Sublattices of lattices of order-convex sets, I.
The main representation theorem

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Abstract

For a partially ordered set $P$, we denote by $\text{Co}(P)$ the lattice of order-convex subsets of $P$. We find three new lattice identities, (S), (U), and (B), such that the following result holds.

\textbf{Theorem.} Let $L$ be a lattice. Then $L$ embeds into some lattice of the form $\text{Co}(P)$ iff $L$ satisfies (S), (U), and (B).

Furthermore, if $L$ has an embedding into some $\text{Co}(P)$, then it has such an embedding that preserves the existing bounds. If $L$ is finite, then one can take $P$ finite, with

$$|P| \leq 2|J(L)|^2 - 5|J(L)| + 4,$$

where $J(L)$ denotes the set of all join-irreducible elements of $L$.

On the other hand, the partially ordered set $P$ can be chosen in such a way that there are no infinite bounded chains in $P$ and the undirected graph of the predecessor relation of $P$ is a tree.

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1. Introduction

For a partially ordered set (from now on poset) \( (P, \preceq) \), a subset \( X \) of \( P \) is order-convex, if \( x \preceq z \preceq y \) and \( \{x, y\} \subseteq X \) implies that \( z \in X \), for all \( x, y, z \in P \). The set \( \text{Co}(P) \) of all order-convex subsets of \( P \) forms a lattice under inclusion. This lattice is algebraic, atomistic, and join-semidistributive (see Section 2 for the definitions), thus it is a special example of a convex geometry, see P.H. Edelman [5], P.H. Edelman and R. Jamison [6], or K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov [2]. Furthermore, it is ‘biatomic’ and satisfies the nonexistence of so-called ‘zigzags’ of odd length on its atoms. Is is proved in G. Birkhoff and M.K. Bennett [3] that these conditions characterize the lattices of the form \( \text{Co}(P) \).

One of the open problems of [2] is the characterization of all sublattices of the lattices of the form \( \text{Co}(P) \).

**Problem 3 (of [2] for \( \text{Co}(P) \)).** Describe the subclass of those lattices that are embeddable into finite lattices of the form \( \text{Co}(P) \).

In the present paper, we solve completely this problem, not only in the finite case but also for arbitrary lattices. Our main result (Theorem 6.7) is that a lattice \( L \) can be embedded into some lattice of the form \( \text{Co}(P) \) iff \( L \) satisfies three completely new identities, that we denote by (S), (U), and (B). Furthermore, \( P \) can be taken either finite in case \( L \) is finite, or tree-like (see Theorem 7.7).

This result is quite surprising, as it yields the unexpected consequence (see Corollary 6.9) that the class of all lattices that can be embedded into some \( \text{Co}(P) \) is a variety, thus it is closed under homomorphic images. However, while it is fairly easy (though not completely trivial) to verify directly that the class is closed under reduced products and substructures (thus it is a quasivariety), we do not know any direct proof that it is closed under homomorphic images.

One of the difficulties of the present work is to guess, for a given \( L \), which poset \( P \) will solve the embedding problem for \( L \) (i.e., \( L \) embeds into \( \text{Co}(P) \)). The first natural guess, that consists of using for \( P \) the set of all join-irreducible elements of \( L \), fails, as illustrated by the two examples of Section 8. We shall construct \( P \) via sequences of join-irreducible elements of \( L \). In fact, we are able to embed \( L \) into \( \text{Co}(P) \) for two different sorts of posets \( P \):

1. \( P \) is finite in case \( L \) is finite; this is the construction of Section 6.
2. \( P \) is tree-like (as defined in Section 2); this is the construction of Section 7.

The two requirements (1) and (2) above can be simultaneously satisfied in case \( L \) has no \( D \)-cycle, see Theorem 7.7(iii). However, the finite lattice \( L \) of Example 8.2 can be embedded into some finite \( \text{Co}(Q) \), but into no \( \text{Co}(R) \), where \( R \) is a finite tree-like poset, see Corollary 10.6. It is used to produce, in Section 10, a quasi-identity that holds in all \( \text{Co}(R) \), where \( R \) is finite and tree-like (or even what we call ‘crown-free’), but not in all finite \( \text{Co}(P) \).

We conclude the paper by a list of open problems.
2. Basic concepts

A lattice \( L \) is join-semidistributive, if it satisfies the axiom
\[
x \lor y = x \lor z \ \Rightarrow \ x \lor y = x \lor (y \land z), \quad \text{for all } x, y, z \in L.
\]
(\( \text{SD}_\lor \))

We denote by \( \text{J}(L) \) the set of join-irreducible elements of \( L \). We say that \( L \) is finitely spatial (respectively, spatial) if every element of \( L \) is a join of join-irreducible (respectively, completely join-irreducible) elements of \( L \).

We say that \( L \) is lower continuous, if the equality
\[
a \lor \bigwedge X = \bigwedge (a \lor X)
\]
holds, for all \( a \in L \) and all downward directed \( X \subseteq L \) such that \( \bigwedge X \) exists (where \( a \lor X = \{ a \lor x \mid x \in X \} \)). It is well known that every dually algebraic lattice is lower continuous—see Lemma 2.3 in P. Crawley and R.P. Dilworth [4], and spatial (thus finitely spatial)—see Theorem I.4.22 in G. Gierz et al. [9] or Lemma 1.3.2 in V.A. Gorbunov [10].

For every element \( x \) in a lattice \( L \), we put
\[
\downarrow x = \{ y \in L \mid y \leq x \}, \quad \uparrow x = \{ y \in L \mid y \geq x \}.
\]

If \( a, b, c \in L \) such that \( a \leq b \lor c \), we say that the (formal) inequality \( a \leq b \lor c \) is a nontrivial join-cover, if \( a \not\leq b, c \). We say that it is minimal in \( b \), if \( a \not\leq x \lor c \), for all \( x < b \), and we say that it is a minimal nontrivial join-cover, if it is a nontrivial join-cover and it is minimal in both \( b \) and \( c \).

The join-dependency relation \( D = D_L \) (see R. Freese, J. Ježek, and J.B. Nation [7]) is defined on the join-irreducible elements of \( L \) by putting
\[
pDq, \text{ if } p \not= q \text{ and } \exists x \text{ such that } p \leq q \lor x \text{ holds and is minimal in } q.
\]
It is important to observe that \( pDq \) implies that \( p \not< q \), for all \( p, q \in \text{J}(L) \).

For a poset \( P \) endowed with a partial ordering \( \preceq \), we shall denote by \( < \) the corresponding strict ordering. The set of all order-convex subsets of \( P \) forms a lattice under inclusion, that we shall denote by \( \text{Co}(P) \). The meet in \( \text{Co}(P) \) is the intersection, while the join is given by
\[
X \lor Y = X \cup Y \cup \bigcup \{ z \in P \mid \exists (x, y) \in (X \times Y) \cup (Y \times X) \text{ such that } x < z < y \},
\]
for all \( X, Y \in \text{Co}(P) \). Let us denote by \( < \) the predecessor relation of \( P \). We say that a path of \( P \) is a finite sequence \( d = (x_0, \ldots, x_{n-1}) \) of distinct elements of \( P \) such that either \( x_i < x_{i+1} \) or \( x_{i+1} < x_i \), for all \( i \) with \( 0 \leq i \leq n - 2 \); if \( n > 0 \), we say that \( d \) is a path from \( x_0 \) to \( x_{n-1} \). We say that the path \( d \) is oriented, if \( x_i < x_{i+1} \), for all \( i \) with \( 0 \leq i \leq n - 2 \). We say that \( P \) is tree-like, if the following properties hold:
(i) for all \(a \preceq b\) in \(P\), there are \(n < \omega\) and \(x_0, \ldots, x_n \in P\) such that \(a = x_0 \prec x_1 \prec \cdots \prec x_n = b\);
(ii) for all \(a, b \in P\), there exists at most one path from \(a\) to \(b\).

3. Dually 2-distributive lattices

For a positive integer \(n\), the identity of \(n\)-distributivity is introduced in A.P. Huhn [12]. In this paper we shall only need the dual of 2-distributivity, which is the following identity:

\[
a \land (x \lor y \lor z) = (a \land (x \lor y)) \lor (a \land (x \lor z)) \lor (a \land (y \lor z)).
\]

We omit the easy proof of the following lemma, that expresses how dual 2-distributivity can be read on the join-irreducible elements.

**Lemma 3.1.** Let \(L\) be a dually 2-distributive lattice. For all \(p \in \text{J}(L)\) and all \(a, b, c \in L\), if \(p \leq a \lor b \lor c\), then either \(p \leq a \lor b\) or \(p \leq a \lor c\) or \(p \leq b \lor c\).

We observe that for finitely spatial \(L\), the converse of Lemma 3.1 holds.

The following lemma will be used repeatedly throughout the paper.

**Lemma 3.2.** Let \(L\) be a dually 2-distributive, complete, lower continuous lattice. Let \(p \in \text{J}(L)\) and let \(a, b \in L\) such that \(p \leq a \lor b\) and \(p \not< a, b\). Then the following assertions hold:

(i) There are minimal \(x \leq a\) and \(y \leq b\) such that \(p \leq x \lor y\).
(ii) Any minimal \(x \leq a\) and \(y \leq b\) such that \(p \leq x \lor y\) are join-irreducible.

**Proof.** (i) Let \(X \subseteq \downarrow a\) and \(Y \subseteq \downarrow b\) be chains such that \(p \leq x \lor y\), for all \((x, y) \in X \times Y\). It follows from the lower continuity of \(L\) that \(p \leq (\bigwedge X) \lor (\bigwedge Y)\). The conclusion of (i) follows from a simple application of Zorn’s Lemma.

(ii) From \(p \not< a, b\) it follows that both \(x\) and \(y\) are nonzero. Suppose that \(x = x_0 \lor x_1\) for some \(x_0, x_1 < x\). It follows from the minimality assumption on \(x\) that \(p \not< x_0 \lor y\) and \(p \not< x_1 \lor y\), whence, by Lemma 3.1, \(p \leq x_0 \lor x_1\), thus \(p \leq x \leq a\), a contradiction. Hence \(x\) is join-irreducible. \(\square\)

For \(p, a, b \in \text{J}(L)\), we say that \(\langle a, b \rangle\) is a conjugate pair with respect to \(p\), if \(p \not< a, b\) and \(a\) and \(b\) are minimal such that \(p \leq a \lor b\); we say then that \(b\) is a conjugate of \(a\) with respect to \(p\). Observe that the latter relation is symmetric in \(a\) and \(b\), and that it implies that \(pD a\) and \(pD b\).

**Notation 3.3.** For a lattice \(L\) and \(p \in \text{J}(L)\), we put

\[
[p]^D = \{x \in \text{J}(L) \mid pD x\}.
\]
Corollary 3.4. Let $L$ be a dually 2-distributive, complete, lower continuous lattice, and let $p \in J(L)$. Then every $a \in [p]^D$ has a conjugate with respect to $p$.

**Proof.** By the definition of join-dependency, there exists $c \in L$ such that $p \leq a \lor c$ and $p \not\leq x \lor c$, for all $x < a$. By Lemma 3.2, there are $a' \leq a$ and $b \leq c$ minimal such that $p \leq a' \lor b$, and both $a'$ and $b$ are join-irreducible. It follows that $a' = a$, whence $b$ is a conjugate of $a$ with respect to $p$. □

4. Stirlitz, Udav, and Bond

4.1. The Stirlitz identity $(S)$ and the axiom $(S_j)$

Let $(S)$ be the following identity:

$$a \land (b' \lor c) = (a \land b') \lor \bigvee_{i < 2} (a \land (b_i \lor c) \land ((b' \land (a \lor b_i)) \lor c)),$$

where we put $b' = b \land (b_0 \lor b_1)$.

**Lemma 4.1.** The Stirlitz identity $(S)$ holds in $\text{Co}(P)$, for any poset $(P, \leq)$.

**Proof.** Let $A, B, B_0, B_1, C \in \text{Co}(P)$ and $a \in A \cap (B' \lor C)$, where we put $B' = B \cap (B_0 \lor B_1)$. Denote by $D$ the right-hand side of the Stirlitz identity calculated with these parameters. If $a \in B'$ then $a \in A \cap B' \subseteq D$. If $a \in C$ then $a \in A \cap C \subseteq D$.

Suppose that $a \notin B' \cup C$. There exist $b \in B'$ and $c \in C$ such that, say, $b \leq a \leq c$. Since $b \in B_0 \lor B_1$, there are $i < 2$ and $b' \in B_i$ such that $b' \leq b$, hence $a \in A \cap (B_i \lor C)$. Furthermore, $b \in B' \cap (A \lor B_i)$, thus $a \in (B' \cap (A \lor B_i)) \lor C$, so $a \in D$. □

**Lemma 4.2.** The Stirlitz identity $(S)$ implies dual 2-distributivity.

**Proof.** Take $b_0 = x, b_1 = y, b = x \lor y, a = c = z$. □

Let $(SD^2_\lor)$ be the following identity:

$$x \lor (y \land z) = x \lor (y \land (x \lor (z \land (x \lor y)))) \quad (SD^2_\lor)$$

It is well known that $(SD^2_\lor)$ implies join-semidistributivity (that is, the axiom $(SD_\lor)$), see, for example, P. Jipsen and H. Rose [13, p. 81].

**Lemma 4.3.** The Stirlitz identity $(S)$ implies $(SD^2_\lor)$.

**Proof.** Let $L$ be a lattice satisfying $(S)$, let $x, y, z \in L$. Set $y_2 = y \land (x \lor (z \land (x \lor y)))$. Set $a = b_1 = y, b = z, c = b_0 = x$, and $b' = b \land (b_0 \lor b_1) = z \land (x \lor y)$. Then the following inequalities hold:
This implies that $x \lor y \leq x \lor (y \land z)$. Since the converse inequality holds in any lattice, the conclusion follows.  

We now introduce a lattice-theoretical axiom, the join-irreducible interpretation of (S), that we will denote by ($S_j$):

For all $a, b, b_0, b_1, c \in J(L)$, the inequalities $a \leq b \lor c$, $b \leq b_0 \lor b_1$, and $a \neq b$ imply that either $a \leq b \lor c$ for some $b < b$ or $b \leq a \lor b_1$ and $a \leq b_1 \lor c$ for some $i < 2$.

Throughout the paper we shall make repeated use of the item (i) of the following statement. Item (ii) provides a convenient algorithm for verifying whether a finite lattice satisfies (S).

**Proposition 4.4.** Let $L$ be a lattice. Then the following assertions hold:

(i) If $L$ satisfies (S), then $L$ satisfies ($S_j$).

(ii) If $L$ is complete, lower continuous, finitely spatial, dually $2$-distributive, and satisfies ($S_j$), then $L$ satisfies (S).

**Proof.** (i) Let $a \leq b \lor c$, $b \leq b_0 \lor b_1$, and $a \neq b$ for some $a, b, b_0, b_1, c \in J(L)$. Then the element $b'$ of the Stirling identity is $b' = b \land (b_0 \lor b_1) = b$; observe also that $a \land (b \lor c) = a$. Therefore, applying (S) yields

$$a = a \land (b' \lor c) = (a \land b') \lor \bigvee_{i<2} (a \land (b_i \lor c) \land ((b' \land (a \lor b_i)) \lor c))$$

Since $a$ is join-irreducible, either $a \leq b$ or $a \leq (b_0 \lor c) \land (b \land (a \lor b_i)) \lor c$ for some $i < 2$. If $a \leq b$ then $a \leq a \lor c$ with $a < b$ (because $a \neq b$). Suppose that $a \not< b$. Then $a \leq (b \lor c) \land (b \land (a \lor b_i)) \lor c \leq b_i \lor c$ for some $i < 2$. If $b \not< a \lor b_1$, then $a \leq b \lor c$ for $b \equiv b \land (a \lor b_i) < b$.

