Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 337 (2008) 1002–1009
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On the bilateral series ${}_5\psi_5$

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Received 26 January 2007
Available online 24 April 2007
Submitted by B.C. Berndt

Abstract

The purpose of this paper is to derive two transformation formulae which imply relations between basic and bilateral basic hypergeometric series. Some special cases of them are also discussed.

Keywords: Basic hypergeometric series; Bilateral basic hypergeometric series; Transformation formula

1. Introduction

It is commonly known that one can derive a bilateral basic hypergeometric series from a unilateral series. For example, one can get bilateral basic hypergeometric series from terminating unilateral series by making use of Cauchy's method (see [1,6–8]). In [3], Chen and Fu also present another method of obtaining bilateral basic hypergeometric series. In [5] and [2], the authors used the method of [3] to find new semi-finite forms of bilateral basic hypergeometric series and obtain two expansions of an ${}_r\psi_r$ series in terms of r ${}_r\phi_{r-1}$ series, respectively.

Here, we mainly make use of a method already used by Bailey [1] to obtain two relations between basic and bilateral basic hypergeometric series. The main idea of this method is to decompose one very-well-poised unilateral series into two unilateral well-poised series, and then suitably shift the index and rearrange the order of one summation. In special cases (when the special parameter a of the original very-well-poised series has been specialized as $a \to q$ or $a \to 1$), we can obtain a bilateral basic hypergeometric series by combining these two unilateral series. We will use this method to establish two transformation formulae. In addition, we also discuss special cases of these two relations.

Throughout this paper, we will adopt the following definitions and notations. For $|q| < 1$, q-shifted factorial is defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{i=1}^{n}(1-aq^{i-1}), \quad n = 1, 2, \ldots,$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.04.028

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

Also, we shall frequently use the following more compact notations:

$$(a_1,a_2,\dots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$
$$(a_1,a_2,\dots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty$$

and

$$(a;q)_{-k} = \frac{(-q/a)^k q^{\binom{k}{2}}}{(q/a;q)_k}.$$

The basic hypergeometric series ${}_r\phi_s$ and the bilateral basic hypergeometric series ${}_r\psi_s$ are defined by

$${}_r\phi_s\left[\begin{matrix} a_1, & a_2, & \dots, & a_r \\ b_1, & b_2, & \dots, & b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_n}{(q,b_1,b_2,\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,$$

and

$${}_r\psi_s\left[\begin{matrix} a_1, & a_2, & \dots, & a_r \\ b_1, & b_2, & \dots, & b_s \end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_n}{(b_1,b_2,\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} z^n,$$

respectively. For more details, in particular on convergence of the respective series, and the definitions of well-poised and very-well-poised, see [4].

2. Main results

In this section, we will make use of the following Bailey's three-term transformation formula for VWP-balanced ${}_8\phi_7$ series:

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e, & f \\ & a^{1/2}, & -a^{1/2}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right] \\
&= \frac{(aq,aq/de,aq/df,aq/ef,eq/c,fq/c,b/a,bef/a;q)_\infty}{(aq/d,aq/e,aq/f,aq/def,q/c,efq/c,be/a,bf/a;q)_\infty} \\
&\quad\times {}_8\phi_7\left[\begin{matrix} ef/c, & q(ef/c)^{1/2}, & -q(ef/c)^{1/2}, & aq/bc, & aq/cd, & ef/a, & e, & f \\ & (ef/c)^{1/2}, & -(ef/c)^{1/2}, & bef/a, & def/a, & aq/c, & fq/c, & eq/c \end{matrix}; q, bd/a\right] \\
&\quad + b/a\frac{(aq,bq/a,bq/c,bq/d,bq/e,bq/f,d,e,f,aq/bc,bdef/a^2,a^2q/bdef;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,aq/f,bd/a,be/a,bf/a,def/a,aq/def,q/c,b^2q/a;q)_\infty} \\
&\quad\times {}_8\phi_7\left[\begin{matrix} b^2/a, & qba^{-1/2}, & -qba^{-1/2}, & b, & bc/a, & bd/a, & be/a, & bf/a \\ & ba^{-1/2}, & -ba^{-1/2}, & bq/a, & bq/c, & bq/d, & bq/e, & bq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right] \qquad (1)
\end{aligned}$$

provided $|\frac{a^2q^2}{bcdef}| < 1$ and $|bd/a| < 1$. See [4, Appendix III (III.37)].

