\alpha q_\beta, K = l^p(I)$
and $L = l^p(J)$. Then using [6, Chap. I, Sect. 2, prop. 5] twice the Hilbert space
$H$ is isomorphic to $r(H) \otimes K \otimes L$. The corresponding factorizations of $M$ and
$M'$ are:
\[ M = (rMr) \otimes B(K) \otimes C_1, \]
\[ M' = (rMr)' \otimes C_K \otimes B(L). \]

However, $rMr$ is isomorphic to $p_\alpha M p_\alpha$ and $(rMr)' = rM' r$ is isomorphic to
$q_\alpha M' q_\alpha$. Hence by the first part of the proof there exist positive, normal,
faultful functionals $\phi, \psi$ on $rMr$ and $rM' r$ respectively, and a strongly con-
tinuous one parameter group $(u_t)_{t \in \mathbb{R}}$ of unitary operators on $r(H)$ such that
\[ \sigma_t^\phi(x) = u_t x u_t^*, \quad x \in rMr, \]
\[ \sigma_t^\psi(y) = u_t^* y u_t, \quad y \in rM' r. \]

Let $tr_1$ and $tr_2$ be the traces on $B(K)$ and $B(L)$, and put $v_t = u_t \otimes 1 \otimes 1$ on
$H = r(H) \otimes K \otimes L$, then
\[ u_t^\phi \otimes tr_1(x) = v_t x v_t^*, \quad x \in M, \]
\[ \sigma_t^\psi \otimes tr_2(y) = v_t^* y v_t, \quad y \in M'. \]

This completes the proof.
Lemma 5.11. Let $M$ be a von Neumann algebra on a Hilbert space $H$.

(a) Let $\phi \in P(M)$ and $\psi \in P(M')$. There exists a strongly continuous one parameter group of unitary operators $(u_t)_{t \in \mathbb{R}}$ on $H$, such that

$$u_t^\phi(x) = v_t x v_t^*, \quad x \in M,$$
$$\sigma_t^\phi(y) = v_t^* y v_t, \quad y \in M'.$$

(b) Let $(u_t)_{t \in \mathbb{R}}$ be a strongly continuous one parameter group of unitary operators on $H$. The following conditions are equivalent

1. $\exists \phi \in P(M) : \sigma_t^\phi(x) = u_t x u_t^*, \quad x \in M$,
2. $\exists \psi \in P(M') : \sigma_t^\psi(y) = u_t^* y u_t, \quad y \in M'$.

Proof. (a) According to lemma 5.10 there exists a strongly continuous one parameter group $(u_t)_{t \in \mathbb{R}}$ of unitary operators on $H$, $\phi_0 \in P(M)$, and $\psi_0 \in P(M')$ such that

$$\sigma_t^\phi(x) = u_t x u_t^*, \quad x \in M, \quad \sigma_t^\psi(y) = u_t^* y u_t, \quad y \in M'.$$

Put $v_t = (D\phi : D\phi_0)_t u_t (D\psi : D\psi_0)_t^*$, $t \in \mathbb{R}$. Since $(D\phi : D\phi_0)_t \in M$ and $(D\phi : D\phi_0)_t \in M'$ it follows that

$$v_t x v_t^* = (D\phi : D\phi_0)_t \sigma_t^\phi(x) (D\phi : D\phi_0)_t^* = \sigma_t^\phi(x), \quad x \in M,$$

and

$$v_t^* y v_t = (D\psi : D\psi_0)_t \sigma_t^\phi(y) (D\psi : D\psi_0)_t^* = \sigma_t^\phi(y), \quad y \in M'.$$

Using [3, lemma 1.2.2] we have:

$$v_{s+t} = (D\phi : D\phi_0)_{s+t} u_{s+t} (D\psi : D\psi_0)_{s+t}^*.$$

Hence $(v_t)_{t \in \mathbb{R}}$ satisfies the conditions.

(b) It is enough to prove $(1) \Rightarrow (2)$. Assume that there exists $\phi \in P(M)$, such that $\sigma_t^\phi(x) = u_t x u_t^*$, $x \in M$. By (a) we can for a given weight $\omega \in P(M')$ choose a strongly continuous one parameter group $(v_t)_{t \in \mathbb{R}}$ of unitary operators, such that

$$\sigma_t^\phi(x) = v_t x v_t^*, \quad x \in M, \quad \sigma_t^\omega(y) = v_t^* y v_t, \quad y \in M'.$$
It follows that $u_t^* v_t x v_t^* u_t = x$ for $x \in M$ which proves that $w_t = u_t^* v_t \in M'$. Moreover:

$$w_{s+t} = u_s^* u_t^* v_s v_t = (u_s^* v_s) v_s^* (u_t^* v_t) v_s = w_s \sigma_s^w(w_t).$$

Hence by [3, Theorem 1.2.41] there exists a n.f.s. weight $\psi$ on $M'$ such that $(D\psi : D\omega)_t = w_t$. Thus:

$$\sigma_t^\psi(y) = w_t \sigma_t^w(y) w_t^* = u_t^* y u_t, \quad y \in M'.$$

This completes the proof.