(ii) Put $b' = b \land (b_0 \lor b_1)$, and let $d$ denote the right-hand side of the identity (S). Since $d \leq a \lor (b' \lor c)$, we must prove the converse inequality only. Let $a_1 \in J(L)$ with $a_1 \leq a \lor (b' \lor c)$. Then $a_1 \leq a$ and $a_1 \leq b' \lor c$. If $a_1 \leq b'$, then $a_1 \leq a \land b' \leq d$. If $a_1 \leq c$, then $a_1 \leq a \lor c \leq d$. 
Suppose now that \( a_1 \not\leq b', c \). Then, by using Lemma 3.2, we obtain that there are minimal \( b'_1 \leq b' \) and \( c_1 \leq c \) such that the following inequality holds,

\[
a_1 \leq b'_1 \lor c_1
\]

(4.1)

and both \( b'_1 \) and \( c_1 \) are join-irreducible. From \( a_1 \not\leq b' \) it follows that \( a_1 \not\leq b'_1 \). If \( b'_1 \leq b_i \) for some \( i < 2 \), then the inequalities \( b'_1 \leq b' \land b_i \leq b' \land (a \lor b_i) \) and \( a_1 \leq b'_1 \lor c_1 \leq (b' \land (a \lor b_i)) \lor c \) hold; but in this case, we also have \( a_1 \leq a \land (b_i \lor c) \), whence \( a_1 \leq d \).

Suppose that \( b'_1 \not\leq b_0, b_1 \). Then, by Lemma 3.2, there are join-irreducible elements \( d_i \leq b_i \), \( i < 2 \), such that the following inequality

\[
b'_1 \leq d_0 \lor d_1
\]

(4.2)

holds. It follows from (4.1), (4.2), \( a_1 \not\leq b'_1 \), the minimality of \( b'_1 \) in (4.1), and \((S_i)\) that there exists \( i < 2 \) such that \( a_1 \leq a \lor d_i \) and \( a_1 \leq d_i \lor c_1 \). Then the following inequalities hold:

\[
a_1 \leq a \land (d_i \lor c_1) \land (b'_1 \lor c_1) \leq a \land (b_i \lor c) \land ((b' \land (a \lor b_i)) \lor c) \leq d.
\]

In every case, \( a_1 \leq d \). Since \( L \) is finitely spatial, it follows that \( a \land (b' \lor c) \leq d \). □

4.2. The Bond identity (B) and the axiom \((B_i)\)

Let (B) be the following identity:

\[
x \land (a_0 \lor a_1) \land (b_0 \lor b_1) = \bigvee_{i<2} \left( (x \land a_i \land (b_0 \lor b_1)) \lor (x \land b_i \land (a_0 \lor a_1)) \right) \lor \bigvee_{i<2} \left( (x \land (a_0 \lor a_1) \land (b_0 \lor b_1) \land (a_0 \lor b_i) \land (a_1 \lor b_{1-i}) \right).
\]

Lemma 4.5. The Bond identity (B) holds in \( \text{Co}(P) \), for any poset \( \langle P, \leq \rangle \).

Proof. Let \( X, A_0, A_1, B_0, B_1 \) be elements of \( \text{Co}(P) \). Denote by \( C \) the right-hand side of the Bond identity formed from these elements. Let \( x \in X \cap (A_0 \lor A_1) \cap (B_0 \lor B_1) \), we prove that \( x \in C \). The conclusion is obvious if \( x \in A_0 \cup A_1 \cup B_0 \cup B_1 \), so suppose that \( x \not\in A_0 \cup A_1 \cup B_0 \cup B_1 \). Since \( x \in (A_0 \lor A_1) \setminus (A_0 \lor A_1) \), there are \( a_0 \in A_0 \) and \( a_1 \in A_1 \) such that, say, \( a_0 \not\leq x \not\leq a_1 \). Since \( x \in (B_0 \lor B_1) \setminus (B_0 \lor B_1) \), there are \( b_0 \in B_0 \) and \( b_1 \in B_1 \) such that either \( b_0 \not\leq x \not\leq b_1 \) or \( b_1 \not\leq x \not\leq b_0 \). In the first case, \( x \) belongs to \( X \cap (A_0 \lor A_1) \cap (B_0 \lor B_1) \cap (A_0 \lor B_1) \cap (A_1 \lor B_0) \), thus to \( C \). In the second case, \( x \) belongs to \( X \cap (A_0 \lor A_1) \cap (B_0 \lor B_1) \cap (A_0 \lor B_1) \cap (A_1 \lor B_0) \) again to \( C \). □

We now introduce a lattice-theoretical axiom, the join-irreducible interpretation of (B), that we will denote by \((B_i)\):
For all \( x, a_0, a_1, b_0, b_1 \in \text{J}(L) \), the inequalities \( x \leq a_0 \lor a_1, b_0 \lor b_1 \) imply that either \( x \leq a_i \) or \( x \leq b_i \) for some \( i < 2 \) or \( x \leq a_0 \lor b_0, a_1 \lor b_1 \) or \( x \leq a_0 \lor b_1, a_1 \lor b_0 \).

Throughout the paper we shall make repeated use of the item (i) of the following statement. Item (ii) provides a convenient algorithm for verifying whether a finite lattice satisfies (B).

**Proposition 4.6.** Let \( L \) be a lattice. Then the following assertions hold:

(i) If \( L \) satisfies (B), then \( L \) satisfies (B).

(ii) If \( L \) is complete, lower continuous, finitely spatial, dually 2-distributive, and satisfies (B), then \( L \) satisfies (B).

**Proof.** Item (i) is easy to prove by using the (B) identity and the join-irreducibility of \( x \).

(ii) Let \( u \) (respectively, \( v \)) denote the left- (respectively, right-) hand side of the identity (B). It is obvious that \( u \leq v \). Since \( L \) is finitely spatial, in order to prove that \( u \leq v \) it is sufficient to prove that for all \( p \in \text{J}(L) \) such that \( p \leq u \), the inequality \( p \leq v \) holds. This is obvious if either \( p \leq a_i \) or \( p \leq b_i \) for some \( i < 2 \), so suppose that \( p \not\leq a_i, b_i \), for all \( i < 2 \). Then, by Lemma 3.2, there exist \( x_0, x_1, y_0, y_1 \in \text{J}(L) \) such that \( x_i \leq a_i \) and \( y_i \leq b_i \), for all \( i < 2 \), while \( p \leq x_0 \lor x_1, y_0 \lor y_1 \). By assumption, we obtain that one of the following assertions holds:

\[
\begin{align*}
p &\leq (x_0 \lor y_0) \land (x_1 \lor y_1) \leq (a_0 \lor b_0) \land (a_1 \lor b_1), \\
p &\leq (x_0 \lor y_1) \land (x_1 \lor y_0) \leq (a_0 \lor b_1) \land (a_1 \lor b_0).
\end{align*}
\]

In any case, \( p \leq v \), which completes the proof. \( \square \)

**4.3. The Udav identity (U) and the axiom (U)\( _j \)**

Let (U) be the following identity:

\[
\begin{align*}
x \land (x_0 \lor x_1) \land (x_1 \lor x_2) \land (x_0 \lor x_2) \\
= (x \land x_0 \land (x_1 \lor x_2)) \lor (x \land x_1 \land (x_0 \lor x_2)) \lor (x \land x_2 \land (x_0 \lor x_1)).
\end{align*}
\]

**Lemma 4.7.** The Udav identity (U) holds in \( \text{Co}(P) \), for any poset \( \langle P, \sqsubseteq \rangle \).

**Proof.** Let \( X, X_0, X_1, X_2 \) be elements of \( \text{Co}(P) \). Denote by \( U \) (respectively, \( V \)) the left-hand side (respectively, right-hand side) of the Udav identity formed from these elements. It is clear that \( U \) contains \( V \). Conversely, let \( x \in U \), we prove that \( x \) belongs to \( V \). This is clear if \( x \in X_0 \cup X_1 \cup X_2 \), so suppose that \( x \not\in X_0 \cup X_1 \cup X_2 \). Since \( x \in (X_0 \lor X_1) \setminus (X_0 \cup X_1) \), there are \( x_0 \in X_0 \) and \( x_1 \in X_1 \) such that, say, \( x_0 \not\leq x \leq x_1 \). Since \( x \in (X_1 \lor X_2) \setminus (X_1 \cup X_2) \), there are \( x_1' \in X_1 \) and \( x_2 \in X_2 \) such that either \( x_1' \not\leq x \not\leq x_2 \) or \( x_2 \not\leq x \not\leq x_1' \). But since \( x \not\leq x_1 \in X_1 \) and \( x \not\in X_1 \), the first possibility is ruled out, whence \( x_2 \not\leq x \not\leq x_1' \). Since \( x \in (X_0 \lor X_2) \setminus (X_0 \cup X_2) \), there are \( x_0' \in X_0 \) and \( x_2' \in X_2 \) such that
either \( x' \preceq x \preceq x' \) or \( x' \preceq x \preceq x'' \). The first possibility is ruled out by \( x_2 \preceq x \) and \( x \notin X_2 \), while the second possibility is ruled out by \( x_0 \preceq x \) and \( x \notin X_0 \). In any case, we obtain a contradiction.

As we already did for (S) and (B), we now introduce a lattice-theoretical axiom, the \textit{join-irreducible interpretation of } \((U)\), that we will denote by \((U_j)\):

For all \( x, x_0, x_1, x_2 \in J(L) \), the inequalities \( x \preceq x_0 \lor x_1 \lor x_2 \) imply that either \( x \preceq x_0 \) or \( x \preceq x_1 \) or \( x \preceq x_2 \).

Throughout the paper we shall make repeated use of the item (i) of the following statement. Item (ii) provides a convenient algorithm for verifying whether a finite lattice satisfies \((U)\).

**Proposition 4.8.** Let \( L \) be a lattice. Then the following assertions hold:

(i) If \( L \) satisfies \((U)\), then \( L \) satisfies \((U_j)\).

(ii) If \( L \) is complete, lower continuous, finitely spatial, dually 2-distributive, and satisfies both \((B_j)\) and \((U_j)\), then \( L \) satisfies both \((B)\) and \((U)\).

**Proof.** Item (i) is easy to prove by using the \((U)\) identity and the join-irreducibility of \( x \).

(ii) We have already seen in Proposition 4.6 that \( L \) satisfies \((B)\).

Let \( u \) (respectively, \( v \)) be the left-hand side (respectively, right-hand side) of the identity \((U)\). It is clear that \( v \preceq u \). Let \( p \in J(L) \) such that \( p \preceq u \), we prove that \( p \preceq v \).

This is obvious if \( p \preceq x_i \) for some \( i < 3 \), so suppose that \( p \notin x_i \), for all \( i < 3 \). Then, by using Lemma 3.2, we obtain that there are join-irreducible elements \( p_i, p_i' \preceq x_i \) (\( i < 3 \)) of \( L \) such that the following inequalities hold:

\[
p \preceq p_0 \lor p_1', p_1' \lor p_2', p_0' \lor p_2'.
\]

(4.3)

Since \( p \notin x_1 \) and \( L \) satisfies \((B_j)\), it follows from the first two inequalities of (4.3) that \( p \preceq p_0 \lor p_1', p_1 \lor p_2 \). Similarly, from \( p \notin x_2 \), the last two inequalities of (4.3), and \((B_j)\), we obtain the inequalities

\[
p \preceq p_1' \lor p_2', p_0' \lor p_2.
\]

and from the first and the last inequality of (4.3), together with \( p \notin x_0 \) and \((B_j)\), we obtain the inequalities

\[
p \preceq p_0 \lor p_1', p_0' \lor p_1.
\]

In particular, we have obtained the inequalities

\[
p \preceq p_0 \lor p_1', p_1' \lor p_2', p_0 \lor p_2',
\]

whence, by the assumption \((U_j)\), \( p \preceq x_i \) for some \( i < 3 \), a contradiction. \( \square \)
5. First steps together of the identities (S), (U), and (B)

5.1. Udav–Bond partitions

The goal of this subsection is to prove the following partition result of the sets \([p]^D\) (see Notation 3.3).

**Proposition 5.1.** Let \(L\) be a complete, lower continuous, dually 2-distributive lattice that satisfies (U) and (B). Then for every \(p \in J(L)\), there are subsets \(A\) and \(B\) of \([p]^D\) that satisfy the following properties:

(i) \([p]^D = A \cup B\) and \(A \cap B = \emptyset\).

(ii) For all \(x, y \in [p]^D\), \(p \leq x \lor y\) iff \((x, y) \in (A \times B) \cup (B \times A)\).

Moreover, the set \([A, B]\) is uniquely determined by these properties.

The set \([A, B]\) will be called the Udav–Bond partition (of \([p]^D\)) associated with \(p\). We observe that every conjugate with respect to \(p\) of an element of \(A\) (respectively, \(B\)) belongs to \(B\) (respectively, \(A\)).

**Proof.** If \([p]^D = \emptyset\) the result is obvious, so suppose that \([p]^D \neq \emptyset\). By Lemma 3.2, there are \(a, b \in [p]^D\) minimal such that \(p \leq a \lor b\). We define \(A\) and \(B\) by the formulas

\[ A = \{ x \in [p]^D \mid p \leq x \lor b \}, \quad B = \{ y \in [p]^D \mid p \leq a \lor y \}. \]

Let \(x \in [p]^D\). By Corollary 3.4, \(x\) has a conjugate with respect to \(p\), denote it by \(y\). By Lemma 3.2(ii), \(y\) is join-irreducible, thus \(y \in [p]^D\). By applying (B\(_j\)) to the inequalities \(p \leq a \lor b, x \lor y\), we obtain that either \(p \leq a \lor x, b \lor y\) or \(p \leq a \lor y, b \lor x\), thus either \(p \leq a \lor x\) or \(p \leq b \lor x\). If both inequalities hold simultaneously, then, since \(p \leq a \lor b\) and by (U\(_j\)), we obtain that \(p\) lies below either \(a\) or \(b\) or \(x\), a contradiction. Hence we have established (i).

Let \(x, y \in [p]^D\), we shall establish in which case the inequality \(p \leq x \lor y\) holds. Suppose first that \(x \in A\) and \(y \in B\). By applying (B\(_j\)) to the inequalities \(p \leq b \lor x, a \lor y\), we obtain that either \(p \leq x \lor y\) or \(p \leq b \lor y\). In the second case, \(y \in B\), but \(y \in A\), a contradiction by item (i); hence \(p \leq x \lor y\).

Now suppose that \(x, y \in A\). If \(p \leq x \lor y\), then, by applying (U\(_j\)) to the inequalities \(p \leq x \lor y, b \lor x, b \lor y\), we obtain that \(p\) lies below either \(x\) or \(y\) or \(b\), a contradiction. Hence \(p \not\leq x \lor y\). The conclusion is the same for \((x, y) \in B \times B\). This concludes the proof of item (ii).