Theorem 2.1. *For* $(|\frac{q^4}{bcdef}|,\ |bd/q|) < 1$, *we have*

$$\begin{aligned}
&{}_5\psi_5\left[\begin{matrix} b, & c, & d, & e, & f \\ q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^2/f \end{matrix}; q, \frac{q^4}{bcdef}\right] \\
&= \frac{(q,q^2/de,q^2/df,q^2/ef,qe/c,qf/c,b/q,bef/q;q)_\infty}{(q^2/d,q^2/e,q^2/f,q^2/def,q/c,qef/c,be/q,bf/q;q)_\infty} \\
&\quad\times {}_8\phi_7\left[\begin{matrix} ef/c, & q(ef/c)^{1/2}, & -q(ef/c)^{1/2}, & q^2/bc, & q^2/cd, & ef/q, & e, & f \\ & (ef/c)^{1/2}, & -(ef/c)^{1/2}, & bef/q, & def/q, & q^2/c, & fq/c, & eq/c \end{matrix}; q, bd/q\right]
\end{aligned}$$

$$+\frac{b}{q}\frac{(q,b,bq/c,bq/d,bq/e,bq/f,d,e,f,q^2/bc;q)_\infty}{(q^2/b,q^2/c,q^2/d,q^2/e,q^2/f,bd/q,be/q;q)_\infty}\frac{(bdef/q^2,q^3/bdef;q)_\infty}{(bf/q,def/q,q^2/def,q/c,b^2;q)_\infty}$$
$$\times {}_7\phi_6\left[\begin{matrix}b^2/q, & bq^{1/2}, & -bq^{1/2}, & bc/q, & bd/q, & be/q, & bf/q \\ & bq^{-1/2}, & -bq^{-1/2}, & bq/c, & bq/d, & bq/e, & bq/f\end{matrix};q,\frac{q^4}{bcdef}\right]. \tag{2}$$

Proof. Taking $a\to q$ in (1), we may rewrite

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e, & f \\ & a^{1/2}, & -a^{1/2}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f\end{matrix};q,\frac{a^2q^2}{bcdef}\right]\\
&=\frac{1}{1-q}\sum_{k=0}^{\infty}\left(1-q^{2k+1}\right)\frac{(b,c,d,e,f;q)_k}{(q^2/b,q^2/c,q^2/d,q^2/e,q^2/f;q)_k}\left(\frac{q^4}{bcdef}\right)^k\\
&=\frac{1}{1-q}\sum_{k=0}^{\infty}\frac{(b,c,d,e,f;q)_k}{(q^2/b,q^2/c,q^2/d,q^2/e,q^2/f;q)_k}\left(\frac{q^4}{bcdef}\right)^k\\
&\quad-\frac{q}{1-q}\sum_{k=0}^{\infty}\frac{(b,c,d,e,f;q)_k}{(q^2/b,q^2/c,q^2/d,q^2/e,q^2/f;q)_k}\left(\frac{q^6}{bcdef}\right)^k\\
&=\frac{1}{1-q}\sum_{k=0}^{\infty}\frac{(b,c,d,e,f;q)_k}{(q^2/b,q^2/c,q^2/d,q^2/e,q^2/f;q)_k}\left(\frac{q^4}{bcdef}\right)^k\\
&\quad-\frac{q}{1-q}\sum_{k=-1}^{-\infty}\frac{(b,c,d,e,f;q)_{-k-1}}{(q^2/b,q^2/c,q^2/d,q^2/e,q^2/f;q)_{-k-1}}\left(\frac{q^6}{bcdef}\right)^{-k-1}\\
&=\frac{1}{1-q}\sum_{k=0}^{\infty}\frac{(b,c,d,e,f;q)_k}{(q^2/b,q^2/c,q^2/d,q^2/e,q^2/f;q)_k}\left(\frac{q^4}{bcdef}\right)^k\\
&\quad+\frac{1}{1-q}\sum_{k=-1}^{-\infty}\frac{(b,c,d,e,f;q)_k}{(q^2/b,q^2/c,q^2/d,q^2/e,q^2/f;q)_k}\left(\frac{q^4}{bcdef}\right)^k\\
&=\frac{1}{1-q}\,{}_5\psi_5\left[\begin{matrix}b, & c, & d, & e, & f \\ q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^2/f\end{matrix};q,\frac{q^4}{bcdef}\right].
\end{aligned}\tag{3}$$