**Proof of Theorem 5.9.** Because of the symmetry it is enough to prove $\Rightarrow$. Let $T \in P(M, N)$ and choose $\phi \in P(N)$ and $\psi \in P(M')$. By lemma 5.11(a) there exists a strongly continuous one parameter group $(v_t)_{t \in \mathbb{R}}$ of unitary operators on $H$ such that

$$\sigma_t^\phi(x) = v_t x v_t^*, \quad x \in M,$$

$$\sigma_t^\psi(y) = v_t^* y v_t, \quad y \in M'.$$

For $x \in N: \sigma_t^\phi(x) = \sigma_t^\psi(x) = v_t x v_t^*$. Hence by lemma 5.11 (b) there exists $\omega \in P(N')$ such that $\sigma_t^\omega(y) = v_t^* y v_t$ for any $y \in N'$. In particular

$$\sigma_t^\omega(y) = \sigma_t^\psi(y) \quad \text{for} \quad y \in M' \subseteq N'.$$

Hence by theorem 5.1 there exists $S \in P(N', M')$ such that $\omega = \phi \circ S$.

**Remark.** We will prove in §6 that $P(M, N)$ and $P(N', M')$ are antiisomorphic in a certain sense.

A combination of Theorem 2.7, corollary 2.10(3) and the above theorem gives:

**Corollary 5.12.** Let $M$ and $N$ be von Neumann algebras, $N \subseteq M$. Then $P(M, N)$ is not empty in the following cases:

1. $M$ and $N$ are semifinite.
2. $N$ is a direct sum of type I factors.
3. $M$ is a direct sum of type I factors.

It is well known that if $M, N$ are von Neumann algebras, $N \subseteq M$, such that type $(N) > \text{type}(M)$, then there do not exist normal conditional expectations from $M$ to $N$ (cf. [11]). It follows from Theorem 5.9 that no such selection rule exists for operator valued weights. Note also that for any von Neumann algebra $M$ on a Hilbert space $H$, there is a n.f.s. operator valued weight from $B(H)$ to $M$ (Corollary 5.12(3)).
6. MODULAR AUTOMORPHISM GROUP ASSOCIATED WITH AN OPERATOR VALUED WEIGHT

We shall in this section generalize some of the results due to Combes and Delaroche in [2]. Let $M, N$ be von Neumann algebras, $N \subseteq M$. As usual $N^c = M \cap N'$.

**Proposition 6.1.** (cf. [2, lemma 3.1 and proposition 4.1]). (1) Let $T \in P(M, N)$. For any $\phi \in P(N)$: $\sigma_i^{\phi^*T}(N^c) = N^c$, and the restriction of $\sigma_i^{\phi^*T}$ to $N^c$ is independent of the choice of $\phi$.

(2) Let $T_1, T_2 \in P(M, N)$. For any $\phi \in P(N)$: $(D\phi \circ T_1 : D\phi \circ T_2)_t \in N^c$ and $(D\phi \circ T_2 : D\phi \circ T_1)_t$ is independent of the choice of $\phi$.

**Proof.** (1) Since $\sigma_i^{\phi^*T}(x) = \sigma_i^\phi(x)$, $x \in N$ we have $\sigma_i^{\phi^*T}(N) = N$. Hence $\sigma_i^{\phi^*T}(N^c) = N^c$. Let $\phi, \psi \in P(N)$, then $(D\phi \circ T : D\phi \circ T)_t = (D\phi : D\phi)_t \in N$. Hence for $x \in N^c$:

$$\sigma_i^{\phi^*T}(x) = (D\phi \circ T : D\phi \circ T)\sigma_i^{\phi^*T}(x)(D\phi \circ T : D\phi \circ T)_t^* = \sigma_i^{\phi^*T}(x)$$

because $\sigma_i^{\phi^*T}(x) \in N^c$.

(2) Let $T_1, T_2 \in P(M)$, and let $F_2$ be the algebra of $2 \times 2$-matrices with natural basis $(e_{ij})_{i,j=1,2}$.

Put

$$T \left( \sum x_{ij} \otimes e_{ij} \right) = (T_1(x_{11}) + T_2(x_{22})) \otimes 1, \quad \sum x_{ij} \otimes e_{ij} \in (M \otimes F_2)_+.$$ 

Then it is easy to check that $T$ is a n.f.s. operator valued weight from $M \otimes F_2$ to $N \otimes 1$. Let $\phi \in P(N)$ and let $\phi'$ be the corresponding weight on $N \otimes 1$, i.e. $\phi'(x \otimes 1) = \phi(x)$, $x \in N_+$. Then

$$\phi' : T \left( \sum x_{ij} \otimes e_{ij} \right) = \phi \circ T_1(x_{11}) + \phi \circ T_2(x_{22}).$$

Hence by [3, lemma 1.2.2] we have

$$(D\phi \circ T_2 : D\phi \circ T_1)_t \otimes e_{21} = \sigma_i^{\phi^*T}(1 \otimes e_{21}).$$

Since the relative commutant of $N \otimes 1$ in $M \otimes F_2$ is $N^c \otimes F_2$ we get $\sigma_i^{\phi^*T}(N^c \otimes F_2) = N^c \otimes F_2$. Therefore $(D\phi \circ T_2 : D\phi \circ T_1)_t \in N^c$. Let now $\phi$, $\psi \in P(N)$.

