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Sublattices of lattices of order-convex sets, I. The main representation theorem

Marina Semenova [a,1] and Friedrich Wehrung [b,*,2]

[a] *Institute of Mathematics of the Siberian Branch of RAS, Acad. Koptyug prosp. 4, 630090 Novosibirsk, Russia*
[b] *LMNO, CNRS UMR 6139, Département de Mathématiques, Université de Caen, 14032 Caen cedex, France*

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Abstract

For a partially ordered set P, we denote by $\mathbf{Co}(P)$ the lattice of order-convex subsets of P. We find three new lattice identities, (S), (U), and (B), such that the following result holds.

Theorem. *Let L be a lattice. Then L embeds into some lattice of the form $\mathbf{Co}(P)$ iff L satisfies* (S), (U), *and* (B).

Furthermore, if L has an embedding into some $\mathbf{Co}(P)$, then it has such an embedding that preserves the existing bounds. If L is finite, then one can take P finite, with

$$|P| \leqslant 2\big|\mathrm{J}(L)\big|^2 - 5\big|\mathrm{J}(L)\big| + 4,$$

where $\mathrm{J}(L)$ denotes the set of all join-irreducible elements of L.

On the other hand, the partially ordered set P can be chosen in such a way that there are no infinite bounded chains in P and the undirected graph of the predecessor relation of P is a tree.

Keywords: Lattice; Embedding; Poset; Order-convex; Join-semidistributivity; 2-distributivity; Join-irreducible

* Corresponding author.
E-mail addresses: semenova@math.nsc.ru (M. Semenova), wehrung@math.unicaen.fr (F. Wehrung).
URL: http://www.math.unicaen.fr/~wehrung (F. Wehrung).
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1. Introduction

For a partially ordered set (from now on *poset*) $\langle P, \trianglelefteq \rangle$, a subset X of P is *order-convex*, if $x \trianglelefteq z \trianglelefteq y$ and $\{x, y\} \subseteq X$ implies that $z \in X$, for all $x, y, z \in P$. The set $\mathbf{Co}(P)$ of all order-convex subsets of P forms a lattice under inclusion. This lattice is algebraic, atomistic, and join-semidistributive (see Section 2 for the definitions), thus it is a special example of a *convex geometry*, see P.H. Edelman [5], P.H. Edelman and R. Jamison [6], or K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov [2]. Furthermore, it is 'biatomic' and satisfies the nonexistence of so-called 'zigzags' of odd length on its atoms. Is is proved in G. Birkhoff and M.K. Bennett [3] that these conditions *characterize* the lattices of the form $\mathbf{Co}(P)$.

One of the open problems of [2] is the characterization of all *sublattices* of the lattices of the form $\mathbf{Co}(P)$.

Problem 3 (*of* [2] *for* $\mathbf{Co}(P)$). Describe the subclass of those lattices that are embeddable into finite lattices of the form $\mathbf{Co}(P)$.

In the present paper, we solve completely this problem, not only in the finite case but also for arbitrary lattices. Our main result (Theorem 6.7) is that a lattice L can be embedded into some lattice of the form $\mathbf{Co}(P)$ iff L satisfies three completely new identities, that we denote by (S), (U), and (B). Furthermore, P can be taken either finite in case L is finite, or *tree-like* (see Theorem 7.7).

This result is quite surprising, as it yields the unexpected consequence (see Corollary 6.9) that the class of all lattices that can be embedded into some $\mathbf{Co}(P)$ is a *variety*, thus it is closed under homomorphic images. However, while it is fairly easy (though not completely trivial) to verify directly that the class is closed under reduced products and substructures (thus it is a *quasivariety*), we do not know any direct proof that it is closed under homomorphic images.