In the above process, we substitute the index k of the second summation in (3) with $-k-1$. Then, by the formula (1) we get

$$\begin{aligned}
&\frac{1}{1-q}\,{}_5\psi_5\left[\begin{matrix}b, & c, & d, & e, & f \\ q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^2/f\end{matrix};q,\frac{q^4}{bcdef}\right]\\
&=\frac{(q^2,q^2/de,q^2/df,q^2/ef,qe/c,qf/c,b/q,bef/q;q)_\infty}{(q^2/d,q^2/e,q^2/f,q^2/def,q/c,qef/c,be/q,bf/q;q)_\infty}\\
&\quad\times {}_8\phi_7\left[\begin{matrix}ef/c, & q(ef/c)^{1/2}, & -q(ef/c)^{1/2}, & q^2/bc, & q^2/cd, & ef/q, & e, & f \\ & (ef/c)^{1/2}, & -(ef/c)^{1/2}, & bef/q, & def/q, & q^2/c, & fq/c, & eq/c\end{matrix};q,bd/q\right]\\
&\quad+\frac{b}{q}\frac{(q^2,b,bq/c,bq/d,bq/e,bq/f,d,e,f,q^2/bc;q)_\infty}{(q^2/b,q^2/c,q^2/d,q^2/e,q^2/f,bd/q,be/q;q)_\infty}\frac{(bdef/q^2,q^3/bdef;q)_\infty}{(bf/q,def/q,q^2/def,q/c,b^2;q)_\infty}\\
&\quad\times {}_7\phi_6\left[\begin{matrix}b^2/q, & bq^{1/2}, & -bq^{1/2}, & bc/q, & bd/q, & be/q, & bf/q \\ & bq^{-1/2}, & -bq^{-1/2}, & bq/c, & bq/d, & bq/e, & bq/f\end{matrix};q,\frac{q^4}{bcdef}\right],
\end{aligned}$$

which completes the proof of (2). □

Corollary 2.2. *For $|q|<1$ and $bcde=q^{3+n}$, we have*

$$ {}_5\psi_5\left[\begin{matrix} b, & c, & d, & e, & q^{-n} \\ q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^{2+n} \end{matrix}; q, q\right] = \frac{(1-q)(q^2, q^2/bc, q^2/bd, q^2/cd; q)_n}{(q^2/b, q^2/c, q^2/d, q^2/bcd; q)_n}, $$

which is just Ex. 5.18(iv) *in* [4].