Then

$$(D\phi \circ T_2 : D\phi \circ T_1)_t = (D\phi \circ T_2 : D\phi \circ T_1)(D\phi \circ T_2 : D\phi \circ T_1),(D\phi \circ T_2 : D\phi \circ T_1)_t = (D\phi : D\phi)_t(D\phi \circ T_2 : D\phi \circ T_1)(D\phi \circ T_2 : D\phi \circ T_1)_t = (D\phi \circ T_2 : D\phi \circ T_1)_t,$$

because $(D\phi \circ T_2 : D\phi \circ T_1) \in N^c$. This completes the proof.
DEFINITION 6.2. (1) For \( T \in P(M, N) \) we let \( \sigma^T_t \) denote the restriction of \( \sigma^\phi_{t \circ T} \), \( \phi \in P(N) \), to \( N^e \).

(2) For \( T_1, T_2 \in P(M, N) \) we put \( (D_{T_2} : DT_{1})_t = (D_\phi \circ T_2 : D_\phi \circ T_1)_t \), \( \phi \in P(N) \).

By proposition 6.1 these definitions are independent of the choice of \( \phi \).

The following proposition is a trivial consequence of [3, §1].

PROPOSITION 6.3. For \( T_1, T_2, T_3 \in P(M, N) \):

1. \( \sigma^{T_3}_t(x) = (D_{T_2} : DT_{1})_t \sigma^{T_1}_t(x)(D_{T_2} : DT_{1})^*_t \),
2. \( (D_{T_2} : DT_{1})_{s+t} = (D_{T_2} : DT_{1})_s \sigma^{T_1}_t(D_{T_2} : DT_{1})_t \),
3. \( (D_{T_3} : DT_{1})_t = (D_{T_3} : DT_{2})(D_{T_2} : DT_{1})_t \),
4. \( (D_{T_1} : DT_{2})_t = (D_{T_2} : DT_{1})^*_t \).

COROLLARY 6.4. If \( M \) and \( N \) are semifinite von Neumann algebras, \( N \subseteq M \), then \( \sigma^T_t \) is inner for any \( T \in P(M, N) \).

Proof. Let \( \tau_1 \) and \( \tau_2 \) be n.f.s. traces on \( M \) and \( N \) respectively. By Theorem 2.7, there exists \( S \in P(M, N) \) such that \( \tau_2 = \tau_1 \circ S \). Hence for any \( x \in N^e : \)

\[ \sigma^\tau_{1,s}(x) = \sigma^\tau_{1,s}(x) = x. \]

Let \( T \in P(M, N) \) and put \( u_T = (DT : DS)_t \). Then by proposition 6.3(2):

\[ u_{t+s} = u_s \sigma^\tau_s(u_t) = u_s \cdot u_t . \]

Hence \( t \to u_t \) is a strongly continuous one parameter group and by proposition 6.3(1):

\[ \sigma^T_t(x) = u_t \sigma^\tau_s(x) u_t^* = u_t u_t^* , \quad x \in N^e . \]

THEOREM 6.5. Assume that \( P(M, N) \neq \emptyset \) and let \( T_0 \) be a fixed element in \( P(M, N) \). The map \( T \to (DT : DT_0) \) is a bijection of \( P(M, N) \) onto the set of \( \sigma^\tau_\phi \)-cocycles in \( N^e \). (i.e. the set of strongly continuous functions \( t \to u_t \) of \( \mathbb{R} \) into the unitary group in \( N^e \), that satisfy \( u_{s+t} = u_s \sigma^\phi_s(u_t) \)).

Proof. By proposition 6.3(2) it follows that \( (DT : DT_0) \) is a \( \sigma^\tau_\phi \)-cocycle for any \( T \in P(M, N) \). Assume that \( T_1, T_2 \in P(M, N) \), and that \( (DT_1 : DT_0)_t = (DT_2 : DT_0)_t \). Let \( \phi \in P(N) \), then \( (D_\phi \circ T_1 : D_\phi \circ T_0)_t = (D_\phi \circ T_2 : D_\phi \circ T_0)_t \). Hence \( \phi \circ T_1 = \phi \circ T_2 \). By lemma 4.8 this implies that \( T_1 = T_2 \). This proves the injectivity of the map \( T \to (DT : DT_0) \). Let \( (u_t)_{t \in \mathbb{R}} \) be a \( \sigma^\tau_\phi \)-cocycle in \( N^e \), and choose \( \phi \in P(N) \). We have \( u_{s+t} = u_s \sigma^\phi_s(t_o(u_t)). \) By [3, Theorem 1.2.4] there exists \( \omega \in P(M) \) such that \( (D_\omega : D_\phi \circ T_0)_t = u_t \).
For $x \in N$:

$$\sigma^\omega_t(x) = u_t \sigma^\phi_t(x) u_t^* = u_t \sigma^\phi_t(x) u_t^* = \sigma^\phi_t(x)$$

because $u_t \in N^\circ$. Hence by Theorem 5.1 there exists $T \in P(M, N)$ such that $\phi \circ T = \omega$. Moreover

$$(DT : DT_0)_t = (D\phi \circ T : D\phi \circ T_0)_t = (D\omega : D\phi \circ T_0)_t = u_t.$$  

This proves the surjectivity.