One of the difficulties of the present work is to guess, for a given L, which poset P will solve the embedding problem for L (i.e., L embeds into $\mathbf{Co}(P)$). The first natural guess, that consists of using for P the set of all join-irreducible elements of L, fails, as illustrated by the two examples of Section 8. We shall construct P *via sequences* of join-irreducible elements of L. In fact, we are able to embed L into $\mathbf{Co}(P)$ for two different sorts of posets P:

(1) P is finite in case L is finite; this is the construction of Section 6.
(2) P is tree-like (as defined in Section 2); this is the construction of Section 7.

The two requirements (1) and (2) above can be simultaneously satisfied in case L *has no D-cycle*, see Theorem 7.7(iii). However, the finite lattice $\boldsymbol{L}$ of Example 8.2 can be embedded into some finite $\mathbf{Co}(Q)$, but into no $\mathbf{Co}(R)$, where R is a finite tree-like poset, see Corollary 10.6. It is used to produce, in Section 10, a quasi-identity that holds in all $\mathbf{Co}(R)$, where R is finite and tree-like (or even what we call 'crown-free'), but not in all finite $\mathbf{Co}(P)$.

We conclude the paper by a list of open problems.

2. Basic concepts

A lattice L is *join-semidistributive*, if it satisfies the axiom

$$x \vee y = x \vee z \quad \Rightarrow \quad x \vee y = x \vee (y \wedge z), \quad \text{for all } x, y, z \in L. \qquad (\mathrm{SD}_\vee)$$

We denote by $\mathrm{J}(L)$ the set of join-irreducible elements of L. We say that L is *finitely spatial* (respectively, *spatial*) if every element of L is a join of join-irreducible (respectively, completely join-irreducible) elements of L.

We say that L is *lower continuous*, if the equality

$$a \vee \bigwedge X = \bigwedge (a \vee X)$$

holds, for all $a \in L$ and all downward directed $X \subseteq L$ such that $\bigwedge X$ exists (where $a \vee X = \{a \vee x \mid x \in X\}$). It is well known that every dually algebraic lattice is lower continuous—see Lemma 2.3 in P. Crawley and R.P. Dilworth [4], and spatial (thus finitely spatial)—see Theorem I.4.22 in G. Gierz et al. [9] or Lemma 1.3.2 in V.A. Gorbunov [10].

For every element x in a lattice L, we put

$$\downarrow x = \{y \in L \mid y \leqslant x\}, \qquad \uparrow x = \{y \in L \mid y \geqslant x\}.$$

If $a, b, c \in L$ such that $a \leqslant b \vee c$, we say that the (formal) inequality $a \leqslant b \vee c$ is a *nontrivial join-cover*, if $a \nleqslant b, c$. We say that it is *minimal in b*, if $a \nleqslant x \vee c$, for all $x < b$, and we say that it is a *minimal nontrivial join-cover*, if it is a nontrivial join-cover and it is minimal in both b and c.

The *join-dependency* relation $D = D_L$ (see R. Freese, J. Ježek, and J.B. Nation [7]) is defined on the join-irreducible elements of L by putting

$$pDq, \text{ if } p \neq q \text{ and } \exists x \text{ such that } p \leqslant q \vee x \text{ holds and is minimal in } q.$$

It is important to observe that pDq implies that $p \nleqslant q$, for all $p, q \in \mathrm{J}(L)$.

For a poset P endowed with a partial ordering $\trianglelefteq$, we shall denote by $\triangleleft$ the corresponding strict ordering. The set of all order-convex subsets of P forms a lattice under inclusion, that we shall denote by $\mathbf{Co}(P)$. The meet in $\mathbf{Co}(P)$ is the intersection, while the join is given by

$$X \vee Y = X \cup Y \cup \bigcup \bigl\{z \in P \mid \exists \langle x, y\rangle \in (X \times Y) \cup (Y \times X) \text{ such that } x \triangleleft z \triangleleft y\bigr\},$$