Proof. When we take $f=q^{-n}$ in (2), the second term on the right-hand side of the identity (2) vanishes. Therefore, we have

$$ \begin{aligned} &{}_5\psi_5\left[\begin{matrix} b, & c, & d, & e, & q^{-n} \\ q^2/b, & q^2/c, & q^2/d, & q^2/e, & q^{2+n} \end{matrix}; q, q\right] \\ &= \frac{(q, q^2/de, q^{2+n}/d, q^{2+n}/e, qe/c, q^{1-n}/c, b/q, beq^{-1-n}; q)_\infty}{(q^2/d, q^2/e, q^{2+n}, q^{2+n}/de, q/c, q^{1-n}e/c, be/q, bq^{-1-n}; q)_\infty} \\ &\quad\times {}_8\phi_7\left[\begin{matrix} eq^{-n}/c, & q(eq^{-n}/c)^{1/2}, & -q(eq^{-n}/c)^{1/2}, & q^2/bc, & q^2/cd, & eq^{-1-n}, & e, & q^{-n} \\ & (eq^{-n}/c)^{1/2}, & -(eq^{-n}/c)^{1/2}, & beq^{-1-n}, & deq^{-1-n}, & q^2/c, & q^{1-n}/c, & eq/c \end{matrix}; q, bd/q\right] \\ &= \frac{(q, q^2/de, q^{2+n}/d, q^{2+n}/e, qe/c, q^{1-n}/c, b/q, beq^{-1-n}; q)_\infty}{(q^2/d, q^2/e, q^{2+n}, q^{2+n}/de, q/c, q^{1-n}e/c, be/q, bq^{-1-n}; q)_\infty} \frac{(eq^{1-n}/c, q^2/ce; q)_n}{(q^2/c, q^{1-n}/c; q)_n} \\ &\quad\times {}_4\phi_3\left[\begin{matrix} q^{-n}, & e, & eq^{-1-n}, & bcdeq^{-3-n} \\ & beq^{-1-n}, & deq^{-1-n}, & ceq^{-1-n} \end{matrix}; q, q\right] \\ &= \frac{(q, q^2/de, q^{2+n}/d, q^{2+n}/e, qe/c, q^{1-n}/c, b/q, beq^{-1-n}; q)_\infty}{(q^2/d, q^2/e, q^{2+n}, q^{2+n}/de, q/c, q^{1-n}e/c, be/q, bq^{-1-n}; q)_\infty} \frac{(eq^{1-n}/c, q^2/ce; q)_n}{(q^2/c, q^{1-n}/c; q)_n} \\ &= \frac{(1-q)(q^2, q^2/bc, q^2/bd, q^2/cd; q)_n}{(q^2/b, q^2/c, q^2/d, q^2/bcd; q)_n} \end{aligned} $$

with $bcde=q^{3+n}$. This is just the result we want. □

In the above process, we make use of Watson's transformation formula:

$$ \begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e, & q^{-n} \\ & a^{1/2}, & -a^{1/2}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{1+n} \end{matrix}; q, a^2q^{2+n}/bcde\right] \\ &= \frac{(aq, aq/de; q)_n}{(aq/d, aq/e; q)_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, & d, & e, & aq/bc \\ & aq/b, & aq/c, & deq^{-n}/a \end{matrix}; q, q\right], \end{aligned} \tag{4} $$

provided $|a^2q^{2+n}/bcde|<1$. See [4, Appendix III (III.18)].

Corollary 2.3. *For $|q^3/bcef|, |b|<1$, we have*

$$ \begin{aligned} &{}_4\psi_4\left[\begin{matrix} b, & c, & e, & f \\ q^2/b, & q^2/c, & q^2/e, & q^2/f \end{matrix}; q, q^3/bcef\right] \\ &= \frac{(1-q/e)(1-q/f)(eq/c, fq/c, b/q, bef/q; q)_\infty}{(1-q/ef)(q/c, efq/c, be/q, bf/q; q)_\infty} \\ &\quad\times {}_8\phi_7\left[\begin{matrix} ef/c, & q(ef/c)^{1/2}, & -q(ef/c)^{1/2}, & q^2/bc, & q/c, & ef/q, & e, & f \\ & (ef/c)^{1/2}, & -(ef/c)^{1/2}, & bef/q, & ef, & q^2/c, & fq/c, & eq/c \end{matrix}; q, b\right] \\ &\quad+ \frac{b}{q} \frac{(q, b, e, f, q^2/bc, bef/q, q^2/bef, q^2/ce, q^2/cf, q^2/ef; q)_\infty}{(q^2/b, q^2/c, q^2/e, q^2/f, be/q, bf/q, ef, q/ef, q/c, q^3/bcef; q)_\infty}. \end{aligned} \tag{5} $$