Let $M$, $N$ be von Neumann algebras, and let $T$ be a normal operator valued weight from $M$ to $N$. The restriction $T' = T | N^\circ$ of $T$ to $N^\circ$ maps $N_+^\circ$ into $Z(N)_+$ because for any $x \in N_+$ and any unitary $u \in Z(N)$ we have

$$u^* T(x) u = T(u^* x u) = T(x).$$

Hence $T'$ is a normal operator valued weight from $N^\circ$ to $Z(N)$. Unfortunately semifiniteness of $T$ does not in general imply semifiniteness of $T'$.

**Theorem 6.6.** (cf. [2, Theorem 5.3]). Let $M$, $N$ be von Neumann algebras $N \subseteq M$.

1. The following four conditions are equivalent:
   i. There exists $T \in P(M, N)$, such that $T | N^\circ$ is semifinite.
   ii. $P(M, N) \neq \varnothing$, and for any $T \in P(M, N)$ the restriction $T | N^\circ$ is semifinite.
   iii. There is a faithful family of normal bounded operator valued weights from $M$ to $N$.
   iv. There is a faithful family of normal conditional expectations from $M$ to $N$.

2. If one of the above conditions is satisfied, the map $T \mapsto T' = T | N^\circ$ is a bijection of $P(M, N)$ onto $P(N^\circ, Z(N))$ that satisfies

$$\sigma^{T'} = \sigma^T, \quad T \in P(M, N).$$

$$(DT'_2 : DT'_1) = (DT'_2 : DT'_1), \quad T_1, T_2 \in P(M, N).$$

**Lemma 6.7.** Let $M$ and $P$ be von Neumann algebras, $P \subseteq M$, and let $\phi$ be a n.f.s. weight on $M$ with the properties

(a) $\sigma^\phi_t(P) = P$, $t \in \mathbb{R}$,
(b) $\phi' = \phi | P$ is semifinite.
Then

(1) \( \sigma^\phi_i(x) = \sigma_i^\phi(x) \forall x \in P. \)

(2) If \( \psi \in P(M) \) and \( (D\psi : D\phi) \in P \) for any \( t \in \mathbb{R} \), then \( \psi' = \psi | P \) is semifinite and \( (D\psi' : D\phi')_t = (D\psi : D\phi)_t \).

Proof. (cf. [2, lemma 1.6]).

Lemma 6.8. Let \( M \) and \( N \) be von Neumann algebras, \( N \subseteq M \). Let \( T \in P(M,N) \), and let \( (e_i)_{i \in I} \) be a partition of 1 into orthogonal projections in \( Z(N) \), and let \( T_i \) and \( T'_i \) denote the restrictions of \( T \) to \( e_i Me_i \) and \( e_i N e_i \) respectively. Put \( T' = T | N \).

(a) \( T' \) is semifinite iff \( T'_i \) is semifinite for any \( i \in I \).

If these conditions are satisfied, then

(b) \( \sigma^T = \sigma^T \) iff \( \sigma^T_i = \sigma^T_i \) for any \( i \in I \).

Proof. (a) Trivial.

(b) Clearly, \( T_i \in P(e_i Me_i, e_i N) \). Moreover the relative commutant of \( e_i Me_i \) in \( e_i N \) is \( e_i N e_i \). For each \( i \in I \) we let \( \phi_i \) and \( \psi_i \) be n.f.s. weights on \( e_i N \) and \( e_i Z(N) \) respectively. Define n.f.s. weights \( \phi, \psi \) on \( N \) and \( Z(N) \) by

\[
\phi(x) = \sum_{i \in I} \phi_i(e_i x e_i), \quad x \in N^+, \\
\psi(y) = \sum_{i \in I} \psi_i(e_i y), \quad y \in Z(N)^+.
\]

Then

\[
\phi \circ T(x) = \sum_{i \in I} \phi_i \circ T_i(e_i x e_i), \quad x \in M^+, \\
\psi \circ T'(y) = \sum_{i \in I} \psi_i \circ T'_i(e_i y), \quad y \in N_+. \]

Since the weights \( \phi_i \circ T_i \) have orthogonal supports in \( M \), it follows that

\[
\sigma^\phi_i T_i(z) = \sigma_i^\phi T_i(z), \quad z \in e_i Me_i.
\]

Similarly

\[
\sigma^\psi_i T'_i(z) = \sigma_i^\psi T'_i(z), \quad z \in e_i N e_i^c.
\]

Hence for \( x \in N e_i^c \), we get by the assumption \( \sigma^T_i = \sigma^T_i \) for any \( i \in I \), that

\[
\sigma_i^T(x) = \sigma^\phi_i T_i(x) = \sum_{i \in I} \sigma_i^\psi T'_i(e_i x) = \sum_{i \in I} \sigma_i^\psi T'_i(e_i x) = \sigma_i^T(x).
\]

It is easy to prove the converse implication.
Lemma 6.9. Let $M_0$, $N_0$ be von Neumann algebras, $N_0 \subseteq M_0$, and let $F$ be a type I factor. Put $M = M_0 \otimes F$ and $N = N_0 \otimes F$, and let $T \in P(M, N)$.