for all $X, Y \in \mathbf{Co}(P)$. Let us denote by $\prec$ the predecessor relation of P. We say that a *path* of P is a finite sequence $\mathbf{d} = \langle x_0, \dots, x_{n-1}\rangle$ of *distinct* elements of P such that either $x_i \prec x_{i+1}$ or $x_{i+1} \prec x_i$, for all i with $0 \leqslant i \leqslant n-2$; if $n > 0$, we say that $\mathbf{d}$ is a path from x_0 to x_{n-1}. We say that the path $\mathbf{d}$ is *oriented*, if $x_i \prec x_{i+1}$, for all i with $0 \leqslant i \leqslant n-2$. We say that P is *tree-like*, if the following properties hold:

(i) for all $a \trianglelefteq b$ in P, there are $n < \omega$ and $x_0, \dots, x_n \in P$ such that $a = x_0 \prec x_1 \prec \cdots \prec x_n = b$;
(ii) for all $a, b \in P$, there exists at most one path from a to b.

3. Dually 2-distributive lattices

For a positive integer n, the identity of n-distributivity is introduced in A.P. Huhn [12]. In this paper we shall only need the dual of 2-distributivity, which is the following identity:

$$a \wedge (x \vee y \vee z) = \bigl(a \wedge (x \vee y)\bigr) \vee \bigl(a \wedge (x \vee z)\bigr) \vee \bigl(a \wedge (y \vee z)\bigr).$$

We omit the easy proof of the following lemma, that expresses how dual 2-distributivity can be read on the join-irreducible elements.

Lemma 3.1. *Let L be a dually 2-distributive lattice. For all $p \in \mathrm{J}(L)$ and all $a, b, c \in L$, if $p \leqslant a \vee b \vee c$, then either $p \leqslant a \vee b$ or $p \leqslant a \vee c$ or $p \leqslant b \vee c$.*

We observe that for finitely spatial L, the converse of Lemma 3.1 holds.

The following lemma will be used repeatedly throughout the paper.

Lemma 3.2. *Let L be a dually 2-distributive, complete, lower continuous lattice. Let $p \in \mathrm{J}(L)$ and let $a, b \in L$ such that $p \leqslant a \vee b$ and $p \nleqslant a, b$. Then the following assertions hold*:

(i) *There are minimal $x \leqslant a$ and $y \leqslant b$ such that $p \leqslant x \vee y$.*
(ii) *Any minimal $x \leqslant a$ and $y \leqslant b$ such that $p \leqslant x \vee y$ are join-irreducible.*

Proof. (i) Let $X \subseteq {\downarrow}a$ and $Y \subseteq {\downarrow}b$ be chains such that $p \leqslant x \vee y$, for all $\langle x, y\rangle \in X \times Y$. It follows from the lower continuity of L that $p \leqslant (\bigwedge X) \vee (\bigwedge Y)$. The conclusion of (i) follows from a simple application of Zorn's Lemma.

(ii) From $p \nleqslant a, b$ it follows that both x and y are nonzero. Suppose that $x = x_0 \vee x_1$ for some $x_0, x_1 < x$. It follows from the minimality assumption on x that $p \nleqslant x_0 \vee y$ and $p \nleqslant x_1 \vee y$, whence, by Lemma 3.1, $p \leqslant x_0 \vee x_1$, thus $p \leqslant x \leqslant a$, a contradiction. Hence x is join-irreducible. □

For $p, a, b \in \mathrm{J}(L)$, we say that $\langle a, b\rangle$ is a *conjugate pair* with respect to p, if $p \nleqslant a, b$ and a and b are minimal such that $p \leqslant a \vee b$; we say then that b is a *conjugate* of a with respect to p. Observe that the latter relation is symmetric in a and b, and that it implies that pDa and pDb.