Proof. Taking $d=q$ in (2), we get

$$
\begin{aligned}
&{}_4\psi_4\left[\begin{matrix} b, & c, & e, & f \\ q^2/b, & q^2/c, & q^2/e, & q^2/f \end{matrix}; q, q^3/bcef\right] \\
&= \frac{(q, q/e, q/f, q^2/ef, eq/c, fq/c, b/q, bef/q; q)_\infty}{(q, q^2/e, q^2/f, q/ef, q/c, efq/c, be/q, bf/q; q)_\infty} \\
&\quad \times {}_8\phi_7\left[\begin{matrix} ef/c, & q(ef/c)^{1/2}, & -q(ef/c)^{1/2}, & q^2/bc, & q/c, & ef/q, & e, & f \\ & (ef/c)^{1/2}, & -(ef/c)^{1/2}, & bef/q, & ef, & q^2/c, & fq/c, & eq/c \end{matrix}; q, b\right] \\
&\quad + \frac{b}{q}\frac{(q, b, bq/c, bq/e, bq/f, e, f, q^2/bc; q)_\infty}{(q^2/b, q^2/c, q^2/e, q^2/f, be/q; q)_\infty}\frac{(bef/q, q^2/bef; q)_\infty}{(bf/q, ef, q/ef, q/c, b^2; q)_\infty} \\
&\quad \times {}_6\phi_5\left[\begin{matrix} b^2/q, & bq^{1/2}, & -bq^{1/2}, & bc/q, & be/q, & bf/q \\ & bq^{-1/2}, & -bq^{-1/2}, & bq/c, & bq/e, & bq/f \end{matrix}; q, q^3/bcef\right].
\end{aligned}
$$

By certain simplifications and the summation formula of nonterminating ${}_6\phi_5$ series,

$$
{}_6\phi_5\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d \\ & a^{1/2}, & -a^{1/2}, & aq/b, & aq/c, & aq/d \end{matrix}; q, aq/bcd\right] = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}
$$

(see [4, Appendix II (II.20)]), we get

$$
\begin{aligned}
&{}_4\psi_4\left[\begin{matrix} b, & c, & e, & f \\ q^2/b, & q^2/c, & q^2/e, & q^2/f \end{matrix}; q, q^3/bcef\right] \\
&= \frac{(1-q/e)(1-q/f)(eq/c, fq/c, b/q, bef/q; q)_\infty}{(1-q/ef)(q/c, efq/c, be/q, bf/q; q)_\infty} \\
&\quad \times {}_8\phi_7\left[\begin{matrix} ef/c, & q(ef/c)^{1/2}, & -q(ef/c)^{1/2}, & q^2/bc, & q/c, & ef/q, & e, & f \\ & (ef/c)^{1/2}, & -(ef/c)^{1/2}, & bef/q, & ef, & q^2/c, & fq/c, & eq/c \end{matrix}; q, b\right] \\
&\quad + \frac{b}{q}\frac{(q, b, e, f, q^2/bc, bef/q, q^2/bef, q^2/ce, q^2/cf, q^2/ef; q)_\infty}{(q^2/b, q^2/c, q^2/e, q^2/f, be/q, bf/q, ef, q/ef, q/c, q^3/bcef; q)_\infty}. \qquad \square
\end{aligned}
$$