Finally, the uniqueness of \([A, B]\) follows easily from items (i) and (ii). \(\square\)

5.2. Choosing orientation with Stirlitz

In this subsection we shall investigate further the configuration on which (S\(_j\)) is based. The following lemma suggests an ‘orientation’ of the join-irreducible elements in such
a configuration. More specifically, we are trying to embed the given lattice into $\text{Co}(P)$, for some poset $(P, \leq)$. Attempting to define $P$ as $J(L)$, this would suggest to order the elements $a, b, b_0,$ and $b_1$ by $c < a < b$ and $b_{1-1} < b < b_1$. Although the elements of $P$ will be defined via finite sequences of elements of $J(L)$, rather than just elements of $J(L)$, this idea will be crucial in the construction of Section 7.

**Lemma 5.2.** Let $L$ be a lattice satisfying ($S_i$) and ($U_j$). Let $a, b, b_0, b_1, c \in J(L)$ such that $a \neq b$ and satisfying the inequalities $a \leq b \lor c$ with $b$ minimal such, and $b \leq b_0 \lor b_1$ with $b \neq b_0, b_1$. Then the following assertions hold:

(i) The inequalities $b \leq a \lor b_1$ and $a \leq b_1 \lor c$ together are equivalent to the single inequality $b \leq b_1 \lor c$, for all $i < 2$.

(ii) There is exactly one $i < 2$ such that $b \leq b_1 \lor c$.

**Proof.** We first observe that $b \neq c$ (otherwise $a \leq c$). If $b \leq a \lor b_1$ and $a \leq b_1 \lor c$, then obviously $b \leq b_1 \lor c$. Suppose, conversely, that $b \leq b_0 \lor c$. If $b \leq b_1 \lor c$, then, by observing that $b \leq b_0 \lor b_1$ and applying ($U_j$), we obtain that either $b \leq b_0$ or $b \leq b_1$ or $b \leq c$, a contradiction. Hence $b \neq b_1 \lor c$, the uniqueness statement of (ii) follows. Furthermore, by ($S_i$), there exists $i < 2$ such that $b \leq a \lor b_1$ and $a \leq b_1 \lor c$, whence $b \leq b_1 \lor c$, thus $i = 0$. Therefore, $b \leq a \lor b_0$ and $a \leq b_0 \lor c$. □

Next, for a conjugate pair $(b, b')$ of elements of $J(L)$ with respect to some element $a$ of $J(L)$, we define

$$C[b, b'] = \{x \in J(L) \mid bDx \text{ and } b \leq b' \lor x\}. \quad (5.1)$$

**Notation 5.3.** Let $\text{SUB}$ denote the class of all lattices that satisfy the identities ($S$), ($U$), and (B).

Hence $\text{SUB}$ is a variety of lattices. It is finitely based, that is, it is defined by finitely many equations.

**Lemma 5.4.** Let $L$ be a complete, lower continuous, finitely spatial lattice in $\text{SUB}$. Let $a, b \in J(L)$ such that $aDb$. Then the equality $C[b, b_0] = C[b, b_1]$ holds, for all conjugates $b_0$ and $b_1$ of $b$ with respect to $a$.

**Proof.** We prove, for example, that $C[b, b_0]$ is contained in $C[b, b_1]$. Let $x \in C[b, b_0]$ (so $b \leq b_0 \lor x$), and suppose that $x \notin C[b, b_1]$ (so $b \notin b_1 \lor x$). By Corollary 3.4, $x$ has a conjugate, say, $y$, with respect to $b$. Since both relations $a \leq b \lor b_1$ and $b \leq x \lor y$ are minimal nontrivial join-covers, it follows from Lemma 5.2 that either $b \leq b_1 \lor x$ or $b \leq b_1 \lor y$, but the first possibility does not hold. Hence the following inequalities hold:

$$b \leq b_0 \lor x, b_1 \lor y, x \lor y. \quad (5.2)$$
Furthermore, by the uniqueness statement of Lemma 5.2, \( b \not\in b_0 \lor y \). Thus, by (Bj) and the first two inequalities in (5.2) (observe that \( b \not\in b_0, b_1, x, y \)), we obtain that \( b \leq b_0 \lor b_1 \). Hence \( a \leq b \lor b_0 \leq b_0 \lor b_1 \), whence \( a \leq b \lor b_0 \lor b_1 \), a contradiction by (Uj).

For all \( a, b \in J(L) \) such that \( aDb \), there exists, by Corollary 3.4, a conjugate \( b' \) of \( b \) with respect to \( a \). By Lemma 5.4, for fixed \( a \), the value of \( C[b; b'] \) does not depend of \( b' \). This entitles us to define

\[
C(a, b) = C[b, b'], \quad \text{for any conjugate } b' \text{ of } b \text{ with respect to } a. \quad (5.3)
\]

**Lemma 5.5.** Let \( a, b \in J(L) \) such that \( aDb \). Then the set \( \{ C(a, b), [b]^D \setminus C(a, b) \} \) is the Udav–Bond partition of \( [b]^D \) associated with \( b \).

**Proof.** It suffices to prove that the assertions (i) and (ii) of Proposition 5.1 are satisfied by the set \( \{ C(a, b), [b]^D \setminus C(a, b) \} \). We first observe the following immediate consequence of Lemma 5.2.

**Claim.** For any \( x \in [b]^D \) and any conjugate \( x' \) of \( x \), \( x \not\in C(a, b) \) iff \( x' \in C(a, b) \).

From now on we fix a conjugate \( b' \) of \( b \) with respect to \( a \). Let \( x, y \in [b]^D \), let \( x' \) (respectively, \( y' \)) be a conjugate of \( x \) (respectively, \( y \)) with respect to \( b \).

Suppose first that \( x \in C(a, b) \) and \( y \not\in C(a, b) \), we prove that \( b \leq x \lor y \). It follows from Claim above that \( y' \in C(a, b) \), whence the inequalities \( b \leq b' \lor x, b' \lor y \) hold, hence, by (Uj), \( b \not\in x \lor y \). But \( b \leq x \lor x', y \lor y' \), thus, since \( b \not\in x, x', y, y' \) and by (Bj), the inequality \( b \leq x \lor y \) holds.

Suppose next that \( x, y \in C(a, b) \). Since \( b \leq b' \lor x, b' \lor y \), the inequality \( b \leq x \lor y \) would yield, by (Uj), a contradiction; whence \( b \not\in x \lor y \).

Suppose, finally, that \( x, y \not\in C(a, b) \). Thus, by Claim, \( y' \in C(a, b) \), whence, by the above, \( b \leq x \lor y ', y \lor y' \), whence, by (Uj), \( b \not\in x \lor y \). \( \square \)

### 5.3. Stirlitz tracks

Throughout this subsection, we shall fix a lattice \( L \) satisfying the identities (S), (U), and (B). By Lemma 4.2, \( L \) is dually 2-distributive as well. Furthermore, it follows from Propositions 4.4, 4.6, and 4.8 that \( L \) satisfies (Sj), (Uj), and (Bj).

**Definition 5.6.** For a natural number \( n \), a Stirlitz track of length \( n \) is a pair \( \sigma = \langle (a_i : 0 \leq i \leq n), (a'_i : 1 \leq i \leq n) \rangle \), where the elements \( a_i \) for \( 0 \leq i \leq n \) and \( a'_i \) for \( 1 \leq i \leq n \) are join-irreducible and the following conditions are satisfied:

(i) the inequality \( a_i \leq a_{i+1} \lor a'_{i+1} \) holds, for all \( i \in \{0, \ldots, n - 1\} \), and it is a minimal nontrivial join-cover;

(ii) the inequality \( a_i \leq a'_i \lor a_{i+1} \) holds, for all \( i \in \{1, \ldots, n - 1\} \).

We shall call \( a_0 \) the base of \( \sigma \). Observe that \( a_iDa_{i+1} \), for all \( i \in \{0, \ldots, n - 1\} \).
Observe that if \( \sigma \) is a Stirlitz track as above, then, by Lemma 5.2, the following inequalities also hold:

\[
\begin{align*}
    a_{i+1} & \leq a_i \lor a_{i+2}, \\
    a_i & \leq a_{i+1} \lor a_{i+2},
\end{align*}
\]

for all \( i \in \{0, \ldots, n-2\} \).

The main property that we will need about Stirlitz tracks is the following:

**Lemma 5.7.** For a positive integer \( n \), let \( \sigma = \langle \langle a_i \mid 0 \leq i \leq n \rangle, \langle a'_i \mid 1 \leq i \leq n \rangle \rangle \) be a Stirlitz track of length \( n \). Then the inequalities \( a_i \leq a_0 \lor a_n \) and \( a_i \leq a'_0 \lor a'_n \), for all \( i \in \{0, \ldots, n\} \). Furthermore, \( 0 \leq k < l \leq n \) implies that \( a_k \neq a_l \); in particular, the elements \( a_i \), for \( 0 \leq i \leq n \), are distinct.

**Proof.** We argue by induction on \( n \). The result is trivial for \( n = 1 \), and it follows from (5.4) and (5.5) for \( n = 2 \). Suppose that the result holds for \( n \geq 2 \), and let \( \sigma = \langle \langle a_i \mid 0 \leq i \leq n \rangle, \langle a'_i \mid 1 \leq i \leq n \rangle \rangle \) be a Stirlitz track of length \( n + 1 \). We observe that \( \sigma_* = \langle \langle a_i \mid 0 \leq i \leq n \rangle, \langle a'_i \mid 1 \leq i \leq n \rangle \rangle \) is a Stirlitz track of length \( n \), whence, by the induction hypothesis, the following inequalities hold:

\[
\begin{align*}
    a_{n-1} & \leq a_0 \lor a_n, \\
    a_{n-1} & \leq a'_1 \lor a'_n.
\end{align*}
\]

We first prove that \( a_{n-1} \leq a_0 \lor a_{n+1} \). Indeed, suppose that this does not hold. Hence, \( a_{n-1} \neq a_0 \lor a_{n+1} \). Hence, by applying (B) to (5.5) (for \( i = n - 1 \)) and (5.6) and observing that \( a_{n-1} \neq a_n \lor a'_n \), we obtain that \( a_{n-1} \leq a_n \lor a_{n+1} \). Therefore, \( a_{n-1} \leq a_n \lor a'_n \lor a'_n \lor a_{n+1} \), a contradiction by (U). Hence, indeed, \( a_{n-1} \leq a_0 \lor a_{n+1} \).

Consequently, by (5.4), \( a_n \leq a_{n-1} \lor a_{n+1} \leq a_0 \lor a_{n+1} \). Hence, for \( i \in \{0, \ldots, n\} \), it follows from the induction hypothesis (applied to \( \sigma_* \)) that \( a_i \leq a_0 \lor a_i \leq a_0 \lor a_{n+1} \).

The proof of the inequalities \( a_i \leq a'_1 \lor a_{n+1} \), for \( i \in \{0, \ldots, n\} \), is similar, with \( a_0 \) replaced by \( a'_1 \) and (5.6) replaced by (5.7).

Finally, let \( 0 \leq k < l \leq n \), and suppose that \( a_k \leq a_i \). By applying the previous result to the Stirlitz track \( \langle \langle a_{k+i} \mid 0 \leq i \leq l - k \rangle, \langle a'_{k+i} \mid 1 \leq i \leq l - k \rangle \rangle \), we obtain that \( a_{l-1} \leq a_k \lor a_l = a_i \), a contradiction. Hence \( a_k \neq a_i \), in particular, \( a_k \neq a_l \). \( \square \)

**Lemma 5.8.** For positive integers \( m, n > 0 \), let

\[
\sigma = \langle \langle a_i \mid 0 \leq i \leq m \rangle, \langle a'_i \mid 1 \leq i \leq m \rangle \rangle, \quad \tau = \langle \langle b_j \mid 0 \leq j \leq n \rangle, \langle b'_j \mid 1 \leq j \leq n \rangle \rangle
\]

be Stirlitz tracks with the same base \( p = a_0 = b_0 \) and \( p \leq a_1 \lor b_1 \). Then \( a_i, b_j \leq a_m \lor b_n \), for all \( i \in \{0, \ldots, m\} \) and \( j \in \{0, \ldots, n\} \).
Proof. Suppose first that the inequality \( p \leq a_1 \lor b'_1 \) holds. Then \( p \leq a_1 \lor b'_1, b'_1 \lor b_1 \), \( b_1 \lor a_1 \), a contradiction by \((U_j)\). Hence \( p \neq a_1 \lor b'_1 \), thus, by applying \((B_j)\) to the inequalities \( p \leq a_1 \lor a'_1, b_1 \lor b'_1 \), we obtain that \( p \leq a'_1 \lor b'_1 \).

Furthermore, from Lemma 5.7 it follows that \( a_i \leq p \lor a_m \), for all \( i \in \{0, \ldots, m\} \), and \( b_j \leq p \lor b_n \), for all \( j \in \{0, \ldots, n\} \), thus it suffices to prove that \( p \leq a_m \lor b_n \). Again, from Lemma 5.7 it follows that \( p \leq a'_1 \lor a_m, b'_1 \lor b_n \). Suppose that \( p \neq a_m \lor b_n \). Then \( p \neq a'_1, a_m, b'_1, b_n \), thus, by \((B_j)\), \( p \leq a'_1 \lor b'_1 \). Furthermore, we have seen that \( p \leq b'_1 \lor b_n \) and \( p \leq a'_1 \lor b'_1 \). Hence, by \((U_j)\), \( p \) lies below either \( a'_1 \) or \( b'_1 \) or \( b_n \), a contradiction. \( \square \)

6. The small poset associated with a lattice in SUB

Everywhere in this section before Theorem 6.7, we shall fix a complete, lower continuous, finitely spatial lattice \( L \) in \( \text{SUB} \). For every element \( p \in J(L) \), we denote by \( \{A_p, B_p\} \) the Udav–Bond partition of \( \{p\}^p \) associated with \( p \) (see Section 5.1). We let \( + \) and \( - \) be distinct symbols, and we put \( R = R_0 \cup R_- \cup R_+ \), where \( R_0, R_- \), and \( R_+ \) are the sets defined as follows:

\[
R_0 = \{(p) \mid p \in J(L)\},
\]

\[
R_+ = \{(a, b, +) \mid a, b \in J(L), aDb\},
\]

\[
R_- = \{(a, b, -) \mid a, b \in J(L), aDb\}.
\]

We define a map \( e : R \to J(L) \) by putting \( e((p)) = p \), for all \( p \in J(L) \), while \( e((a, b, +)) = e((a, b, -)) = b \), for all \( a, b \in J(L) \) with \( aDb \).

Let \( \prec \) be the binary relation on \( R \) that consists of the following pairs:

\[
(p, a, -) \prec (p, b, +) \quad \text{whenever } a \in A_p \text{ and } b \in B_p,
\]

\[
(b, c, -) \prec (a, b, +) \prec (b, d, +), \quad \text{and}
\]

\[
(b, d, -) \prec (a, b, -) \prec (b, c, +), \quad \text{whenever } c \in [b]^D \setminus C(a, b) \text{ and } d \in C(a, b).
\]

6.3

**Lemma 6.1.** Let \( \varepsilon \in \{+, -\} \), let \( n < \omega \), and let \( a_0, \ldots, a_n, b_0, \ldots, b_n \in J(L) \) such that \( a_iDb_i \), for all \( i \in \{0, \ldots, n\} \) and \( (a_0, b_0, \varepsilon) \prec \cdots \prec (a_n, b_n, \varepsilon) \). Then exactly one of the following cases occurs:

(i) \( \varepsilon = + \) and, putting \( a_{n+1} = b_n \), the equality \( a_{i+1} = b_i \) holds, for all \( i \in \{0, \ldots, n\} \), while there are join-irreducible elements \( a'_1, \ldots, a'_{n+1} \) of \( L \) such that \( \langle a_i \mid 0 \leq i \leq n+1 \rangle \) is a Stirlitz track.