Notation 3.3. For a lattice L and $p \in \mathrm{J}(L)$, we put

$$[p]^D = \bigl\{x \in \mathrm{J}(L) \mid pDx\bigr\}.$$

Corollary 3.4. *Let L be a dually 2-distributive, complete, lower continuous lattice, and let $p \in \mathrm{J}(L)$. Then every $a \in [p]^D$ has a conjugate with respect to p.*

Proof. By the definition of join-dependency, there exists $c \in L$ such that $p \leqslant a \vee c$ and $p \nleqslant x \vee c$, for all $x < a$. By Lemma 3.2, there are $a' \leqslant a$ and $b \leqslant c$ minimal such that $p \leqslant a' \vee b$, and both a' and b are join-irreducible. It follows that $a' = a$, whence b is a conjugate of a with respect to p. □

4. Stirlitz, Udav, and Bond

4.1. *The Stirlitz identity* (S) *and the axiom* ($\mathrm{S_j}$)

Let (S) be the following identity:

$$a \wedge \bigl(b' \vee c\bigr) = \bigl(a \wedge b'\bigr) \vee \bigvee_{i<2} \bigl(a \wedge (b_i \vee c) \wedge \bigl(\bigl(b' \wedge (a \vee b_i)\bigr) \vee c\bigr)\bigr),$$

where we put $b' = b \wedge (b_0 \vee b_1)$.

Lemma 4.1. *The Stirlitz identity* (S) *holds in* $\mathbf{Co}(P)$, *for any poset* $\langle P, \trianglelefteq\rangle$.

Proof. Let $A, B, B_0, B_1, C \in \mathbf{Co}(P)$ and $a \in A \cap (B' \vee C)$, where we put $B' = B \cap (B_0 \vee B_1)$. Denote by D the right-hand side of the Stirlitz identity calculated with these parameters. If $a \in B'$ then $a \in A \cap B' \subseteq D$. If $a \in C$ then $a \in A \cap C \subseteq D$.

Suppose that $a \notin B' \cup C$. There exist $b \in B'$ and $c \in C$ such that, say, $b \lhd a \lhd c$. Since $b \in B_0 \vee B_1$, there are $i < 2$ and $b' \in B_i$ such that $b' \trianglelefteq b$, hence $a \in A \cap (B_i \vee C)$. Furthermore, $b \in B' \cap (A \vee B_i)$, thus $a \in (B' \cap (A \vee B_i)) \vee C$, so $a \in D$. □

Lemma 4.2. *The Stirlitz identity* (S) *implies dual* 2*-distributivity.*

Proof. Take $b_0 = x$, $b_1 = y$, $b = x \vee y$, and $c = z$. □

Let ($\mathrm{SD}^2_\vee$) be the following identity:

$$x \vee (y \wedge z) = x \vee \bigl(y \wedge \bigl(x \vee \bigl(z \wedge (x \vee y)\bigr)\bigr)\bigr). \qquad (\mathrm{SD}^2_\vee)$$

It is well known that ($\mathrm{SD}^2_\vee$) implies join-semidistributivity (that is, the axiom ($\mathrm{SD}_\vee$)), see, for example, P. Jipsen and H. Rose [13, p. 81].

Lemma 4.3. *The Stirlitz identity* (S) *implies* ($\mathrm{SD}^2_\vee$).

Proof. Let L be a lattice satisfying (S), let $x, y, z \in L$. Set $y_2 = y \wedge (x \vee (z \wedge (x \vee y)))$. Set $a = b_1 = y$, $b = z$, $c = b_0 = x$, and $b' = b \wedge (b_0 \vee b_1) = z \wedge (x \vee y)$. Then the following inequalities hold:

$$\begin{aligned}
y_2 &= y \wedge \bigl(x \vee \bigl(z \wedge (x \vee y)\bigr)\bigr) = a \wedge \bigl(\bigl(b \wedge (b_0 \vee b_1)\bigr) \vee c\bigr) \\
&\leqslant \bigl(a \wedge b'\bigr) \vee \bigvee_{i<2} \bigl(a \wedge (b_i \vee c) \wedge \bigl(\bigl(b' \wedge (a \vee b_i)\bigr) \vee c\bigr)\bigr) \\
&= (y \wedge z) \vee (y \wedge x) \vee \bigl(y \wedge \bigl((z \wedge y) \vee x\bigr)\bigr) = (y \wedge z) \vee \bigl(y \wedge \bigl(x \vee (y \wedge z)\bigr)\bigr) \\
&= y \wedge \bigl(x \vee (y \wedge z)\bigr) \leqslant x \vee (y \wedge z).
\end{aligned}$$