Corollary 2.4. *For $|q^{3+n}/bce|, |q|<1$, we have*

$$
\begin{aligned}
&{}_4\psi_4\left[\begin{matrix} b, & c, & e, & q^{-n} \\ q^2/b, & q^2/c, & q^2/e, & q^{2+n} \end{matrix}; q, q^{3+n}/bce\right] \\
&= \frac{(1-q/e)(1-q^{1+n})(q^2/be, q^2/ce; q)_n}{(1-q^{1+n}/e)(q^2/b, q^2/c; q)_n} e^n {}_4\phi_3\left[\begin{matrix} q^{-n}, & eq^{-1-n}, & e, & bceq^{-2-n} \\ & beq^{-1-n}, & eq^{-n}, & ceq^{-1-n} \end{matrix}; q, q\right]. \qquad (6)
\end{aligned}
$$

Proof. Let $f=q^{-n}$ in (5). Then, the second term on the right-hand side of (5) vanishes and by (4) the first term on the right side of (5) equals

$$
\begin{aligned}
&\frac{(1-q/e)(1-q^{1+n})(eq/c, q^{1-n}/c, b/q, beq^{-1-n}; q)_\infty}{(1-q^{1+n}/e)(q/c, eq^{1-n}/c, be/q, bq^{-1-n}; q)_\infty}\frac{(eq^{1-n}/c, q^2/ce; q)_n}{(q^2/c, q^{1-n}/c; q)_n} \\
&\quad \times {}_4\phi_3\left[\begin{matrix} q^{-n}, & eq^{-1-n}, & e, & bceq^{-2-n} \\ & beq^{-1-n}, & eq^{-n}, & ceq^{-1-n} \end{matrix}; q, q\right] \\
&= \frac{(1-q/e)(1-q^{1+n})(q^2/be, q^2/ce; q)_n}{(1-q^{1+n}/e)(q^2/b, q^2/c; q)_n} e^n {}_4\phi_3\left[\begin{matrix} q^{-n}, & eq^{-1-n}, & e, & bceq^{-2-n} \\ & beq^{-1-n}, & eq^{-n}, & ceq^{-1-n} \end{matrix}; q, q\right],
\end{aligned}
$$

as desired. $\square$

Note. The symmetry of the right-hand side of Eq. (6) with respect to interchanging b and e is also a consequence of the Sears transformation (Eq. (8.3) in [9]).

When $bce = q^{n+2}$ in (6), we get the following summation formula:

Corollary 2.5.

$$ {}_4\psi_4\left[\begin{matrix} b, & c, & e, & q^{-n} \\ q^2/b, & q^2/c, & q^2/e, & q^{2+n} \end{matrix}; q, q\right] = \frac{(1-q/e)(1-q^{1+n})(q^2/be, q^2/ce; q)_n}{(1-q^{1+n}/e)(q^2/b, q^2/c; q)_n} e^n, $$

where $bce = q^{n+2}$.

Corollary 2.6. *For* $|q^2/def| < 1$, *we have*

$$ {}_3\psi_3\left[\begin{matrix} d, & e, & f \\ q^2/d, & q^2/e, & q^2/f \end{matrix}; q, q^2/def\right] = \frac{(q, q^2/de, q^2/df, q^2/ef; q)_\infty}{(q^2/d, q^2/e, q^2/f, q^2/def; q)_\infty}, $$

which is just Ex. 5.18(ii) *in* [4].

Proof. If taking $bc = q^2$ in (2), the second term on the right-hand side of (2) reduces to zero and ${}_8\phi_7$ term equals one. Then we can get

$$ \begin{aligned} {}_3\psi_3\left[\begin{matrix} d, & e, & f \\ q^2/d, & q^2/e, & q^2/f \end{matrix}; q, q^2/def\right] &= \frac{(q, q^2/de, q^2/df, q^2/ef, eq/c, fq/c, b/q, bef/q; q)_\infty}{(q^2/d, q^2/e, q^2/f, q^2/def, q/c, efq/c, be/q, bf/q; q)_\infty} \\ &= \frac{(q, q^2/de, q^2/df, q^2/ef, be/q, bf/q, q/c, efq/c; q)_\infty}{(q^2/d, q^2/e, q^2/f, q^2/def, q/c, efq/c, be/q, bf/q; q)_\infty} \\ &= \frac{(q, q^2/de, q^2/df, q^2/ef; q)_\infty}{(q^2/d, q^2/e, q^2/f, q^2/def; q)_\infty} \end{aligned} $$

as desired. □

Similarly, when choosing $a \to 1$ in (1), we get the following transformation formula:

Theorem 2.7. *For* $|q^2/bcdef|, |bd| < 1$, *we have*

$$ \begin{aligned} &{}_5\psi_5\left[\begin{matrix} b, & c, & d, & e, & f \\ q/b, & q/c, & q/d, & q/e, & q/f \end{matrix}; q, \frac{q^2}{bcdef}\right] \\ &= \frac{(q, q/de, q/df, q/ef, qe/c, qf/c, b, bef; q)_\infty}{(q/d, q/e, q/f, q/def, q/c, qef/c, be, bf; q)_\infty} \\ &\quad \times {}_8\phi_7\left[\begin{matrix} ef/c, & q(ef/c)^{1/2}, & -q(ef/c)^{1/2}, & q/bc, & q/cd, & ef, & e, & f \\ & (ef/c)^{1/2}, & -(ef/c)^{1/2}, & bef, & def, & q/c, & fq/c, & eq/c \end{matrix}; q, bd\right] \\ &\quad + b\frac{(q, bq, bq/c, bq/d, bq/e, bq/f, d, e, f, q/bc; q)_\infty}{(q/b, q/c, q/d, q/e, q/f, bd, be; q)_\infty} \frac{(bdef, q/bdef; q)_\infty}{(bf, def, q/def, q/c, b^2q; q)_\infty} \\ &\quad \times {}_6\phi_5\left[\begin{matrix} b^2, & -bq, & bc, & bd, & be, & bf \\ & -b, & bq/c, & bq/d, & bq/e, & bq/f \end{matrix}; q, \frac{q^2}{bcdef}\right]. \end{aligned} \tag{7} $$

Proof. We specialize (1) by choosing $a \to 1$. The series on the left-hand side becomes

$$ \begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, & qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e, & f \\ & a^{1/2}, & -a^{1/2}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right] \\ &= 1 + \sum_{k=1}^{\infty}(1+q^k)\frac{(b, c, d, e, f; q)_k}{(q/b, q/c, q/d, q/e, q/f; q)_k}(q^2/bcdef)^k \\ &= \sum_{k=0}^{\infty}\frac{(b, c, d, e, f; q)_k}{(q/b, q/c, q/d, q/e, q/f; q)_k}(q^2/bcdef)^k + \sum_{k=1}^{\infty}\frac{(b, c, d, e, f; q)_k}{(q/b, q/c, q/d, q/e, q/f; q)_k}(q^3/bcdef)^k \end{aligned} $$

$$=\sum_{k=0}^{\infty}\frac{(b,c,d,e,f;q)_k}{(q/b,q/c,q/d,q/e,q/f;q)_k}\left(q^2/bcdef\right)^k+\sum_{k=-1}^{-\infty}\frac{(b,c,d,e,f;q)_{-k}}{(q/b,q/c,q/d,q/e,q/f;q)_{-k}}\left(q^3/bcdef\right)^{-k}$$
$$=\sum_{k=0}^{\infty}\frac{(b,c,d,e,f;q)_k}{(q/b,q/c,q/d,q/e,q/f;q)_k}\left(q^2/bcdef\right)^k+\sum_{k=-1}^{-\infty}\frac{(b,c,d,e,f;q)_k}{(q/b,q/c,q/d,q/e,q/f;q)_k}\left(q^2/bcdef\right)^k.$$

The above two series are combined to the bilateral ${}_5\psi_5$ series in (7). The corresponding specialization of the right-hand side of (1) to the right-hand side of (7) is immediate. □

Note. The ${}_6\phi_5$ series in Eq. (7) is not very-well-poised.