(ii) \( \varepsilon = - \) and, putting \( a_{i-1} = b_0 \), the equality \( a_{i-1} = b_1 \) holds, for all \( i \in \{0, \ldots, n\} \), while there are join-irreducible elements \( a'_{n-1}, \ldots, a'_{n-1} \) of \( L \) such that \( \langle a_{n-i} \mid 0 \leq i \leq n+1 \rangle \) is a Stirlitz track.
Proof. Suppose that $\varepsilon = +$ (the proof for $\varepsilon = -$ is similar). We argue by induction on $n$. If $n = 0$, then, from the assumption that $a_0D_0$ and by using Corollary 3.4, we obtain a conjugate $a'_1$ of $b_0$ with respect to $a_0$, and $\langle a_0, a_1, \langle a'_1 \rangle \rangle$ is obviously a Stirlitz track.

Suppose that $n > 0$. From the assumption that $\langle a_{n-1}, b_{n-1}, + \rangle < \langle a_n, b_n, + \rangle$ and the definition of $\prec$, we obtain that $a_n = b_{n-1}$. Furthermore, from the induction hypothesis it follows that there exists a Stirlitz track of the form

$$\{\langle a_i \mid 0 \leq i \leq n \rangle, \langle a'_i \mid 1 \leq i \leq n \rangle\}.$$ 

Put $a_{n+1} = b_n$, and let $a'_{n+1}$ be a conjugate of $a_{n+1}$ with respect to $a_n$. Using again the assumption that $\langle a_{n-1}, b_{n-1}, + \rangle < \langle a_n, b_n, + \rangle$, we obtain the inequality $a_n \prec a'_n \cup a_{n+1}$. Therefore, $\langle a_i \mid 0 \leq i \leq n+1 \rangle, \langle a'_i \mid 1 \leq i \leq n+1 \rangle$ is a Stirlitz track. □

Let $\preceq$ denote the reflexive and transitive closure of $\prec$.

Lemma 6.2. The relation $\preceq$ is a partial ordering on $R$, and $\prec$ is the predecessor relation of $\preceq$.

Proof. We need to prove that for any $n > 0$, if $r_0 \prec \cdots \prec r_n$ in $R$, then $r_0 \neq r_n$. We have three cases to consider.

Case 1. $r_0 \in R_+$. In this case, $r_i = \langle a_i, b_i, + \rangle \in R_+$, for all $i \in \{1, \ldots, n\}$. By Lemma 6.1, if we put $a_{n+1} = b_n$, then $a_{i+1} = b_i$, for all $i \in \{0, \ldots, n\}$, and there are join-irreducible elements $a'_1, \ldots, a'_{n+1}$ of $L$ such that

$$\{\langle a_i \mid 0 \leq i \leq n+1 \rangle, \langle a'_i \mid 1 \leq i \leq n+1 \rangle\}$$

is a Stirlitz track. In particular, by Lemma 5.7, $a_0 \neq a_n$, whence $r_0 \neq r_n$.

Case 2. $r_0 \in R_0$. Then $r_i \in R_+$, for all $i \in \{1, \ldots, n\}$, thus $r_0 \neq r_n$.

Case 3. $r_0 \in R_-$. If $r_n \notin R_-$, then $r_0 \neq r_n$. Suppose that $r_n \in R_-$. Then $r_i = \langle a_i, b_i, - \rangle$ belongs to $R_-$, for all $i \in \{0, \ldots, n\}$. By Lemma 6.1, if we put $a_{-1} = b_0$, then $a_{i-1} = b_i$, for all $i \in \{0, \ldots, n\}$, and there are join-irreducible elements $a'_1, \ldots, a'_{n-1}$ of $L$ such that $\langle a_{-1} \mid 0 \leq i \leq n+1 \rangle, \langle a'_{n-1} \mid 1 \leq i \leq n+1 \rangle$ is a Stirlitz track. In particular, by Lemma 5.7, $a_0 \neq a_n$, whence $r_0 \neq r_n$. □

Definition 6.3.

(i) Two finite sequences $r = (r_0, \ldots, r_{n-1})$, $s = (s_0, \ldots, s_{n-1})$ of the same length of $R$ are isotype, if either $e(r_i) = e(s_i)$, for all $i \in \{0, \ldots, n-1\}$, or $e(r_i) = e(s_{n-1-i})$, for all $i \in \{0, \ldots, n-1\}$.

(ii) An oriented path (see Section 2) $r = (r_0, \ldots, r_{n-1})$ of elements of $R$ is

* positive (respectively, negative), if there are elements $a_i, b_i$ (for $0 \leq i < n$) of $J(L)$ such that $r_i = \langle a_i, b_i, + \rangle$ (respectively, $r_i = \langle a_i, b_i, - \rangle$), for all $i \in \{0, \ldots, n-1\}$,
• **reduced**, if either it is positive or has the form

\[ \langle u_0, \ldots, u_{k-1}, (p), v_0, \ldots, v_{l-1} \rangle, \]

where \( p \in J(L) \), \( \langle u_0, \ldots, u_{k-1} \rangle \) is negative, and \( \langle v_0, \ldots, v_{l-1} \rangle \) is positive.

**Lemma 6.4.** Every oriented path of \( R \) is isotype to a reduced oriented path.

**Proof.** Let \( r \) be an oriented path of \( R \), we prove that \( r \) is isotype to a reduced oriented path. If \( r \) is either positive or reduced there is nothing to do. Suppose that \( r \) is neither positive nor reduced. Then \( r \) has the form

\[ \langle \langle a_{k-1}, a_k, - \rangle, \ldots, \langle a_0, a_1, - \rangle, \langle b_0, b_1, + \rangle, \ldots, \langle b_{l-1}, b_l, + \rangle \rangle \]

for some integers \( k > 0 \) and \( l \geq 0 \). If \( l = 0 \), then \( r \) is isotype to the positive path

\[ \langle \langle a_0, a_1, + \rangle, \ldots, \langle a_{k-1}, a_k, + \rangle \rangle. \]

Suppose now that \( l > 0 \). Since \( \langle a_0, a_1, - \rangle \prec \langle b_0, b_1, + \rangle \), two cases can occur.

**Case 1.** \( a_0 = b_1 \) and \( a_1 \notin C(b_0, b_1) \) (see (6.2)). Observe that \( \langle a_0, a_1, - \rangle \prec \langle a_0 \rangle \) if \( a_1 \in A_{a_0} \) while \( \langle a_0 \rangle \prec \langle a_0, a_1, + \rangle \) if \( a_1 \in B_{a_0} \) (see (6.1)). In the first case, it follows from Lemma 5.5 (applied to \( C(a_0, a_1) \)) that the sequence

\[ \langle \langle a_{k-1}, a_k, - \rangle, \ldots, \langle a_0, a_1, - \rangle, \langle a_0 \rangle, \langle b_1, b_2, + \rangle, \ldots, \langle b_{l-1}, b_l, + \rangle \rangle \]

is an oriented path, isotype to \( r \). Similarly, in the second case, the oriented path

\[ \langle \langle b_{l-1}, b_l, - \rangle, \ldots, \langle b_1, b_2, - \rangle, \langle a_0 \rangle, \langle a_0, a_1, + \rangle, \ldots, \langle a_{k-1}, a_k, + \rangle \rangle \]

is isotype to \( r \).

**Case 2.** \( a_1 = b_0 \) and \( b_1 \notin C(a_0, a_1) \) (see (6.3)). Observe that \( \langle b_0 \rangle \prec \langle b_0, b_1, + \rangle \) if \( b_1 \in B_{b_0} \) while \( \langle b_0, b_1, - \rangle \prec \langle b_0 \rangle \) if \( b_1 \in A_{b_0} \) (see (6.1)). In the first case, the oriented path

\[ \langle \langle a_{k-1}, a_k, - \rangle, \ldots, \langle a_1, a_2, - \rangle, \langle b_0 \rangle, \langle b_0, b_1, + \rangle, \ldots, \langle b_{l-1}, b_l, + \rangle \rangle \]

is isotype to \( r \). Similarly, in the second case, the oriented path

\[ \langle \langle b_{l-1}, b_l, - \rangle, \ldots, \langle b_0, b_1, - \rangle, \langle b_0 \rangle, \langle a_1, a_2, + \rangle, \ldots, \langle a_{k-1}, a_k, + \rangle \rangle \]

is isotype to \( r \). This concludes the proof. \( \Box \)

We define a map \( \varphi \) from \( L \) into the powerset of \( R \) as follows:

\[ \varphi(x) = \{ r \in R \mid e(r) \leq x \}, \quad \text{for all } x \in L. \quad (6.4) \]
Lemma 6.5. The set \( \varphi(x) \) belongs to \( \text{Co}(R, \preceq) \), for all \( x \in L \).

Proof. It is sufficient to prove that if \( r_0 \prec \cdots \prec r_n \) in \( R \) such that \( e(r_k) \preceq x \), the relation \( e(r_k) \preceq x \) holds whenever \( 0 < k < n \). By Lemma 6.4, it is sufficient to consider the case where the oriented path \( r = (r_0, \ldots, r_n) \) is reduced. If it is positive, then, by Lemma 6.1, there exists a Stirlitz track of the form

\[
\langle \{a_i \mid 0 \leq i \leq n\} \rangle, \langle a'_{i} \mid 1 \leq i \leq n+1 \rangle
\]

for join-irreducible elements \( a_i, a'_i \) of \( L \) with \( a_i, a'_i \in \text{J}(L) \) and \( a_i \cap a'_i = \emptyset \). From Lemma 6.1 it follows that there are Stirlitz tracks of the form

\[
\langle \{a_i \mid 0 \leq i \leq n\} \rangle, \langle a'_{i} \mid 1 \leq i \leq n+1 \rangle
\]

hence, by Lemma 5.7, \( e(r_k) = a_k \preceq a_0 \lor a_n \preceq x \).

Case 1. \( r = \langle (a_0), (a_0, a_1, +), \ldots, (a_{n-1}, a_n, +) \rangle \) for some \( a_0, \ldots, a_n \in J(L) \). It follows from Lemma 6.1 that there exists a Stirlitz track of the form

\[
\langle \{a_i \mid 0 \leq i \leq n\} \rangle, \langle a'_{i} \mid 1 \leq i \leq n+1 \rangle
\]

hence, by Lemma 5.7, \( e(r_k) = a_k \preceq a_0 \lor a_n \preceq x \).

Case 2. \( r = \langle (a_{n-1}, a_n, -), \ldots, (a_0, a_1, -), (a_0) \rangle \) for some \( a_0, \ldots, a_n \in J(L) \). The argument is similar to the one for Case 1.

Case 3. \( r = \langle (a_{n-1}, a_n, +), \ldots, (a_0, a_1, -), (a_0), (b_0, b_1, +), \ldots, (b_{n-1}, b_n, +) \rangle \) for some positive integers \( n' \) and \( n'' \) and join-irreducible \( a_0 = b_0, a_1, \ldots, a_{n'}, b_1, \ldots, b_{n''} \). From \( \langle a_0, a_1, - \rangle \prec \langle a_0 \prec \langle b_0, b_1, + \rangle \) it follows that \( a_0 = b_0 \preceq a_1 \lor b_1 \). From Lemma 6.1 it follows that there are Stirlitz tracks of the form

\[
\sigma = \langle \{a_i \mid 0 \leq i \leq n\} \rangle, \langle a'_{i} \mid 1 \leq i \leq n' \rangle
\]

\[
\tau = \langle \{b_j \mid 0 \leq j \leq n''\} \rangle, \langle b'_{j} \mid 1 \leq j \leq n'' \rangle
\]

with the same base \( a_0 = b_0 \preceq a_1 \lor b_1 \). Since \( e(r_k) \) has either the form \( a_i \), where \( 0 \leq i \leq n' \), or \( b_j \), where \( 0 \leq j < n'' \), it follows from Lemma 5.8 that \( e(r_k) \preceq a_{n'} \lor b_{n''} \preceq x \). This concludes the proof.

\[\square\]

Lemma 6.6. The map \( \varphi \) is a \( \langle 0, 1 \rangle \)-lattice embedding from \( L \) into \( \text{Co}(R) \).

Proof. It is obvious that \( \varphi \) is a \( \langle 0, 1 \rangle \)-homomorphism. Let \( x, y \in L \) such that \( x \nleq y \). Since \( L \) is finitely spatial, there exists \( p \in J(L) \) such that \( p \leq x \) and \( p \nleq y \). Hence, \( \langle p \rangle \in \varphi(x) \setminus \varphi(y) \), so \( \varphi(x) \nleq \varphi(y) \). Therefore, \( \varphi \) is a \( \langle 0, 1 \rangle \)-embedding.

Now let \( x, y \in L \) and let \( r \in \varphi(x) \lor \varphi(y) \), we prove that \( r \in \varphi(x) \lor \varphi(y) \). The conclusion is trivial if \( r \in \varphi(x) \cup \varphi(y) \), so suppose that \( r \notin \varphi(x) \cup \varphi(y) \). We need to consider two cases:
Case 1. \( r = \langle p \rangle \), for some \( p \in J(L) \). So \( p \leq x \lor y \) while \( p \not\leq x, y \). By Lemma 3.2, there are minimal \( u \leq x \) and \( v \leq y \) such that \( p \leq u \lor v \), hence \( u \) and \( v \) are join-irreducible and they do not belong to the same side of the Udav–Bond partition of \([p]^{\Omega}\) associated with \( p \) (see Proposition 5.1). Hence, by the definition of \( \prec \), either \( \langle p, u, - \rangle \prec \langle p \rangle \prec \langle p, v, + \rangle \) or \( \langle p, v, - \rangle \prec \langle p \rangle \prec \langle p, u, + \rangle \). Since \( \langle p, u, \varepsilon \rangle \in \varphi(x) \) and \( \langle p, v, \varepsilon \rangle \in \varphi(y) \), for all \( \varepsilon \in \{+, -\} \), it follows from this that \( \langle p \rangle \in \varphi(x) \lor \varphi(y) \).

Case 2. \( r = \langle a, b, + \rangle \) for some \( a, b \in J(L) \) such that \( a Db \). So \( b \leq x \lor y \) while \( b \not\leq x, y \). By Lemma 3.2, there are minimal \( u \leq x \) and \( v \leq y \) such that \( b \leq u \lor v \), hence \( u \) and \( v \) are join-irreducible and they do not belong to the same side of the Udav–Bond partition of \([b]^{\Omega}\) associated with \( b \) (see Proposition 5.1). Hence, it follows from Lemma 5.5 that either \( u \not\in C(a, b) \) and \( v \in C(a, b) \) or \( u \in C(a, b) \) and \( v \not\in C(a, b) \). In the first case,

\[ \langle b, u, - \rangle \prec \langle a, b, + \rangle \prec \langle b, v, + \rangle, \]

while in the second case,

\[ \langle b, v, - \rangle \prec \langle a, b, + \rangle \prec \langle b, u, + \rangle. \]

Since \( \langle b, u, \varepsilon \rangle \in \varphi(x) \) and \( \langle b, v, \varepsilon \rangle \in \varphi(y) \), for all \( \varepsilon \in \{+, -\} \), it follows from this that \( r \in \varphi(x) \lor \varphi(y) \).