This implies that $x \vee y_2 \leqslant x \vee (y \wedge z)$. Since the converse inequality holds in any lattice, the conclusion follows. □

We now introduce a lattice-theoretical axiom, the *join-irreducible interpretation of* (S), that we will denote by $(\mathrm{S_j})$:

> For all $a, b, b_0, b_1, c \in \mathrm{J}(L)$, the inequalities $a \leqslant b \vee c$, $b \leqslant b_0 \vee b_1$, and $a \neq b$ imply that either $a \leqslant \bar{b} \vee c$ for some $\bar{b} < b$ or $b \leqslant a \vee b_i$ and $a \leqslant b_i \vee c$ for some $i < 2$.

Throughout the paper we shall make repeated use of the item (i) of the following statement. Item (ii) provides a convenient algorithm for verifying whether a finite lattice satisfies (S).

Proposition 4.4. *Let L be a lattice. Then the following assertions hold*:

(i) *If L satisfies* (S), *then L satisfies* $(\mathrm{S_j})$.
(ii) *If L is complete, lower continuous, finitely spatial, dually* 2*-distributive, and satisfies* $(\mathrm{S_j})$, *then L satisfies* (S).

Proof. (i) Let $a \leqslant b \vee c$, $b \leqslant b_0 \vee b_1$, and $a \neq b$ for some $a, b, b_0, b_1, c \in \mathrm{J}(L)$. Then the element b' of the Stirlitz identity is $b' = b \wedge (b_0 \vee b_1) = b$; observe also that $a \wedge (b \vee c) = a$. Therefore, applying (S) yields

$$\begin{aligned}
a = a \wedge \bigl(b' \vee c\bigr) &= \bigl(a \wedge b'\bigr) \vee \bigvee_{i<2} \bigl(a \wedge (b_i \vee c) \wedge \bigl(\bigl(b' \wedge (a \vee b_i)\bigr) \vee c\bigr)\bigr) \\
&= (a \wedge b) \vee \bigvee_{i<2} \bigl(a \wedge (b_i \vee c) \wedge \bigl(\bigl(b \wedge (a \vee b_i)\bigr) \vee c\bigr)\bigr).
\end{aligned}$$

Since a is join-irreducible, either $a \leqslant b$ or $a \leqslant (b_i \vee c) \wedge ((b \wedge (a \vee b_i)) \vee c)$ for some $i < 2$. If $a \leqslant b$ then $a \leqslant a \vee c$ with $a < b$ (because $a \neq b$). Suppose that $a \nleqslant b$. Then $a \leqslant (b_i \vee c) \wedge ((b \wedge (a \vee b_i)) \vee c) \leqslant b_i \vee c$ for some $i < 2$. If $b \nleqslant a \vee b_i$, then $a \leqslant \bar{b} \vee c$ for $\bar{b} = b \wedge (a \vee b_i) < b$.

(ii) Put $b' = b \wedge (b_0 \vee b_1)$, and let d denote the right-hand side of the identity (S). Since $d \leqslant a \wedge (b' \vee c)$, we must prove the converse inequality only. Let $a_1 \in \mathrm{J}(L)$ with $a_1 \leqslant a \wedge (b' \vee c)$. Then $a_1 \leqslant a$ and $a_1 \leqslant b' \vee c$. If $a_1 \leqslant b'$, then $a_1 \leqslant a \wedge b' \leqslant d$. If $a_1 \leqslant c$, then $a_1 \leqslant a \wedge c \leqslant d$.