(a) There is a unique $T_0 \in P(M_0, N_0)$, such that

$$T(x \otimes 1) = T_0(x) \otimes 1, \quad x \in (M_0)_+.$$  

(b) For any $\phi \in P(N_0)$:

$$(\phi \otimes \text{tr}) \circ T = (\phi \circ T_0) \otimes \text{tr}$$

where $\text{tr}$ is the trace on $F$.

(c) Put $N_0^c = N_0' \cap M_0$. Then $N_0^c = N_0^c \otimes 1$ and

$$\sigma^T_\epsilon(x \otimes 1) = \sigma^T_{\epsilon \phi}(x) \otimes 1, \quad x \in N_0^c.$$  

(d) Put $T' = T | \ N^c$ and $T_0' = T_0 | N_0^c$, then $T'(x \otimes 1) = T'_0(x) \otimes 1$ for any $x \in (N_0^c)_+$. In particular

$$T' \text{ semifinite} \iff T'_0 \text{ semifinite}.$$  

(e) If the conditions in (d) are satisfied, then

$$\sigma^\epsilon_{T'}(y \otimes 1) = \sigma^\epsilon_{T_0'}(y) \otimes 1, \quad y \in N_0^c.$$  

Proof. (a) For any unitary $u \in F$ we have $1 \otimes u \in N$. Hence for $x \in M_0^+$

$$(1 \otimes u)T(x \otimes 1)(1 \otimes u^*) = T((1 \otimes u)(x \otimes 1)(1 \otimes u^*)) = T(x \otimes 1).$$

Hence $T(x \otimes 1)$ is affiliated with $N \cap (M_0 \otimes 1) = N_0 \otimes 1$. Thus there exists $T_0x \in (N_0)_+^\wedge$, such that

$$T(x \otimes 1) = T_0x \otimes 1.$$  

It is easy to check that the map $x \rightarrow T_0x$ is a normal, faithful operator valued weight from $M_0$ to $N_0$. The semifiniteness of $T_0$ will follow, when (b) is proved.

(b) Let $\phi \in P(N_0)$ and let $\text{tr}$ be the trace on $F$. For $x \in F$ we have $1 \otimes x \in N$. Hence by Theorem 4.7

$$\sigma^\epsilon_{T_0} \otimes \lambda(1 \otimes x) = \sigma^\epsilon_{T_0} \otimes \lambda(1 \otimes x) = 1 \otimes x.$$  

This shows that $1 \otimes F$ is contained in the centralizer of $(\phi \otimes \text{tr}) \circ T$. Hence by [5, Chap. I, lemma 1.7] there exists $\psi \in P(M_0)$ such that

$$(\phi \otimes \text{tr}) \circ T = \psi \otimes \text{tr}. $$
Let $x \in (M_0)_{+}$ and let $e$ be a minimal projection in $F$. Since $1 \otimes e \in N$ we have

$$T(x \otimes e) = T((1 \otimes e)(x \otimes 1)(1 \otimes e))$$

$$= (1 \otimes e) T(x \otimes 1)(1 \otimes e)$$

$$= (1 \otimes e)(T_0 x \otimes 1)(1 \otimes e).$$

Let $y_n$ be an increasing sequence in $N_0^+$, such that

$$T_0 x = \sup_{n \in N} y_n \text{ (cf. corollary 1.6)}.$$

Then

$$T(x \otimes e) = \sup_{n \in N} (1 \otimes e)(y_n \otimes 1)(1 \otimes e) = \sup_{n \in N} (y_n \otimes e).$$

Hence

$$\psi(x) = (\psi \otimes \text{tr})(x \otimes e) = (\phi \otimes \text{tr}) \circ T(x \otimes e)$$

$$= \sup_{n \in N} (\phi \otimes \text{tr})(y_n \otimes e)$$

$$= \sup_{n \in N} \phi(y_n) = \phi(T_0 x),$$

where $\phi$ is extended to $(N_0^+)_+$ in the usual way. Thus $\psi = \phi \circ T_0$, and therefore $(\phi \otimes \text{tr}) \circ T = (\phi \circ T_0) \otimes \text{tr}$. It follows now from lemma 2.6 that $T_0$ is semifinite.

(c) Clearly $N_0^c = N^c \otimes 1$. For $x \in N_0^c$:

$$\sigma_i^T(x \otimes 1) = \sigma_i^{(\phi \otimes \text{tr}) \circ T}(x \otimes 1) = \sigma_i^{(\phi \circ T_0) \otimes \text{tr}}(x \otimes 1)$$

$$= \sigma_i^{\phi \circ T_0}(x) \otimes 1 = \sigma_i^{T_0}(x) \otimes 1.$$

(d) Trivial.

(e) Since the map $x \mapsto x \otimes 1$ is an isomorphism of $N_0^c$ onto $N^c$ we get by (d) that $\sigma_i^T(y \otimes 1) = \sigma_i^{T_0}(y) \otimes 1$, $y \in N_0^c$.

Proof of Theorem 6.6. Part 1. (iv) $\Rightarrow$ (i) follows from the proof of proposition 2.9.