Case 3. \( r = \langle a, b, - \rangle \) for some \( a, b \in J(L) \) such that \( a Db \). The proof is similar to the proof of Case 2. □

We can now formulate the main theorem of this paper.

Theorem 6.7. Let \( L \) be a lattice. Then the following are equivalent:

(i) \( L \) embeds into a lattice of the form \( \text{Co}(P) \), for some poset \( P \);
(ii) \( L \) satisfies the identities \((S), (U), \) and \((B) \) (i.e., it belongs to the class \( \text{SUB} \));
(iii) \( L \) has a lattice embedding into a lattice of the form \( \text{Co}(R) \), for some poset \( R \), that preserves the existing bounds. Furthermore, if \( L \) is finite, then \( R \) is finite, with

\[ |R| \leq 2|J(L)|^2 - 5|J(L)| + 4. \]

Proof. (i) \( \Rightarrow \) (ii) follows immediately from Lemmas 4.1, 4.5, and 4.7.

(ii) \( \Rightarrow \) (iii). Denote by \( \text{Fil} L \) the lattice of all dual ideals (= filters) of \( L \), ordered by reverse inclusion; if \( L \) has no unit element, then we allow the empty set in \( \text{Fil} L \), otherwise we require filters to be nonempty. This way, \( \text{Fil} L \) is complete and the canonical lattice embedding \( x \mapsto \uparrow x \) from \( L \) into \( \text{Fil} L \) preserves the existing bounds. It is well known that \( \text{Fil} L \) is a dually algebraic lattice that extends \( L \) and that satisfies the same identities as \( L \) (see, for example, G. Grätzer [11]), in particular, it belongs to \( \text{SUB} \). Furthermore, \( \text{Fil} L \) is dually algebraic, thus lower continuous and spatial, thus it is a fortiori finitely spatial.
We consider the poset \( \langle R, \preceq \rangle \) constructed above from Fil \( L \). By Lemmas 6.5 and 6.6, the canonical map \( \phi \) defines a \( \langle 0, 1 \rangle \)-embedding from Fil \( L \) into \( \text{Co}(R) \).

(iii) \( \Rightarrow \) (i) is trivial.

In case \( L \) is finite, put \( n = |J(L)| \), we verify that \( |R| \leq 2n^2 - 5n + 4 \) for the poset \( \langle R, \preceq \rangle \) constructed above, in the case where \( n \geq 2 \) (for \( n \leq 1 \) then one can take for \( P \) a singleton). Indeed, it follows from the join-semidistributivity of \( L \) (that itself follows from Lemma 4.3) that \( L \) has at least two \( D \)-maximal (= join-prime) elements, hence the number of pairs \( \langle a, b \rangle \) of elements of \( J(L) \) such that \( a \preceq b \) is at most \( (n - 1)(n - 2) \), whence

\[
|R| \leq 2(n - 1)(n - 2) + n = 2n^2 - 5n + 4.
\]

\( \square \)

Remark 6.8. The upper bound \( 2|J(L)|^2 - 5|J(L)| + 4 \) of Theorem 6.7(iii), obtained for the particular poset \( R \) constructed above, is reached for \( L \) defined as the lattice of all order-convex subsets of a finite chain.

Corollary 6.9. The class of all lattices that can be embedded into some \( \text{Co}(P) \) coincides with \( \text{SUB} \); it is a finitely based variety. In particular, it is closed under homomorphic images.

Of course, we proved more, for example, the class of all lattices that can be embedded into some finite \( \text{Co}(P) \) forms a pseudovariety (see [10]), thus it is closed under homomorphic images.

7. The tree-like poset associated with a lattice in \( \text{SUB} \)

Everywhere in this section before Theorem 7.7, we shall fix a complete, lower continuous, finitely spatial lattice \( L \) in \( \text{SUB} \). The goal of the present section is to define a tree-like poset \( \Gamma \) and a lattice embedding from \( L \) into \( \text{Co}(\Gamma) \) that preserves the existing bounds, see Theorem 7.7.

The idea to use \( D \)-increasing finite sequences of join-irreducible elements is introduced in K.V. Adaricheva [1], where it is proved that every finite lattice without \( D \)-cycle can be embedded into the lattice of subsemilattices of some finite meet-semilattice; see also [2].

We denote by \( \Gamma \) the set of all finite, nonempty sequences \( \alpha = \langle \alpha(0), \ldots, \alpha(n) \rangle \) of elements of \( J(L) \) such that \( \alpha(i) \preceq \alpha(i + 1) \), for all \( i < n \). We put \( |\alpha| = n \) (the length of \( \alpha \)), and we extend this definition by putting \( |\emptyset| = -1 \). We further put \( \tilde{\alpha} = \langle \alpha(0), \ldots, \alpha(n - 1) \rangle \) (the truncation of \( \alpha \)) and \( e(\alpha) = \alpha(n) \) (the extremity of \( \alpha \)). If \( \alpha = \beta \), we say that \( \beta \) is a one-step extension of \( \alpha \). Furthermore, for all \( n \geq 0 \), we put

\[
\Gamma_n = \{ \alpha \in \Gamma \mid |\alpha| \leq n \} \quad \text{and} \quad E_n = \Gamma_n \setminus \Gamma_{n-1} \quad \text{for} \ n > 0.
\]

For \( \alpha \in \Gamma_n \setminus \Gamma_0 \), we say that a conjugate of \( \alpha \) is an element \( \beta \) of \( \Gamma \) such that \( \tilde{\alpha} = \tilde{\beta} \) and \( e(\alpha) \) and \( e(\beta) \) are conjugate with respect to \( e(\tilde{\alpha}) \). It follows from Corollary 3.4 that every element of \( \Gamma \) has a conjugate. Furthermore, for \( \alpha, \beta \in \Gamma \), we write \( \alpha \sim \beta \), if either \( \alpha = \beta \) or \( \beta = \tilde{\alpha} \).
For all \( n > 0 \), we define inductively a binary relation \( \prec_n \) on \( \Gamma_n \), together with subsets \( A_n \) and \( B_n \) of \([e(\alpha)]^D\) for \( \alpha \in \Gamma_{n-1} \).

The induction hypothesis to be satisfied consists of the following two assertions:

1. (S1) \( \prec_n \) is acyclic.
2. (S2) For all \( \alpha, \beta \in \Gamma_n \), \( \alpha \sim \beta \) iff either \( \alpha \prec_n \beta \) or \( \beta \prec_n \alpha \).

For \( n = 0 \), let \( \prec_n \) be empty.

The case \( n = 1 \) is the only place where we have some freedom in the choice of \( \prec_n \). We suppose that we have already used this freedom for the construction of the poset \( \langle R, \sqsubseteq \rangle \) of Section 6, that is, for each \( p \in J(L) \), let \( A_p, B_p \) such that \( \{A_p, B_p\} \) is the Udav–Bond partition of \([p]^D\) associated with \( p \) (see Section 5.1), and we let \( R \) be the poset associated with this choice that we constructed in Section 6. Then we put \( A(p) = A_p \) and \( B(p) = B_p \), and we define

\[
\prec_1 = \big\{ \langle \langle p, a \rangle, \langle p \rangle \rangle \mid p \in J(L), \ a \in A(p) \big\} \cup \big\{ \langle \langle p, \langle p, b \rangle \rangle \rangle \mid p \in J(L), \ b \in B(p) \big\}.
\]

It is obvious that \( \prec_1 \) satisfies both (S1) and (S2).

Now suppose having defined \( \prec_n \), for \( n \geq 1 \), that satisfies both (S1) and (S2). For all \( \alpha \in E_n \), we define subsets \( A_\alpha \) and \( B_\alpha \) of \([e(\alpha)]^D\) as follows:

**Case 1.** \( \bar{\alpha} \prec_n \alpha \). Then we put \( A_\alpha = [e(\alpha)]^D \setminus C(e(\bar{\alpha}), e(\alpha)) \) and \( B_\alpha = C(e(\bar{\alpha}), e(\alpha)) \).

**Case 2.** \( \alpha \prec_n \bar{\alpha} \). Then we put \( A_\alpha = C(e(\bar{\alpha}), e(\alpha)) \) and \( B_\alpha = [e(\alpha)]^D \setminus C(e(\bar{\alpha}), e(\alpha)) \).

Then we define \( \prec_{n+1} \) as

\[
\prec_{n+1} = \prec_n \cup \big\{ \langle \alpha^\sim \langle x \rangle, \alpha \rangle \mid \alpha \in E_n \text{ and } x \in A_\alpha \big\} \\
\cup \big\{ \langle \alpha, \alpha^\sim \langle y \rangle \rangle \mid \alpha \in E_n \text{ and } y \in B_\alpha \big\},
\]

where \( \langle \alpha, \beta \rangle \mapsto \alpha^\sim \beta \) denotes concatenation of finite sequences.

**Lemma 7.1.** The relation \( \prec_{n+1} \) satisfies both (S1) and (S2).

**Proof.** It is obvious that \( \prec_{n+1} \) satisfies (S2). Now let us prove (S1), and suppose that \( \prec_{n+1} \) has a cycle, say, \( \alpha_0 \prec_{n+1} \alpha_1 \prec_{n+1} \cdots \prec_{n+1} \alpha_k = \alpha_0 \), where \( k \geq 2 \). We pick \( k \) minimal with this property. As \( A_\alpha \cap B_\alpha = \emptyset \), for all \( \alpha \), we cannot have \( k = 2 \) as well, so \( k \geq 3 \).

By the induction hypothesis, one of the elements of the cycle belongs to \( E_{n+1} \), without loss of generality we may assume that it is the case for \( \alpha_0 \). Hence, by (7.1), \( \alpha_1 = \alpha_0^\sim \) belongs to \( \Gamma_n \). Let \( l \) be the smallest element of \( \{1, \ldots, k-1\} \) such that \( \alpha_{l+1} \notin \Gamma_n \) (it exists since \( \alpha_k = \alpha_0 \notin \Gamma_n \)). Suppose that \( l < k-1 \). By (S2) for \( \prec_{n+1} \), \( \alpha_{l+2} = \alpha_{l+1} \), a contradiction by the minimality of \( k \). Hence \( l = k-1 \), which means that \( \alpha_1, \ldots, \alpha_{k-1} \in \Gamma_n \). Hence, since \( k-1 \geq 2 \), we obtain that \( \alpha_1 \prec_{n} \cdots \prec_{n} \alpha_{k-1} = \alpha_1 \) is a \( \prec_n \)-cycle, a contradiction. \( \square \)
Lemma 7.1 completes the definition of $\prec_n$, for all $n > 0$. We define $\prec$ as the union over all $n < \omega$ of $\prec_n$. Hence $\prec$ is an acyclic binary relation on $\Gamma$ such that $\alpha \prec \beta$ iff either $\alpha < \beta$ or $\beta < \alpha$, for all $\alpha, \beta \in \Gamma$. Since $\prec$ is acyclic, the reflexive and transitive closure $\equiv$ of $\prec$ is a partial ordering on $\Gamma$, for which $\prec$ is exactly the predecessor relation. For the sake of clarity, we rewrite below the inductive definition of $\prec$ and the sets $A_{\alpha}$ and $B_{\alpha}$ for $\alpha \in \Gamma$.

(a) For $|\alpha| = 0$, $A_{\alpha}$ and $B_{\alpha}$ are chosen such that $\{A_{\alpha}, B_{\alpha}\}$ is the Uvd–Bond partition of $\{e(\alpha)\}^D$ associated with $e(\alpha)$.

(b) Suppose that $|\alpha| > 1$. Then we define $A_{\alpha}$ and $B_{\alpha}$ by

$$
\langle A_{\alpha}, B_{\alpha} \rangle = \begin{cases}
\{ \langle \nu \rangle \cap \{e(\alpha)\} \setminus C(e(\alpha)), e(\alpha) \} & \text{if } \bar{a} < \alpha, \\
\{ \langle C(e(\alpha)), e(\alpha) \rangle, \nu \} & \text{if } \alpha \prec \alpha, \\
\{ \langle e(\alpha), e(\alpha) \rangle \} & \text{if } \alpha = \bar{\alpha}.
\end{cases}
$$

(c) $\alpha < \beta$ implies that $\alpha \sim \beta$.

(d) $\alpha \prec (x) \prec \alpha$ iff $x \in A_{\alpha}$ and $\alpha \prec \alpha \prec (x)$ iff $x \in B_{\alpha}$, for all $\alpha \in \Gamma$ and all $x \in \{e(\alpha)\}^D$.

By Lemma 5.5, the set $\{A_{\alpha}, B_{\alpha}\}$ is the Uvd–Bond partition of $\{e(\alpha)\}^D$ associated with $\alpha$, for all $\alpha \in \Gamma$. Therefore, by Proposition 5.1 and the definition of $\prec$, we obtain immediately the following consequence.

**Corollary 7.2.** For all $\alpha \in \Gamma$ and all $x, y \in \{e(\alpha)\}^D$, $e(\alpha) \lessgtr x \lor y$ iff either $\alpha \prec (x) \prec \alpha \prec \alpha \lessgtr (y)$ or $\alpha \prec (x)$ or $\alpha \prec (y) \prec \alpha \prec \alpha \lessgtr (x)$.

For $\alpha, \beta \in \Gamma$, we denote by $\alpha \star \beta$ the largest common initial segment of $\alpha$ and $\beta$. Observe that $\alpha \star \beta$ belongs to $\Gamma \cup \{\emptyset\}$ and that $\alpha \star \beta = \beta \star \alpha$. Put $m = |\alpha| - |\alpha \star \beta|$ and $n = |\beta| - |\alpha \star \beta|$. We let $P(\alpha, \beta)$ be the finite sequence $\langle \gamma_0, \gamma_1, \ldots, \gamma_{m+n} \rangle$, where the $\gamma_i$, for $0 \leq i \leq m + n$, are defined by $\gamma_0 = \alpha$, $\gamma_{i+1} = \gamma_i$, for all $i < m$, $\gamma_{m+n} = \beta$, and $\gamma_{m+n-j} = \gamma_{m+n-j-1}$, for all $j < n$. Hence the $\gamma_i$-s first decrease from $\gamma_0 = \alpha$ to $\gamma_m = \alpha \star \beta$ by successive truncations, then they increase again from $\gamma_m$ to $\gamma_{m+n} = \beta$ by successive one-step extensions.

For $\alpha, \beta \in \Gamma$, we observe that a path (see Section 2) from $\alpha$ to $\beta$ is a finite sequence $c = \langle \gamma_0, \gamma_1, \ldots, \gamma_k \rangle$ of distinct elements of $\Gamma$ such that $\gamma_0 = \alpha$, $\gamma_k = \beta$, and $\gamma_i \sim \gamma_{i+1}$, for all $i < k$.

**Proposition 7.3.** For all $\alpha, \beta \in \Gamma$, there exists at most one path from $\alpha$ to $\beta$, and then this path is $P(\alpha, \beta)$. Furthermore, such a path exists iff $\alpha(0) = \beta(0)$.