Suppose now that $a_1 \nleqslant b', c$. Then, by using Lemma 3.2, we obtain that there are minimal $b'_1 \leqslant b'$ and $c_1 \leqslant c$ such that the following inequality holds,

$$a_1 \leqslant b'_1 \vee c_1 \tag{4.1}$$

and both b'_1 and c_1 are join-irreducible. From $a_1 \nleqslant b'$ it follows that $a_1 \nleqslant b'_1$. If $b'_1 \leqslant b_i$ for some $i < 2$, then the inequalities $b'_1 \leqslant b' \wedge b_i \leqslant b' \wedge (a \vee b_i)$ and $a_1 \leqslant b'_1 \vee c_1 \leqslant (b' \wedge (a \vee b_i)) \vee c$ hold; but in this case, we also have $a_1 \leqslant a \wedge (b_i \vee c)$, whence $a_1 \leqslant d$. Suppose that $b'_1 \nleqslant b_0, b_1$. Then, by Lemma 3.2, there are join-irreducible elements $d_i \leqslant b_i$, $i < 2$, such that the following inequality

$$b'_1 \leqslant d_0 \vee d_1 \tag{4.2}$$

holds. It follows from (4.1), (4.2), $a_1 \nleqslant b'_1$, the minimality of b'_1 in (4.1), and ($\mathrm{S_j}$) that there exists $i < 2$ such that $b'_1 \leqslant a_1 \vee d_i$ and $a_1 \leqslant d_i \vee c_1$. Then the following inequalities hold:

$$a_1 \leqslant a \wedge (d_i \vee c_1) \wedge \bigl(b'_1 \vee c_1\bigr) \leqslant a \wedge (b_i \vee c) \wedge \bigl(\bigl(b' \wedge (a \vee b_i)\bigr) \vee c\bigr) \leqslant d.$$

In every case, $a_1 \leqslant d$. Since L is finitely spatial, it follows that $a \wedge (b' \vee c) \leqslant d$. □

4.2. *The Bond identity* (B) *and the axiom* ($\mathrm{B_j}$)

Let (B) be the following identity:

$$x \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) = \bigvee_{i<2} \bigl(\bigl(x \wedge a_i \wedge (b_0 \vee b_1)\bigr) \vee \bigl(x \wedge b_i \wedge (a_0 \vee a_1)\bigr)\bigr)$$
$$\vee \bigvee_{i<2} \bigl(x \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) \wedge (a_0 \vee b_i) \wedge (a_1 \vee b_{1-i})\bigr).$$

Lemma 4.5. *The Bond identity* (B) *holds in* $\mathbf{Co}(P)$, *for any poset* $\langle P, \trianglelefteq \rangle$.

Proof. Let X, A_0, A_1, B_0, B_1 be elements of $\mathbf{Co}(P)$. Denote by C the right-hand side of the Bond identity formed from these elements. Let $x \in X \cap (A_0 \vee A_1) \cap (B_0 \vee B_1)$, we prove that $x \in C$. The conclusion is obvious if $x \in A_0 \cup A_1 \cup B_0 \cup B_1$, so suppose that $x \notin A_0 \cup A_1 \cup B_0 \cup B_1$. Since $x \in (A_0 \vee A_1) \setminus (A_0 \cup A_1)$, there are $a_0 \in A_0$ and $a_1 \in A_1$ such that, say, $a_0 \lhd x \lhd a_1$. Since $x \in (B_0 \vee B_1) \setminus (B_0 \cup B_1)$, there are $b_0 \in B_0$ and $b_1 \in B_1$ such that either $b_0 \lhd x \lhd b_1$ or $b_1 \lhd x \lhd b_0$. In the first case, x belongs to $X \cap (A_0 \vee A_1) \cap (B_0 \vee B_1) \cap (A_0 \vee B_1) \cap (A_1 \vee B_0)$, thus to C. In the second case, x belongs to $X \cap (A_0 \vee A_1) \cap (B_0 \vee B_1) \cap (A_0 \vee B_0) \cap (A_1 \vee B_1)$, thus again to C. □