(i) $\Rightarrow$ (ii). Let $T \in P(M, N)$ be chosen such that $T' = T \mid N^c$ is semifinite, and let $S \in P(M, N)$ be arbitrary. We shall prove that $S \mid N^c$ is semifinite. Since $N$ can be decomposed in the form $N = \sum_{i \in I} N_0 \otimes F_i$ where $N_\ell$ are $\sigma$-finite and $F_i$ are type I factors (cf. [6, Chap. III, §2 prop. 5]) it follows from lemma 6.8(a) that it is enough to treat the case $N = N_0 \otimes F$ where $N_0$ is $\sigma$-finite, and $F$ is a type I factor. Let $M = M_0 \otimes F$ be the corresponding factorization of $M$.

We have $N_0 \subseteq M_0$. Thus by lemma 6.9 (a) and (d) it is sufficient to treat the case where $N$ is $\sigma$-finite. Let $\omega$ be a normal faithful functional on $N$, and
let \( \omega' \) be the restriction of \( \omega \) to \( Z(N) \). Note that the restriction of \( \omega \circ T \) to \( N^c_+ \) is \( \omega' \circ T' \), which is semifinite. Moreover \( \sigma_{\omega T}(N^c_t) = N^c \) for any \( t \in \mathbb{R} \), and \( (D\omega \circ S : D\omega \circ T)_t = (DS : DT)_t \in N^c \) for any \( t \in \mathbb{R} \). Hence by lemma 6.7 (b) it follows that the restriction of \( \omega \circ S \) to \( N^c \) is semifinite, or equivalently \( \omega' \circ S' \) is semifinite. Therefore \( S' \) is semifinite by lemma 2.6.

(ii) \( \Rightarrow \) (iii). Let \( R \in P(M, N) \). Since \( T \mid N^c \) is semifinite there exists a net \((a_i)_{i \in I}\) of operators in \( n_T \cap N^c \) that converges \( \sigma \)-strongly to 1. Put

\[
T_i(x) = T(a_i^* x a_i), \quad x \in M_+, \quad i \in I.
\]

Clearly \( (T_i)_{i \in I} \) is a faithful family of bounded operator valued weights from \( M \) to \( N \).

(iii) \( \Rightarrow \) (iv). Let \( x_0 \in M_+ \setminus \{0\} \). We shall show that there exists a normal conditional expectation \( \epsilon \) from \( M \) to \( N \), such that \( \epsilon(x_0) \neq 0 \). By the assumptions there exists a bounded, normal operator valued weight \( S_0 \) from \( M \) to \( N \), such that \( S_0 x_0 \neq 0 \). We can assume that \( S_0(1) \leq 1 \). Choose a maximal family \( (S_i)_{i \in I} \), containing \( S_0 \), of non zero operator valued weights from \( M \) to \( N \), such that \( S_i(1) \leq 1 \) for any \( i \in I \), and \( S_i \) have pairwise orthogonal supports. Since \( 1 \in N^c \) we have \( S_i(1) \in Z(N) \) for any \( i \in I \). Assume that \( \Sigma_{i \in I} [S_i(1)] < 1 \) and put \( q = 1 - \Sigma_{i \in I} [S_i(1)] \in Z(N). \) (\( [\cdot] = \) support projection). Then by the assumptions there exists a bounded, normal operator valued weight \( T \) from \( M \) to \( N \), such that \( T(q) \neq 0 \). We can assume that \( T(1) \leq 1 \). Put \( R(x) = T(q x) \), \( x \in M_+ \). Then \( R \) is a non zero normal operator valued weight from \( M \) to \( N \), such that \( R(1) = q \). This contradicts the maximality of \( (S_i)_{i \in I} \). Hence \( \Sigma_{i \in I} [S_i(1)] = 1 \). Put \( h = \Sigma_{i \in I} S_i(1) \in Z(N). \) Then \( h \leq 1 \) and \( h \) is injective. Let \( h^{-1} = \int \lambda d\epsilon_h \) be the spectral resolution of \( h^{-1} \). Put \( k_n = \int_1^n \lambda d\epsilon_h + \int_n^\infty n d\epsilon_h \). Then \( (k_n)_{n \in \mathbb{N}} \) is an increasing sequence and \( 1 \leq k_n \leq n \) for any \( n \in \mathbb{N} \). Put

\[
\epsilon_+(x) = \sup_{n \in \mathbb{N}} \left( k_n \sum_{i \in I} S_i(x) \right), \quad x \in M_+.
\]

It is easy to check that \( \epsilon_+ \) is a normal operator valued weight from \( M \) to \( N \), and that \( \epsilon(1) = 1 \). Hence \( \epsilon_+ \) is the positive part of a conditional expectation \( \epsilon \). Moreover \( \epsilon(x_0) \geq \Sigma_{i \in I} S_i(x_0) \geq S_0 x_0 \). Hence \( \epsilon(x_0) \neq 0 \). This proves (iv).