Hence, by using the terminology of Section 2: the poset $(\Gamma, \equiv)$ is tree-like.

**Proof.** Put again $m = |\alpha| - |\alpha \star \beta|$ and $n = |\beta| - |\alpha \star \beta|$, and $P(\alpha, \beta) = \langle \gamma_0, \ldots, \gamma_{m+n} \rangle$. Let $d = \langle \delta_0, \ldots, \delta_k \rangle$ (for $k < \omega$) be a path from $\alpha$ to $\beta$. We begin with the following essential observation.

**Claim.** The path $d$ consists of a sequence of truncations followed by a sequence of one-step extensions.
Proof of Claim. Suppose that there exists an index \( i \in \{1, \ldots, k - 1\} \) such that \( \delta_i \) extends both \( \delta_{i-1} \) and \( \delta_{i+1} \). Then \( \delta_{i-1} = \delta_i = \delta_{i+1} \), which contradicts the fact that all entries of \( d \) are distinct.

Hence, either \( d \) consists of a sequence of truncations, or there exists a least index \( l \in \{0, \ldots, k - 1\} \) such that \( \delta_{l+1} \) is an extension of \( \delta_l \). If \( \delta_{l+1} \) is not an extension of \( \delta_l \) for some \( i \in \{l, \ldots, k - 1\} \), then, taking the least such \( i \), we obtain that \( \delta_l \) extends both \( \delta_{l-1} \) and \( \delta_{l+1} \), a contradiction by the first paragraph of the present proof. Hence \( \delta_{l+1} \) is a one-step extension of \( \delta_l \), for all \( i \in \{l, \ldots, k - 1\} \).

Let \( l \) denote the least element of \( \{0, \ldots, k\} \) such that \( l < k \) implies that \( \delta_{l+1} \) extends \( \delta_l \). In particular, \( \delta_l \) is a common initial segment of both \( \alpha \) and \( \beta \), thus of \( \alpha \star \beta \). Furthermore,

\[
|\alpha| - l = |\delta_0| - l = |\delta_l| \leq |\alpha \star \beta| = |\alpha| - m,
\]

thus \( l \geq m \). Similarly,

\[
|\beta| - (k - l) = |\delta_l| \leq |\alpha \star \beta| = |\beta| - n,
\]

thus \( k - l \geq n \). In addition, both \( \alpha \star \beta \) and \( \delta_m \) are initial segments of \( \alpha \) of the same length \( |\alpha| - m \), thus \( \alpha \star \beta = \delta_m \). Similarly, both \( \alpha \star \beta \) and \( \delta_{k-n} \) are initial segments of \( \beta \) of the same length \( |\beta| - n \), whence \( \alpha \star \beta = \delta_{k-n} \). Therefore, \( \delta_m = \delta_{k-n} \), whence, since all entries of \( d \) are distinct, \( m = k - n \), so \( k = m + n \), whence \( l = m \) since \( m \leq l < k - n \). It follows then from the claim that \( d = P(\alpha, \beta) \).

Furthermore, from \( \alpha \sim \beta \) it follows that \( \alpha(0) = \beta(0) \), thus the same conclusion follows from the assumption that there exists a path from \( \alpha \) to \( \beta \). Conversely, if \( \alpha(0) = \beta(0) \), then \( \alpha \star \beta \) is nonempty, thus so are all entries of \( P(\alpha, \beta) \). Hence \( P(\alpha, \beta) \) is a path from \( \alpha \) to \( \beta \).

Now we define a map \( \pi : \Gamma \rightarrow R \) by the following rule:

\[
\pi(\alpha) = \begin{cases} 
\alpha & \text{if } |\alpha| = 0, \\
(e(\bar{\alpha}), e(\alpha), +) & \text{if } \bar{\alpha} \prec \alpha, \\
(e(\bar{\alpha}), e(\alpha), -) & \text{if } \alpha \prec \bar{\alpha}, 
\end{cases}
\]

Lemma 7.4. \( \alpha \prec \beta \) in \( \Gamma \) implies that \( \pi(\alpha) < \pi(\beta) \) in \( R \), for all \( \alpha, \beta \in \Gamma \). In particular, \( \pi \) is order-preserving.

Proof. We argue by induction on the least integer \( n \) such that \( \alpha, \beta \in \Gamma_n \). We need to consider first the case where \( p, a, b \in 1(L) \), \( a \in A_p, b \in B_p \) (so that \( \langle p, a \rangle < \langle p \rangle < \langle p, b \rangle \) in \( \Gamma \)), and prove that \( \pi(\langle p, a \rangle) < \pi(\langle p \rangle) < \pi(\langle p, b \rangle) \) in \( R \). But by the definition of \( \pi \), the following equalities hold,

\[
\pi(\langle p, a \rangle) = \langle p, a, - \rangle, \quad \pi(\langle p \rangle) = \langle p \rangle, \quad \text{and} \quad \pi(\langle p, b \rangle) = \langle p, b, + \rangle,
\]

while, by the definition of \( < \) on \( R \),

\[
\langle p, a, - \rangle < \langle p \rangle < \langle p, b, + \rangle,
\]
which solves the case where \( n = 1 \).

The remaining case to consider is where \( \alpha \prec (x) < \alpha \prec \alpha \prec (y) \) in \( \Gamma \), for \( |\alpha| > 0 \). Thus \( x \in A_\alpha \) and \( y \in B_\alpha \), whence

\[
\pi(\alpha \prec (x)) = \langle e(\alpha), x, - \rangle,
\pi(\alpha \prec (y)) = \langle e(\alpha), y, + \rangle.
\]

Suppose first that \( \bar{\alpha} \prec \alpha \). Then

\[
A_\alpha = [e(\alpha)]^D \setminus C(e(\bar{\alpha}), e(\alpha)) \quad \text{while} \quad B_\alpha = C(e(\bar{\alpha}), e(\alpha)).
\]

Furthermore, \( \pi(\alpha) = \langle e(\bar{\alpha}), e(\alpha), + \rangle \), while, by the definition of \( \prec \) on \( R \),

\[
\{e(\alpha), x, -\} < \{e(\bar{\alpha}), e(\alpha), +\} < \{e(\alpha), y, +\},
\]
in other words,

\[
\pi(\alpha \prec (x)) < \pi(\alpha) < \pi(\alpha \prec (y)).
\]

Suppose now that \( \alpha \prec \bar{\alpha} \). Then

\[
A_\alpha = C(e(\bar{\alpha}), e(\alpha)) \quad \text{while} \quad B_\alpha = [e(\alpha)]^D \setminus C(e(\bar{\alpha}), e(\alpha)).
\]

Furthermore, \( \pi(\alpha) = \langle e(\bar{\alpha}), e(\alpha), - \rangle \), while, by the definition of \( \prec \) on \( R \),

\[
\{e(\alpha), x, -\} < \{e(\bar{\alpha}), e(\alpha), -\} < \{e(\alpha), y, +\},
\]
in other words,

\[
\pi(\alpha \prec (x)) < \pi(\alpha) < \pi(\alpha \prec (y)),
\]
which completes the proof. \( \square \)

We observe the following immediate consequence of Lemma 7.4.

**Corollary 7.5.** One can define a zero-preserving complete meet homomorphism \( \pi^* : \operatorname{Co}(R) \to \operatorname{Co}(\Gamma) \) by the rule

\[
\pi^*(X) = \pi^{-1}[X], \quad \text{for all} \ X \in \operatorname{Co}(R).
\]

We put \( \psi = \pi^* \circ \varphi \), where \( \varphi : L \hookrightarrow \operatorname{Co}(R) \) is the canonical map defined in Section 6. Hence \( \psi \) is a zero-preserving meet homomorphism from \( L \) into \( \operatorname{Co}(\Gamma) \). For any \( x \in L \), the value \( \psi(x) \) is calculated by the same rule as \( \varphi(x) \), see (6.4):

\[
\psi(x) = \{ \alpha \in \Gamma \mid e(\alpha) \leq x \}.
\]
Lemma 7.6. The map $\psi$ is a lattice embedding from $L$ into $\text{Co}(\Gamma)$. Moreover, $\psi$ preserves the existing bounds.

Proof. The statement about preservation of bounds is obvious. We have already seen (and it is obvious) that $\psi$ is a meet homomorphism. Let $x, y \in L$ such that $x \not\leq y$. Since $L$ is finitely spatial, there exists $p \in J(L)$ such that $p \leq x$ and $p \not\leq y$; whence $\langle p \rangle \in \psi(x) \setminus \psi(y)$. Hence $\psi$ is a meet embedding from $L$ into $\text{Co}(\Gamma)$.

Let $x, y \in L$, let $\alpha \in \psi(x \lor y)$, we prove that $\alpha \in \psi(x) \lor \psi(y)$. This is obvious if $\alpha \in \psi(x) \lor \psi(y)$, so suppose that $\alpha \not\in \psi(x) \lor \psi(y)$. Hence $e(\alpha) \leq x \lor y$ while $e(\alpha) \not\leq x, y$, thus, by Lemma 3.2, there are minimal $u \leq x$ and $v \leq y$ such that $e(\alpha) \leq u \lor v$, and both $u$ and $v$ belong to $[e(\alpha)]^\uparrow$. Therefore, by Corollary 7.2, either $\alpha \lor u \lor v \lor \alpha \lor \alpha \lor \alpha$ or $\alpha \lor v \lor \alpha \lor v \lor \alpha \lor u$. In both cases, since $\alpha \lor (u \lor v) \in \psi(x)$ and $\alpha \lor (v \lor u) \in \psi(y)$, we obtain that $\alpha \in \psi(x) \lor \psi(y)$. Therefore, $\psi$ is a join homomorphism.

Now we can state the main embedding theorem of the present section.

Theorem 7.7. Let $L$ be a lattice. Then the following assertions are equivalent:

(i) there exists a poset $P$ such that $L$ embeds into $\text{Co}(P)$;
(ii) $L$ satisfies the identities $(S)$, $(U)$, and $(B)$ (i.e., it belongs to the class $\text{SUB}$);
(iii) there exists a tree-like (see Section 2) poset $\Gamma$ such that $L$ has an embedding into $\text{Co}(\Gamma)$ that preserves the existing bounds. Furthermore, if $L$ is finite without $D$-cycle, then $\Gamma$ is finite.

Proof. (i) $\Rightarrow$ (ii) has already been established, see Theorem 6.7.

(ii) $\Rightarrow$ (iii). As in the proof of Theorem 6.7, we denote by Fil $L$ the lattice of all filters of $L$, ordered by reverse inclusion; if $L$ has no unit element, then we allow the empty set in Fil $L$, otherwise we require filters to be nonempty. We consider the poset $\Gamma$ constructed from Fil $L$ as in Section 7. By Lemma 7.6, $L$ embeds into $\text{Co}(\Gamma)$. The finiteness statement of (iii) is obvious.

(iii) $\Rightarrow$ (i) is trivial. □

Even in case $L = \text{Co}(P)$, for a finite totally ordered set $P$, the poset $\Gamma$ constructed in Theorem 7.7 is not isomorphic to $P$ as a rule. As it is constructed from finite sequences of elements of $P$, it does not lend itself to easy graphic representation. However, many of its properties can be seen on the simpler poset represented on Fig. 5, which is tree-like.

As we shall see in Sections 9 and 10, the assumption in Theorem 7.7(iii) that $L$ be without $D$-cycle cannot be removed.

8. Non-preservation of atoms

The posets $R$ and $\Gamma$ that we constructed in Sections 6 and 7 are defined via sequences of join-irreducible elements of $L$. This is to be put in contrast with the main result of O. Frink [8] (see also [11]), that embeds any complemented modular lattice into a geometric lattice:
namely, this construction preserves atoms. Hence the question of the necessity of the complication of the present paper, that is, using sequences of join-irreducible elements rather than just join-irreducible elements, is natural. In the present section we study two examples that show that this complication is, indeed, necessary.

Example 8.1. A finite, atomistic lattice in SUB without D-cycle that cannot be embedded atom-preservingly into any Co(T).

Proof. Let \( P \) be the nine-element poset represented on the left-hand side of Fig. 1, together with order-convex subsets \( P_0, P_1, P_2, Q_0, Q_1, Q_2 \).

We let \( K \) be the set of all elements \( X \) of \( \text{Co}(P) \) such that \( p_i \in X \iff p'_i \in X \), for all \( i < 3 \). It is obvious that \( K \) is a meet-subsemilattice of \( \text{Co}(P) \) which contains \( \{\emptyset, P\} \cup \Omega \), where \( \Omega = \{P_0, P_1, P_2, Q_0, Q_1, Q_2\} \). We prove that \( K \) is a join-subsemilattice of \( \text{Co}(P) \). Indeed, for all \( i < 3 \), both \( p_i \) and \( p'_i \) are either maximal or minimal in \( P \), hence, for all \( X, Y \in \text{Co}(P) \), \( p_i \in X \lor Y \iff p_i \in X \cup Y \), and, similarly, \( p'_i \in X \lor Y \iff p'_i \in X \cup Y \). Hence \( X, Y \in K \) implies that \( X \lor Y \in K \).

Therefore, \( K \) is a sublattice of \( \text{Co}(P) \). It follows immediately that the atoms of \( K \) are the elements of \( \Omega \), that \( K \) is atomistic, and the atoms of \( K \) satisfy the following relations (see the right half of Fig. 1):

\[
Q_0 \leq P_1 \lor P_2, \quad Q_1 \leq P_0 \lor P_2, \quad Q_2 \leq P_0 \lor P_1, \\
P_0 \not\leq P_1 \lor P_2, \quad P_1 \not\leq P_0 \lor P_2, \quad P_2 \not\leq P_0 \lor P_1.
\]

Hence, the sequence \( P_0 P_1 P_2 P_0 P_1 \) is a zigzag of length 5 (in the sense of [3]). It follows from this and the easy direction of the main theorem of [3] that \( K \) cannot be embedded atom-preservingly into any \( \text{Co}(T) \). \( \Box \)

By contrast, our second example is subdirectly irreducible, but it has D-cycles. We shall see in a subsequent paper [15] that the latter condition is unavoidable, that is, any finite, subdirectly irreducible atomistic lattice without D-cycle that can be embedded into some \( \text{Co}(P) \) can be embedded atom-preservingly into some finite \( \text{Co}(P) \) without D-cycle.

Example 8.2. A finite, atomistic, subdirectly irreducible lattice in SUB that cannot be embedded into \( \text{Co}(T) \), for any poset \( T \), in an atom-preserving way.
Proof. Let $Q$ be the 12-element poset represented on the left hand side of Fig. 2, together with order-convex subsets $A, B, C, A', B', C'$.

We let $\sigma$ be the anti-automorphism of $Q$ defined by $\sigma(a_i) = a_{1-i}$, $\sigma(a'_i) = a'_{1-i}$, $\sigma(b_i) = b_{1-i}$, $\sigma(b'_i) = b'_{1-i}$, $\sigma(c_i) = c_{1-i}$, $\sigma(c'_i) = c'_{1-i}$, for all $i < 2$, and we let $L$ be the set of all elements $X$ of $\text{Co}(Q)$ such that $\sigma(X) = X$. It is obvious that $L$ is a meet-subsemilattice of $\text{Co}(Q)$ which contains $\{\emptyset, Q\} \cup \Omega$, where $\Omega = \{A, B, C, A', B', C'\}$. We prove that $L$ is a join-subsemilattice of $\text{Co}(Q)$. Let $X, Y \in L$, we prove that $X \vee Y \in L$.