We now introduce a lattice-theoretical axiom, the *join-irreducible interpretation of* (B), that we will denote by ($\mathrm{B_j}$):

For all $x, a_0, a_1, b_0, b_1 \in \mathrm{J}(L)$, the inequalities $x \leqslant a_0 \vee a_1, b_0 \vee b_1$ imply that either $x \leqslant a_i$ or $x \leqslant b_i$ for some $i < 2$ or $x \leqslant a_0 \vee b_0, a_1 \vee b_1$ or $x \leqslant a_0 \vee b_1, a_1 \vee b_0$.

Throughout the paper we shall make repeated use of the item (i) of the following statement. Item (ii) provides a convenient algorithm for verifying whether a finite lattice satisfies (B).

Proposition 4.6. *Let L be a lattice. Then the following assertions hold*:

(i) *If L satisfies* (B), *then L satisfies* ($\mathrm{B_j}$).
(ii) *If L is complete, lower continuous, finitely spatial, dually* 2*-distributive, and satisfies* ($\mathrm{B_j}$), *then L satisfies* (B).

Proof. Item (i) is easy to prove by using the (B) identity and the join-irreducibility of x.

(ii) Let u (respectively, v) denote the left- (respectively, right-) hand side of the identity (B). It is obvious that $v \leqslant u$. Since L is finitely spatial, in order to prove that $u \leqslant v$ it is sufficient to prove that for all $p \in \mathrm{J}(L)$ such that $p \leqslant u$, the inequality $p \leqslant v$ holds. This is obvious if either $p \leqslant a_i$ or $p \leqslant b_i$ for some $i < 2$, so suppose that $p \nleqslant a_i, b_i$, for all $i < 2$. Then, by Lemma 3.2, there exist $x_0, x_1, y_0, y_1 \in \mathrm{J}(L)$ such that $x_i \leqslant a_i$ and $y_i \leqslant b_i$, for all $i < 2$, while $p \leqslant x_0 \vee x_1, y_0 \vee y_1$. By assumption, we obtain that one of the following assertions holds:

$$p \leqslant (x_0 \vee y_0) \wedge (x_1 \vee y_1) \leqslant (a_0 \vee b_0) \wedge (a_1 \vee b_1),$$
$$p \leqslant (x_0 \vee y_1) \wedge (x_1 \vee y_0) \leqslant (a_0 \vee b_1) \wedge (a_1 \vee b_0).$$

In any case, $p \leqslant v$, which completes the proof. □

4.3. *The Udav identity* (U) *and the axiom* ($\mathrm{U_j}$)

Let (U) be the following identity:

$$x \wedge (x_0 \vee x_1) \wedge (x_1 \vee x_2) \wedge (x_0 \vee x_2)$$
$$= \big(x \wedge x_0 \wedge (x_1 \vee x_2)\big) \vee \big(x \wedge x_1 \wedge (x_0 \vee x_2)\big) \vee \big(x \wedge x_2 \wedge (x_0 \vee x_1)\big).$$

Lemma 4.7. *The Udav identity* (U) *holds in* $\mathbf{Co}(P)$, *for any poset* $\langle P, \trianglelefteq \rangle$.