Part 2. It follows from condition (ii) that the restriction map \( T \rightarrow T' = T \mid N^c \) maps \( P(M, N) \) into \( P(N^c, Z(N)) \). We shall prove that \( \sigma^{T'} = \sigma^T \). Since \( N \) can be decomposed in the form \( N = \sum_{i \in I} N_i \otimes F_i \), where \( N_i \) are \( \sigma \)-finite, and \( F_i \) are type I factors, it follows from lemma 6.8 (b) and lemma 6.9 (c) and (e) that it is enough to treat the case where \( N \) is \( \sigma \)-finite. Let \( \omega \) be a normal faithful functional on \( N \), and let \( \omega' \) be its restriction to \( Z(N) \). The restriction of \( \omega \circ T \) to \( N^c \) is \( \omega' \circ T' \). Hence by Lemma 6.7 (a) we have

\[
\sigma_{\omega T}(x) = \sigma_{\omega' T'}(x) \quad \forall x \in N^c
\]
or equivalently
\[ \sigma^T_t(x) = \sigma^T_t(x) \quad \forall x \in N^c. \]

Let now \( T_1, T_2 \in P(M, N) \) and define \( T \in P(M \otimes F_2, N \otimes 1) \) by
\[
T \left( \sum x_{ij} \otimes e_{ij} \right) = (T_1(x_{11}) + T_2(x_{22})) \otimes 1, \quad \sum x_{ij} \otimes e_{ij} \in (M \otimes F_2)_+. \]
as in the proof of proposition 6.1. Clearly the relative commutant of \( N \otimes 1 \) in \( M \otimes F_2 \) is \( N^c \otimes F_2 \), and the restriction \( T' = T \mid N^c \otimes F_2 \) is given by
\[
T' \left( \sum y_{ij} \otimes e_{ij} \right) = (T'_1(y_{11}) + T'_2(y_{22})) \otimes 1, \quad \sum y_{ij} \otimes e_{ij} \in (N^c \otimes F_2)_+. \]
In particular \( T' \) is semifinite. Then using \( \sigma^{T'} = \sigma^T \) on \( N^c \otimes F_2 \) we get
\[
(DT_2 : DT_1)_1 \otimes e_{21} = \sigma^{T'}_t(1 \otimes e_{21}) = \sigma^T_t(1 \otimes e_{21}) = (DT_2 : DT_1)_1 \otimes e_{21}. \]
That the map \( T \rightarrow T \mid N^c \) is a bijection of \( P(M, N) \) onto \( P(N^c, Z(N)) \) can now be proved as in the proof of [2, Theorem 5.3].

**Corollary 6.10.** Let \( M \) and \( N \) be von Neumann algebras, \( N \subseteq M \), such that there is a faithful family of conditional expectations from \( M \) to \( N \), then for any \( T \in P(M, N) \) there exists \( \phi \in P(N^c) \) such that \( \sigma^T = \sigma^\phi \).

**Proof.** Let \( \psi \) be a n.f.s. weight on \( Z(N) \) and put \( \phi = \psi \circ T' \) where \( T' = T \mid N^c \). Then for any \( x \in N^c \):
\[
\sigma^T_t(x) = \sigma^{T'}_t(x) = \sigma^\phi_t(x) = \sigma^T_t(x). \]

**Remark.** Let \( R \) be the hyperfinite type II\(_\infty\) factor. By [8, lemma 2] there exists an abelian subalgebra \( A \) of \( R \) such that the relative commutant \( A^c = A' \cap R \) is of type III. Since \( R \) and \( A \) are semifinite, \( P(R, A) \) is not empty, and for any \( T \in P(R, A) \) we have \( \sigma^T \) inner (corollary 6.4). Hence in this case \( \sigma^T \) cannot be equal to the modular automorphism group of some n.f.s. weight on \( A^c \).

**Corollary 6.11.** Let \( M = R(N, \theta) \) be the continuous decomposition of a properly infinite von Neumann algebra, as a crossed product of a semifinite subalgebra \( N \), and a one parameter group of automorphisms \( (\theta_s)_{s \in \mathbb{R}} \) on \( N \) (cf. [15]). Then \( P(M, N) \neq \emptyset \), but there is no normal conditional expectation from \( M \) to \( N \).

**Proof.** By [5, Chap. II] there exists an integrable weight \( \phi \in P(M) \), such that the centralizer \( M_\phi \) is equal to \( N \). Thus by the remark following Theorem 5.7
\[
T_\phi x = \int_{-\infty}^\infty \sigma_t^\phi x \, dt, \quad x \in M_+, \]
defines a n.f.s. operator valued weight from $M$ to $N$. Hence $P(M, N) \neq \emptyset$. (See also [10]). Assume that there exists a normal conditional expectation from $M$ to $N$. By Connes’ and Takesaki’s relative commutant theorem [5, Chap. II, Theorem 5.1] we have

$$N^c = M_\phi \cap M \subseteq M_\phi = N.$$  

Hence the support $[\epsilon]$ of $\epsilon$ belongs to $N$, and thus $1 - [\epsilon] = \epsilon(1 - [\epsilon]) = 0$, which proves that $\epsilon$ is faithful. Hence using theorem 6.6 (a) any $T \in P(M, N)$ has a semifinite restriction to $N^c$. However, for any $x \in N_+ \subseteq N_+$ we get

$$T_\phi x = \int_{-\infty}^{\infty} \sigma_t^\phi(x) \, dt = \left( \int_{-\infty}^{\infty} dt \right) x = \infty \cdot x$$

which contradicts that $T_\phi \mid N^c$ is semifinite.