Since both $a'_0$ and $a'_1$ are either maximal or minimal in $Q$, the equivalence $a'_i \in X \vee Y \Leftrightarrow a'_i \in X \cup Y$ holds, for all $i < 2$, whence $a'_i \in X \vee Y \Leftrightarrow a'_i \in X \vee Y$. Similarly, $b'_0 \in X \vee Y \Leftrightarrow b'_1 \in X \vee Y$ and $c'_0 \in X \vee Y \Leftrightarrow c'_1 \in X \vee Y$.

Suppose now that $a'_0 \in X \vee Y$, we prove that $a'_1 \in X \vee Y$. If $a'_0 \in X \cup Y$ this is obvious, so suppose that $a'_0 \notin X \cup Y$. Without loss of generality, there are $x \in X$ and $y \in Y$ such that $x < a'_0 \leq y$, whence $x \in \{b'_1, b_1, c'_1, c_1\}$ and $y = a'_0$. From $Y \in L$ it follows that $a'_1 \in Y$, thus $A' \subseteq Y$. Similarly, from $X \in L$ it follows that either $B \subseteq X$ or $C \subseteq X$ or $B' \subseteq X$ or $C' \subseteq X$.

If $B \subseteq X$, then $b_0 \in X$, thus, since $a'_1 < a'_0 < a_1 < b'_0$ and $a'_1 \in Y$, we obtain that $a_1 \in X \vee Y$. If $B' \subseteq X$, then $b'_0 \in X$, thus, since $a'_1 < a'_0 < a'_1 < b'_0$ and $a'_1 \in Y$, we obtain again that $a_1 \in X \vee Y$. Similar results hold for either $C \subseteq X$ or $C' \subseteq X$. Therefore, $a'_0 \in X \vee Y$ implies that $a_1 \in X \vee Y$. By symmetry, we obtain the converse. Similarly, $b_0 \in X \vee Y \Leftrightarrow b'_1 \in X \vee Y$ and $c_0 \in X \vee Y \Leftrightarrow c'_1 \in X \vee Y$. Therefore, $X \vee Y$ belongs to $L$, which completes the proof that $L$ is a sublattice of $\text{Co}(Q)$.

It follows immediately that the atoms of $L$ are the elements of $\Omega$, that $L$ is atomistic, and the atoms of $L$ satisfy the following relations:

\[
\begin{align*}
A, B & \leq A' \vee B', & A & \leq A' \vee B, & B & \leq A \vee B', \\
B, C & \leq B' \vee C', & B & \leq B' \vee C, & C & \leq B \vee C', \\
A, C & \leq A' \vee C', & A & \leq A' \vee C, & C & \leq A \vee C'.
\end{align*}
\]

Hence, $L$ is subdirectly irreducible, with monolith (i.e., smallest nonzero congruence) the smallest congruence $\Theta(\emptyset, A)$ identifying $\emptyset$ and $A$, also equal to $\Theta(\emptyset, B)$ and to $\Theta(\emptyset, C)$. Furthermore, the sequence $A'B'C'A'B'$ is a zigzag of length 5 (in the sense of [3]). It follows from this and the easy direction of the main theorem of [3] that $L$ cannot be embedded atom-preservingly into any $\text{Co}(T)$. □
9. Crowns in posets

We first recall the following classical definition.

**Definition 9.1.** For an integer \( n \geq 2 \), we denote by \( \mathbb{Z}/n\mathbb{Z} \) the set of integers modulo \( n \). The \( n \)-crown \( C_n \) is the poset with underlying set \((\mathbb{Z}/n\mathbb{Z}) \times \{0, 1\}\) and ordering defined by \((i, 0), (i + 1, 0) < (i, 1)\), for all \( i \in \mathbb{Z}/n\mathbb{Z} \).

The crown \( C_n \) is illustrated on Fig. 3.

We shall mostly deal with sub-crowns of posets.

**Definition 9.2.** For \( n \geq 2 \) and a poset \((T, \preceq)\), a \( n \)-crown of \( T \) is a finite sequence \( \langle \langle a_i, b_i \rangle | i \in \mathbb{Z}/n\mathbb{Z} \rangle \) of elements of \( T \times T \) such that there exists an order-embedding \( f : C_n \hookrightarrow T \) with \( f(i, 0) = a_i \) and \( f(i, 1) = b_i \), for all \( i \in \mathbb{Z}/n\mathbb{Z} \).

We shall sometimes identify an integer modulo \( n \) with its unique representative in \([0, 1, \ldots, n - 1]\) and a \( n \)-crown \( \langle \langle a_i, b_i \rangle | i \in \mathbb{Z}/n\mathbb{Z} \rangle \) with the finite sequence \( \langle \langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle, \ldots, \langle a_{n-1}, b_{n-1} \rangle \rangle \).

The following lemma makes it possible to identify crowns within posets.

**Lemma 9.3.** Let \((T, \preceq)\) be a poset, let \( n \geq 3 \), and let \( a_i, b_i \) (\( i \in \mathbb{Z}/n\mathbb{Z} \)) be elements of \( T \). Then the following are equivalent:

(i) \( \langle \langle a_i, b_i \rangle | i \in \mathbb{Z}/n\mathbb{Z} \rangle \) is a \( n \)-crown.

(ii) \( a_i \preceq b_j \) iff \( i \in \{j, j + 1\} \), for all \( i, j \in \mathbb{Z}/n\mathbb{Z} \).

Proof. (i) \( \Rightarrow \) (ii) is trivial. Conversely, suppose (ii) satisfied, we prove that \( f : C_n \hookrightarrow T \) defined by \( f(i, 0) = a_i \) and \( f(i, 1) = b_i \), for all \( i \in \mathbb{Z}/n\mathbb{Z} \), is an order-embedding. We need to prove the following assertions:

(i) \( a_i \preceq a_j \) implies that \( i = j \), for all \( i, j \in \mathbb{Z}/n\mathbb{Z} \). Indeed, if \( a_i \preceq a_j \), then \( a_i \preceq b_j, b_{j-1} \) (because \( a_j \preceq b_j, b_{j-1} \)), thus, by assumption, \( i \in \{j, j + 1\} \cap \{j, j - 1\} = \{j\} \) (we use here the inequality \( n \geq 3 \)), that is, \( i = j \).

(ii) \( b_i \preceq b_j \) implies that \( i = j \), for all \( i, j \in \mathbb{Z}/n\mathbb{Z} \). The proof is similar to the one of (i).

(iii) \( b_j \preceq a_i \) occurs for no \( i, j \in \mathbb{Z}/n\mathbb{Z} \). Indeed, suppose that \( b_j \preceq a_i \). Then \( b_j \preceq b_i, b_{i-1} \) (because \( a_i \preceq b_i, b_{i-1} \)), thus, by (ii), \( j = i = i - 1 \), a contradiction.

Fig. 3. The crown \( C_n \).
This concludes the proof.

**Definition 9.4.** A poset $T$ is *crown-free*, if it has no $n$-crown for any $n \geq 3$.

Strictly speaking, the 2-crown $C_2$ is crown-free since we are requiring $n \geq 3$ in the definition above. The motivation why we are putting this slight restriction on $n$ lies in the following observation. First, the poset of Fig. 4(i) is tree-like, but it contains the 2-crown represented on Fig. 4(ii); observe also that the $n$-crown, for any $n \geq 2$, is never tree-like.

On the other hand, we shall now prove the following result.

**Proposition 9.5.** Every tree-like poset is crown-free.

As witnessed by the square $2^2$, the converse of Proposition 9.5 does not hold.

**Proof.** Let $(T, \leq)$ be a tree-like poset. For $x, y \in T$, we denote by $d(x, y)$ the length of the unique path from $x$ to $y$ if there is such a path, $\infty$ otherwise. Observe that $x \leq y$ implies that $d(x, y) < \infty$ (but the converse does not hold as a rule), and then the unique path from $x$ to $y$ is oriented (see Section 2).

For a $n$-crown $\gamma = \langle \langle a_i, b_i \rangle | i \in \mathbb{Z}/n\mathbb{Z} \rangle$ in $T$, we put

$$\ell(\gamma) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} d(a_i, b_i).$$

Suppose that $T$ has a $n$-crown, for some integer $n \geq 3$. We pick such a crown $\gamma = \langle \langle a_i, b_i \rangle | i \in \mathbb{Z}/n\mathbb{Z} \rangle$ with $\ell(\gamma)$ minimum. For all $i \in \mathbb{Z}/n\mathbb{Z}$, we let

$$a_i = x_{i,0} < x_{i,1} < \cdots < x_{i,p_i} = b_i,$$

$$a_{i+1} = y_{i,0} < y_{i,1} < \cdots < y_{i,q_i} = b_i$$

be the paths from $a_i$ (respectively, $a_{i+1}$) to $b_i$, where $<$ denotes the predecessor relation of $T$.

**Claim 1.** $\{x_{i,p} | 0 \leq p < p_i\} \cap \{y_{i,q} | 0 \leq q < q_i\} = \emptyset$, for all $i \in \mathbb{Z}/n\mathbb{Z}$.

**Proof of Claim.** Suppose, to the contrary, that $x_{i,p} = y_{i,q}$ for some $p \in \{0, \ldots, p_i - 1\}$ and $q \in \{0, \ldots, q_i - 1\}$. We put $b'_j = b_j$, for all $j \neq i$ in $\mathbb{Z}/n\mathbb{Z}$, while $b'_i = x_{i,p}$. Since

![Fig. 4. A tree-like poset which contains the crown $C_2$.](image)
Case 3. a contradiction.

From Claim 2 it follows that 

\[ \xi, k \]

which contradicts the minimality of \( \ell(\gamma) \).

The proof of the following claim is symmetric.

Claim 2. \( \{y_i, q | 0 < q < q_i\} \cap \{x_i, p | 0 < p < p_i+1\} = \emptyset \), for all \( i \in \mathbb{Z}/n\mathbb{Z} \).

We define a walk of \( T \) to be a finite sequence \( e = \langle c_0, c_1, \ldots, c_m \rangle \) of elements of \( T \) such that either \( c_i < c_{i+1} \) or \( c_{i+1} < c_i \), for all \( i < m \), we say then that \( e \) is a walk from \( c_0 \) to \( c_m \).

Hence, a nonempty path of \( T \) is a walk with all distinct entries.

Now we let \( d \) be the finite sequence defined by

\[
\begin{align*}
d &= (x_0, k | 0 < k < p_0) \prec (y_{0, q_0} - l | 0 < l < q_0) \prec (x_1, k | 0 \leq k \leq p_1) \\
&\quad \cdots \prec (x_{n-1, k} | 0 \leq k \leq p_{n-1}).
\end{align*}
\]

It is obvious that \( d \) is a walk from \( x_0, 0 = d_0 \) to \( x_{n-1, p_{n-1}} = b_{n-1} \). We shall now prove that \( d \) is a path.

Suppose, indeed, that \( d \) is not a path. Then one of the following cases occurs:

**Case 1.** There are distinct \( i, j \in \mathbb{Z}/n\mathbb{Z} \), together with \( k \in \{0, \ldots, p_i\} \) and \( l \in \{0, \ldots, p_j\} \), such that \( x_i, k = x_j, l \). Then \( a_i \triangleq x_i, k \leq b_j \), thus \( i \in \{j, j+1\} \), while \( a_j \triangleq x_j, l = x_i, k \triangleq b_l \), thus \( i \in \{i, i+1\} \). Since \( n \geq 3 \), we obtain that \( i = j \), a contradiction.

**Case 2.** There are distinct \( i, j \in (\mathbb{Z}/n\mathbb{Z}) \setminus \{n-1\} \), together with \( k \in \{1, \ldots, q_i - 1\} \) and \( l \in \{1, \ldots, q_j - 1\} \), such that \( y_i, k = y_j, l \). Then \( a_{i+1} \triangleq y_i, k \leq b_j \), thus \( i \in \{j, j-1\} \), while \( a_{j+1} \triangleq y_j, l = y_i, k \triangleq b_l \), thus \( j \in \{i, i-1\} \), whence, since \( n \geq 3 \), \( i = j \), a contradiction.

**Case 3.** There are \( i \in \mathbb{Z}/n\mathbb{Z} \) and \( j \in (\mathbb{Z}/n\mathbb{Z}) \setminus \{n-1\} \), together with \( k \in \{0, \ldots, p_i\} \) and \( l \in \{1, \ldots, q_j - 1\} \), such that \( x_i, k = y_j, l \). Then from Claim 1 it follows that \( i \neq j \), while from Claim 2 it follows that \( i \neq j + 1 \). On the other hand, \( a_i \triangleq x_i, k = y_j, l \triangleq b_j \), thus \( i \in \{j, j+1\} \), a contradiction.

Therefore, we have proved that \( d \) is, indeed, a path from \( a_0 \) to \( b_{n-1} \). However, the finite sequence

\[
d' = (y_{n-1, l} | 0 \leq l < q_{n-1})
\]
is a path from \( y_{n-1,0} = a_n = a_0 \) (the indices are modulo \( n \)) to \( y_{n-1,qn-1} = b_{n-1} \), thus, by the uniqueness of the path from \( a_0 \) to \( b_{n-1} \), \( d = \mathbf{d}' \). Thus every entry \( x \) of \( \mathbf{d} \) satisfies that \( x \subseteq b_{n-1} \). In particular, \( b_0 = x_{0,p_0} \subseteq b_{n-1} \), a contradiction since \( n \neq 1 \). \( \square \)

10. A quasi-identity for \( \text{Co}(T) \), for finite and crown-free \( T \)

Let \( (T, \subseteq) \) be a finite crown-free poset. Then \( \text{Co}(T) \) satisfies \( (\theta) \).

Let us begin with an arbitrary (not necessarily finite, not necessarily crown-free) poset \( (T, \subseteq) \) and convex subsets \( A, B, C, A', B', C' \) of \( T \) that satisfy the premise of \( (\theta) \), that is,

\[
A \subseteq A' \lor B, \quad A \subseteq A' \lor C, \\
B \subseteq B' \lor A, \quad B \subseteq B' \lor C, \\
C \subseteq C' \lor A, \quad C \subseteq C' \lor B,
\]

\[
A \cap A' \subseteq B' \cap C', \quad B \cap B' \subseteq A' \cap C', \quad C \cap C' \subseteq A' \cap B', \\
A \cap B \subseteq A' \cap B', \quad B \cap C \subseteq B' \cap C', \quad A \cap C \subseteq A' \cap C'.
\]

We shall put \( \widehat{A} = A \setminus A' \), \( \widehat{B} = B \setminus B' \), and \( \widehat{C} = C \setminus C' \). Observe that

\[
\widehat{A} \cap (B \cup C) = \widehat{B} \cap (A \cup C) = \widehat{C} \cap (A \cup B) = \emptyset, \\
\widehat{A} \cap \widehat{B} = \widehat{A} \cap \widehat{C} = \widehat{B} \cap \widehat{C} = \emptyset.
\]

We shall later perform a construction whose key argument is provided by the following lemma.

**Lemma 10.2.** Let \( a \in \widehat{A} \) and let \( a' \in A' \) with \( a \nless a' \). Then there exists \( (b, b') \in \widehat{B} \times B' \) such that \( b' \nless b \nless a \).