Next, from (7) we are led to get two results which are just Ex. 5.18(i) and Ex. 5.18(iii) in [4].

Corollary 2.8. *For* $|q/def|<1$, *we have*

$${}_3\psi_3\left[\begin{matrix}d, & e, & f\\ q/d, & q/e, & q/f\end{matrix};q,q/def\right]=\frac{(q,q/de,q/df,q/ef;q)_\infty}{(q/d,q/e,q/f,q/def;q)_\infty},$$

which is just Ex. 5.18(i) *in* [4].

Proof. We simply take $bc=q$ in (7). The first series on the right-hand side becomes one, while the prefactor of the second series vanishes. □

Corollary 2.9. *When* $bcde=q^{1+n}$, *we have*

$${}_5\psi_5\left[\begin{matrix}b, & c, & d, & e, & q^{-n}\\ q/b, & q/c, & q/d, & q/e, & q^{n+1}\end{matrix};q,q\right]=\frac{(q,q/bc,q/bd,q/cd;q)_n}{(q/b,q/c,q/d,q/bcd;q)_n},$$

which is just Ex. 5.18(iii) *in* [4].

Proof. When taking $f=q^{-n}$ and $bcde=q^{1+n}$ in (7), we get

$${}_5\psi_5\left[\begin{matrix}b, & c, & d, & e, & q^{-n}\\ q/b, & q/c, & q/d, & q/e, & q^{1+n}\end{matrix};q,q\right]$$
$$=\frac{(q,q/de,q^{1+n}/d,q^{1+n}/e,qe/c,q^{1-n}/c,b,beq^{-n};q)_\infty}{(q/d,q/e,q^{1+n},q^{1+n}/de,q/c,q^{1-n}e/c,be,bq^{-n};q)_\infty}$$
$$\times{}_8\phi_7\left[\begin{matrix}eq^{-n}/c, & q(eq^{-n}/c)^{1/2}, & -q(eq^{-n}/c)^{1/2}, & q/bc, & q/cd, & eq^{-n}, & e, & q^{-n}\\ & (eq^{-n}/c)^{1/2}, & -(eq^{-n}/c)^{1/2}, & beq^{-n}, & deq^{-n}, & q/c, & q^{1-n}/c, & eq/c\end{matrix};q,bd\right]$$
$$=\frac{(q,q/de,q^{1+n}/d,q^{1+n}/e,qe/c,q^{1-n}/c,b,beq^{-n};q)_\infty}{(q/d,q/e,q^{1+n},q^{1+n}/de,q/c,q^{1-n}e/c,be,bq^{-n};q)_\infty}\frac{(eq^{1-n}/c,q/ce;q)_n}{(q/c,q^{1-n}/c;q)_n}$$
$$\times{}_4\phi_3\left[\begin{matrix}q^{-n}, & e, & eq^{-n}, & 1\\ & beq^{-n}, & deq^{-n}, & ceq^{-n}\end{matrix};q,q\right]$$
$$=\frac{(q,q/de,q^{1+n}/d,q^{1+n}/e,qe/c,q^{1-n}/c,b,beq^{-n};q)_\infty}{(q/d,q/e,q^{1+n},q^{1+n}/de,q/c,q^{1-n}e/c,be,bq^{-n};q)_\infty}\frac{(eq^{1-n}/c,q/ce;q)_n}{(q/c,q^{1-n}/c;q)_n}$$
$$=\frac{(q,q/bc,q/bd,q/cd;q)_n}{(q/b,q/c,q/d,q/bcd;q)_n},$$

where we have used formula (4). □

Acknowledgments

The authors would like to thank the anonymous referee for his/her valuable suggestions. This research is supported by the National Natural Science Foundation of China (Grant No. 10471016) and the Natural Science Foundation of Henan Province (Grant No. 0511010300).

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