In [2] it is proved that two normal, faithful conditional expectations $\epsilon_1$ and $\epsilon_2$ have the same modular automorphism group, if and only if there exists a positive selfadjoint operator $h$ affiliated with $Z(N^c)$ such that $\epsilon_2 = \epsilon_1(h^\ast)$. (cf. [2, proposition 4.11 and remark 4.12]). It is easy to see that this result can be generalized to operator valued weights, if the solution of the following problem is affirmative.

**Problem 6.12.** Let $M$, $N$ be von Neumann algebras, $N \subseteq M$, and let $T \in P(M, N)$. Does $\sigma^T$ leave the center of $N^c$ pointwise fixed? Clearly the answer is affirmative if $Z(N^c) = Z(N)$ in particular if $N^c \subseteq N$ or if $N^{cc} = N$. Moreover it follows from corollary 6.4 and corollary 6.10 that the solution is affirmative if $M$ and $N$ are both semifinite, or if there is a faithful family of normal, conditional expectations from $M$ to $N$.

**Theorem 6.13.** Let $M$, $N$ be von Neumann algebras on a Hilbert space $H$. There exists a bijection $\alpha$ of $P(M, N)$ onto $P(N', M')$ such that

1. $\sigma_t^{\alpha(T)} = \sigma_{-t}^T$, $T \in P(M, N), \ t \in \mathbb{R},$
2. $(D_\lambda(T_2) : D_\lambda(T_1))_t = (DT_2 : DT_1)_t$, $T_1, T_2 \in P(M, N), \ t \in \mathbb{R}.$

**Proof.** We have already proved in Section 5 that $P(N', M') \not\cong \emptyset$ if $P(M, N) \not\cong \emptyset$. Assume that $P(M, N) \not\cong \emptyset$.

Let $T_0 \in P(M, N)$ and let $\phi \in P(N)$ and $\psi \in P(M')$. By the proof of Theorem 5.9 there exists a strongly continuous one parameter group of unitaries on $H$, and an operator valued weight $S_0 \in P(N', M')$, such that

$$\sigma_t^{\psi \circ \phi(T)}(x) = u_t x u_t^*, \quad x \in M,$$
$$\sigma_t^{\phi \circ \psi}(y) = u_t^* y u_t, \quad y \in N'.$$
Hence for $x \in N_c = N' \cap M = (M')' \cap N'$ we have

$$\sigma_t^f(x) = \sigma_t^o(x) = v_t^* x v_t = \sigma_t^{o_s o}(x) = \sigma_t^{o}(x).$$

Let now $T \in P(M, N)$ be arbitrary, and put

$$u_t = (DT : DT_0)_t.$$

Since $u_t$ is a $\sigma_t^{o}$-cocycle, it follows that $u_{-t}$ is a $\sigma_t^{o}$-cocycle in $N_c$. Hence by Theorem 6.5 there exists a unique operator valued weight $\alpha(T) \in P(N', M')$ such that

$$(D\alpha(T) : DS_0)_t = u_{-t}. $$

Using Theorem 6.5 we get that the map $\alpha$ is a bijection of $P(M, N)$ onto $P(N', M')$. Moreover for $x \in N_c$:

$$\sigma_t^{o(T)}(x) = (D\alpha(T) : DS_0)_t \sigma_t^o(x)(D\alpha(T) : DS_0)_t^* = (DT : DT_0)_t^\tau(x).$$

Hence (1). Clearly $\alpha(T_0) = S_0$. Thus for any $T \in P(M, N)$

$$(D\alpha(T) : D\alpha(T_0))_t = (DT : DT_0)_t^\tau.$$ 

Hence using the chainrule we get (2).

Remark. Let $\phi$ and $\psi$ be n.f.s. weights on a von Neumann algebra. By [4] we have $\phi \leq \psi$ iff the map $t \mapsto (D\phi : D\phi)_t$ has a (unique) bounded $\sigma$-weakly continuous extension to the strip $-\frac{1}{2} \leq \text{Im} z \leq 0$, analytic in the interior of the strip, such that $\| (D\phi : D\phi)_{-1/2} \| \leq 1$. By the same method as in the proof of lemma 4.8, one can prove that for $T_1, T_2 \in P(M, N)$:

$$T_1 \leq T_2 \iff \phi \circ T_1 \leq \phi \circ T_2 \quad \forall \phi \in P(N).$$

Hence $T_1 \leq T_2$ iff the map $t \mapsto (DT_2 : DT_1)_t$ has a bounded, $\sigma$-weakly continuous extension to the strip $-\frac{1}{2} \leq \text{Im} z \leq 0$, analytic in the interior, such that $\| (DT_2 : DT_1)_{-1/2} \| \leq 1$. Let $\alpha$ be as in Theorem 6.12. Then using

$$(D\alpha(T_1) : D\alpha(T_2))_t = (DT_1 : DT_2)_t = (DT_2 : DT_1)_t, \quad t \in \mathbb{R},$$

it follows that $\alpha$ reverses the order:

$$T_1 \leq T_2 \iff \alpha(T_1) \geq \alpha(T_2).$$
REFERENCES