**Proof.** Observe first that \( a \in A \subseteq A' \lor B \). Since \( a \notin A' \lor B \), there exists \( (\widehat{a}', b) \in A' \times B \) such that either \( \widehat{a}' \nless a \nless b \) or \( b \nless a \nless \widehat{a}' \). In the first case, \( \widehat{a}' \nless a \nless a' \), thus, by the convexity of \( A', a \in A' \), a contradiction; whence \( b \nless a \). If \( b \in B' \), then \( b \in B \cap B' \subseteq A' \), but...
Then, since \( b \triangleleft a \triangleleft a' \), thus \( a \in A' \), a contradiction; whence \( b \in \widehat{B} \). If there exists \( x \in A \) with \( x \leq b \), then, since \( b \triangleleft a \), we obtain that \( b \in A \cap B \subseteq A' \), a contradiction again. But \( b \in B \subseteq A \vee B' \) and \( b \not\in B' \), thus there exists \( b' \in B' \) such that \( b' \triangleleft b \).

In particular, we observe the following corollary.

**Corollary 10.3.** The sets \( \widehat{A}, \widehat{B}, \) and \( \widehat{C} \) are either simultaneously empty or simultaneously nonempty.

**Proof.** If \( \widehat{A} \) is nonempty, we pick \( a \in \widehat{A} \). So \( a \in A' \vee B \) while \( a \not\in A' \cup B \), thus there is \( (a', b) \in A' \times B \) such that either \( b \triangleleft a \) or \( b \not\triangleleft a' \). In the first case, we apply Lemma 10.2 to deduce that \( B \neq \emptyset \). In the second case, we apply the dual of Lemma 10.2 to reach the same conclusion. \( \square \)

Now we observe that \( \widehat{A} \) is nonempty, and we pick \( a_0 \in \widehat{A} \). As in the proof of Corollary 10.3, there exists \( a_0' \in A' \) such that either \( a_0 \triangleleft a_0' \) or \( a_0' \triangleleft a_0 \); by replacing \( \leq \) with its dual if needed, we may assume without loss of generality that \( a_0 \triangleleft a_0' \).

By Lemma 10.2, there are \( (b_0, b_0') \in \widehat{B} \times B' \) and \( (c_1, c_1') \in \widehat{C} \times C' \) such that \( b_0 \triangleleft b_0' \triangleleft a_0 \) and \( c_1 \triangleleft c_1' \triangleleft a_0 \). By applying the dual of Lemma 10.2 to \( c_1' \triangleleft c_1 \), we obtain \( (b_1, b_1') \in \widehat{B} \times B' \) such that \( c_1 \triangleleft b_1 \triangleleft b_1' \). By applying Lemma 10.2 to \( b_1 \triangleleft b_1' \), we obtain \( (a_2, a_2') \in \widehat{A} \times A' \) such that \( a_2 \triangleleft a_2' \triangleleft b_1 \). By applying in the same fashion Lemma 10.2 and its dual, we obtain \( (c_2, c_2') \in \widehat{C} \times C' \), \( (b_3, b_3') \in \widehat{B} \times B' \), and \( (a_3, a_3') \in \widehat{A} \times A' \) such that \( a_2 \triangleleft c_2 \triangleleft c_2' \triangleleft b_3 \triangleleft c_2 \triangleleft b_3 ' \triangleleft a_3 \triangleleft a_3' \).

Now we observe that \( b_0' \triangleleft b_0 \triangleleft a_0 \) and \( b_3' \triangleleft a_3 \triangleleft a_3' \), that is, we can start the process again. Arguing by induction, we obtain elements \( (a_i, a_i') \in \widehat{A} \times A' \) for \( i \not\equiv 1 \pmod{3} \), elements \( (b_i, b_i') \in \widehat{B} \times B' \) for \( i \not\equiv 2 \pmod{3} \), and elements \( (c_i, c_i') \in \widehat{C} \times C' \) for \( i \not\equiv 0 \pmod{3} \) such that the following relations hold, for all \( i < \omega \):

\[
\begin{align*}
  b_{3i} &\triangleleft b_{3i+1} \triangleleft a_{3i} \triangleleft a_{3i+1}, \\
  c_{3i+1} &\triangleleft c_{3i+2} \triangleleft b_{3i+1} \triangleleft b_{3i+2}, \\
  a_{3i+2} &\triangleleft a_{3i+3} \triangleleft c_{3i+2} \triangleleft c_{3i+3}.
\end{align*}
\]

(10.1) \hspace{1cm} (10.2) \hspace{1cm} (10.3)

This can be illustrated by Fig. 5.

Now we define subsets of \( T \) as follows:

Fig. 5. A pattern in \( T \).
\[
\begin{align*}
\Omega^+ &= \{a_{3i} \mid i < \omega\} \cup \{b_{3i+1} \mid i < \omega\} \cup \{c_{3i+2} \mid i < \omega\}; \\
\Omega^- &= \{a_{3i+2} \mid i < \omega\} \cup \{b_{3i} \mid i < \omega\} \cup \{c_{3i+1} \mid i < \omega\}; \\
\Omega &= \Omega^+ \cup \Omega^-.
\end{align*}
\]

Since \(\hat{A}, \hat{B},\) and \(\hat{C}\) are mutually disjoint and their union contains \(\Omega\), we can define a map \(\chi : \Omega \rightarrow 3\) by the rule

\[
\chi(x) = \begin{cases} 
0 & (x \in \hat{A}), \\
1 & (x \in \hat{B}), \\
2 & (x \in \hat{C}). 
\end{cases}
\]

**Lemma 10.4.** For all \(\langle x, y \rangle \in \Omega^- \times \Omega^+\), \(\chi(x) = \chi(y)\) implies that \(x \not\sim y\). In particular, \(\Omega^- \cap \Omega^+ = \emptyset\).

**Proof.** We need to prove that for all natural numbers \(i\) and \(j\), the following inequalities hold:

- \(a_{3i+2} \not\in a_{3j}\). Otherwise, by (10.1) and (10.3), \(a_{3i+2} \not\prec a_{3j}\), thus \(a_{3i+2} \in A'\), a contradiction.
- \(b_{3i} \not\in b_{3j+1}\). Otherwise, by (10.1) and (10.2), \(b_{3i} \not\prec b_{3j+1}\), thus \(b_{3i} \in B'\), a contradiction.
- \(c_{3i+1} \not\in c_{3j+2}\). Otherwise, by (10.2) and (10.3), \(c_{3i+1} \not\prec c_{3j+2}\), thus \(c_{3i+1} \in C'\), a contradiction.

This concludes the proof. \(\square\)

For an integer \(m \geq 2\), we define a \(m\)-pre-crown to be a finite sequence \(\langle \langle x_i, y_i \rangle \mid i \in \mathbb{Z}/m\mathbb{Z} \rangle\) of elements of \(\Omega^- \times \Omega^+\) such that the following conditions hold, for all \(i \in \mathbb{Z}/m\mathbb{Z}\):

- (C1) \(x_i, x_{i+1} \prec y_i\);
- (C2) \(\chi(x_i) \neq \chi(x_{i+1})\) and \(\chi(y_i) \neq \chi(y_{i+1})\) if \(i \neq m - 1\).

If \(m = 2\), then, by (C1), \(x_0, x_1 \prec y_0, y_1\). Furthermore, by (C2), \(\chi(x_0) \neq \chi(x_1)\), thus it follows from \(x_0, x_1 \prec y_0\) and Lemma 10.4 that \(\chi(y_0)\) is the unique element of \(3 \setminus \{\chi(x_0), \chi(x_1)\}\). The same holds for \(\chi(y_1)\), whence \(\chi(y_0) = \chi(y_1)\), which contradicts (C2). Therefore, if there exists a \(m\)-pre-crown, then \(m \geq 3\).

We can now prove the main lemma of this section.

**Lemma 10.5.** Suppose that \(T\) is crown-free. Then there are no pre-crowns in \(T\).

**Proof.** Otherwise, let \(m\) be the least positive integer such that there exists a \(m\)-pre-crown, and let \(c = \langle \langle x_i, y_i \rangle \mid i \in \mathbb{Z}/m\mathbb{Z} \rangle\) be such a pre-crown. As observed before, \(m \geq 3\). By assumption on \(T\), in order to get a contradiction, it suffices to prove that \(c\) is a crown of \(T\).

By (C1) and Lemma 9.3, it suffices to prove that for all \(i, j \in \mathbb{Z}/m\mathbb{Z}\) such that \(i \notin \{j, j+1\}, \)
the inequality \( x_i \preceq y_j \) does not hold. Suppose otherwise; by Lemma 10.4, \( x_i \prec y_j \). Two cases can occur.

**Case 1.** \( i < j \). Then the finite sequence

\[
\langle \langle x_i, y_i \rangle, \langle x_{i+1}, y_{i+1} \rangle, \ldots, \langle x_j, y_j \rangle \rangle
\]

is a \((j - i + 1)\)-pre-crown (see Fig. 6(i)), with \( 1 \leq j - i \leq m - 1 \). By the minimality assumption on \( m \), this cannot happen unless \( i = 0 \) and \( j = m - 1 \), in which case \( i = j + 1 \) (modulo \( m \) as usual), a contradiction.

**Case 2.** \( j < i \). Then the finite sequence

\[
\langle \langle x_i, y_{i-1} \rangle, \ldots, \langle x_{j+2}, y_{j+1} \rangle, \langle x_{j+1}, y_j \rangle \rangle
\]

is a \((i - j)\)-pre-crown (see Fig. 6(ii)), with \( 2 \leq i - j < m \), which contradicts again the minimality of \( m \).

Hence \( c \) is a \( m \)-crown of \( T \), a contradiction. □

Now we have all the necessary tools to conclude the proof of Theorem 10.1.

**Proof of Theorem 10.1.** Suppose that \( T \) is finite and crown-free. There are \( i < j \) such that \( b_{3i} = b_{3j} \). Then the finite sequence

\[
\{ \langle b_{3i}, a_{3i} \rangle, \langle c_{3i+1}, b_{3i+1} \rangle, \ldots, \langle a_{3j-1}, c_{3j-1} \rangle \}
\]

is a \((3j - 3i)\)-pre-crown in \( T \) (see Fig. 7), a contradiction.

Hence we have proved that \( \hat{A} = \emptyset \), that is, \( A \subseteq A' \). Therefore, \( Co(T) \) satisfies \((\theta)\). □
Corollary 10.6. Let $Q$ be the finite poset and $L$ the finite lattice of Example 8.2. Then, although $L$ embeds into $\text{Co}(Q)$, there is no finite, tree-like poset $R$ such that $L$ embeds into $\text{Co}(R)$.

Proof. It follows from Proposition 9.5 that $R$ is crown-free, thus, by Theorem 10.1, $\text{Co}(R)$ satisfies $(\theta)$. On the other hand, the lattice $L$ of Example 8.2 does not satisfy $(\theta)$ (consider the atoms $A, B, C, A', B', C'$ of $L$), therefore it cannot be embedded into $\text{Co}(R)$.

On the other hand, it follows from Theorem 7.7(iii) that if a finite lattice $L$ without $D$-cycle embeds into some $\text{Co}(P)$, then it embeds into $\text{Co}(R)$ for some finite, tree-like poset $R$. In the presence of $D$-cycles anything can happen, for example, take $L = \text{Co}(4)$, the lattice of all order-convex subsets of a four-element chain; it embeds into $\text{Co}(4)$ for the finite, tree-like poset $4$, however it has $D$-cycles.

11. Finite generation and word problem in $\text{SUB}$

For a lattice term $s(x_1,\ldots,x_n)$, a poset $P$, and convex subsets $X_1,\ldots,X_n$ of $P$, we denote by $s^P(X_1,\ldots,X_n)$ the evaluation of the term $s(x_1,\ldots,x_n)$ at $(X_1,\ldots,X_n)$ in the lattice $\text{Co}(P)$.

The present section rests on the following lemma. Its proof is an easy induction argument on the length of $s$, that we leave to the reader.

Lemma 11.1. Let $n$ be a positive integer, let $s(x_1,\ldots,x_n)$ be a lattice term, and let $X_1,\ldots,X_n$ be convex subsets of a poset $P$. Then $s^P(X_1,\ldots,X_n)$ is the directed union of all subsets of the form $s^Q(X_1\cap Q,\ldots,X_n\cap Q)$, for $Q \subseteq P$ finite.

As immediate corollaries, we get the following:

Corollary 11.2. Let $P$ be a poset. Any lattice-theoretical identity valid in all $\text{Co}(Q)$, for $Q$ a finite subset of $P$, is also valid in $\text{Co}(P)$.

Corollary 11.3. A lattice-theoretical identity is valid in $\text{SUB}$ iff it holds in $\text{Co}(P)$ for every finite poset $P$.

Consequently, the variety $\text{SUB}$ is generated by its finite members. By using the results of J.C.C. McKinsey [14], we obtain the following consequence.

Corollary 11.4. The word problem in the variety $\text{SUB}$ is decidable.

This means that it is decidable whether a given lattice identity $s(x_1,\ldots,x_m) = t(x_1,\ldots,x_m)$ holds in all lattices of the form $\text{Co}(P)$. A closer look at the proof of Lemma 11.1 shows that it is sufficient to verify whether the given identity holds in all $\text{Co}(P)$ for $|P| \leq n$, where $n$ is the supremum of the lengths of the terms $s$ and $t$. 
12. Open problems

We know that the class $\text{SUB}$ is generated, as a variety, by its finite members (see Corollary 11.3). We also know that any finite lattice in $\text{SUB}$ can be embedded into some finite $\text{Co}(P)$ (see Theorem 6.7). Nevertheless we do not know whether the latter generate the whole quasi-variety.

**Problem 1.** Is the class $\text{SUB}$ generated, as a quasivariety, by its finite members?

Equivalently, does there exist a lattice quasi-identity that holds in all finite $\text{Co}(P)$-s but not in all $\text{Co}(P)$-s?

**Problem 2.** Is the universal theory of all lattices of the form $\text{Co}(P)$ decidable?

A positive answer to Problem 1 would yield a positive answer to Problem 2.

**Problem 3.** Is the class $\mathcal{C}$ of all lattices that can be embedded into a product of the form $\prod_{i \in I} \text{Co}(C_i)$, where the $C_i$ are chains, a variety?

Problem 3 is solved by the authors in [16].

**Problem 4.** Can the embedding problem of a lattice in $\text{SUB}$ into some $\text{Co}(P)$ be solved by a functor (that, say, sends any $L$ to some $\text{Co}(P)$)? Can such a functor be idempotent?

Our next problem has a more computational nature.

**Problem 5.** For each positive integer $n$, denote by $\xi(n)$ the least positive integer such that every finite lattice $L$ in $\text{SUB}$ with $n$ join-irreducible elements embeds into some $\text{Co}(P)$, where $|P| \leq \xi(n)$. Compute $\xi(n)$, for all $n > 0$. Does $\xi(n) = O(n)$ as $n$ goes to infinity?

For a sublattice $K$ of a finite lattice $L$, the inequality $|J(K)| \leq |J(L)|$ holds, see [1, Lemma 2]. In particular, if a finite lattice $L$ embeds into $\text{Co}(P)$ for some finite poset $P$, then $|J(L)| \leq |P|$. By combining this with the result of Theorem 6.7, we obtain the inequalities

$$n \leq \xi(n) \leq 2n^2 - 5n + 4.$$  

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