Proof. Let X, X_0, X_1, X_2 be elements of $\mathbf{Co}(P)$. Denote by U (respectively, V) the left-hand side (respectively, right-hand side) of the Udav identity formed from these elements. It is clear that U contains V. Conversely, let $x \in U$, we prove that x belongs to V. This is clear if $x \in X_0 \cup X_1 \cup X_2$, so suppose that $x \notin X_0 \cup X_1 \cup X_2$. Since $x \in (X_0 \vee X_1) \setminus (X_0 \cup X_1)$, there are $x_0 \in X_0$ and $x_1 \in X_1$ such that, say, $x_0 \lhd x \lhd x_1$. Since $x \in (X_1 \vee X_2) \setminus (X_1 \cup X_2)$, there are $x_1' \in X_1$ and $x_2 \in X_2$ such that either $x_1' \lhd x \lhd x_2$ or $x_2 \lhd x \lhd x_1'$. But since $x \lhd x_1 \in X_1$ and $x \notin X_1$, the first possibility is ruled out, whence $x_2 \lhd x \lhd x_1'$. Since $x \in (X_0 \vee X_2) \setminus (X_0 \cup X_2)$, there are $x_0' \in X_0$ and $x_2' \in X_2$ such that either $x'_0 \lhd x \lhd x'_2$ or $x'_2 \lhd x \lhd x'_0$. The first possibility is ruled out by $x_2 \lhd x$ and $x \notin X_2$, while the second possibility is ruled out by $x_0 \lhd x$ and $x \notin X_0$. In any case, we obtain a contradiction. □

As we already did for (S) and (B), we now introduce a lattice-theoretical axiom, the *join-irreducible interpretation of* (U), that we will denote by $(\mathrm{U_j})$:

> For all $x, x_0, x_1, x_2 \in \mathrm{J}(L)$, the inequalities $x \leqslant x_0 \vee x_1, x_0 \vee x_2, x_1 \vee x_2$ imply that either $x \leqslant x_0$ or $x \leqslant x_1$ or $x \leqslant x_2$.

Throughout the paper we shall make repeated use of the item (i) of the following statement. Item (ii) provides a convenient algorithm for verifying whether a finite lattice satisfies (U).

Proposition 4.8. *Let L be a lattice. Then the following assertions hold*:

(i) *If L satisfies* (U), *then L satisfies* $(\mathrm{U_j})$.
(ii) *If L is complete, lower continuous, finitely spatial, dually* 2*-distributive, and satisfies both* $(\mathrm{B_j})$ *and* $(\mathrm{U_j})$, *then L satisfies both* (B) *and* (U).

Proof. Item (i) is easy to prove by using the (U) identity and the join-irreducibility of x.

(ii) We have already seen in Proposition 4.6 that L satisfies (B).

Let u (respectively, v) be the left-hand side (respectively, right-hand side) of the identity (U). It is clear that $v \leqslant u$. Let $p \in \mathrm{J}(L)$ such that $p \leqslant u$, we prove that $p \leqslant v$. This is obvious if $p \leqslant x_i$ for some $i < 3$, so suppose that $p \nleqslant x_i$, for all $i < 3$. Then, by using Lemma 3.2, we obtain that there are join-irreducible elements $p_i, p'_i \leqslant x_i$ $(i < 3)$ of L such that the following inequalities hold:

$$p \leqslant p_0 \vee p_1, p'_1 \vee p_2, p'_0 \vee p'_2. \tag{4.3}$$

Since $p \nleqslant x_1$ and L satisfies $(\mathrm{B_j})$, it follows from the first two inequalities of (4.3) that $p \leqslant p_0 \vee p'_1, p_1 \vee p_2$. Similarly, from $p \nleqslant x_2$, the last two inequalities of (4.3), and $(\mathrm{B_j})$, we obtain the inequalities

$$p \leqslant p'_1 \vee p'_2, p'_0 \vee p_2,$$

and from the first and the last inequality of (4.3), together with $p \nleqslant x_0$ and $(\mathrm{B_j})$, we obtain the inequalities

$$p \leqslant p_0 \vee p'_2, p'_0 \vee p_1.$$

In particular, we have obtained the inequalities

$$p \leqslant p_0 \vee p'_1, p'_1 \vee p'_2, p_0 \vee p'_2,$$

whence, by the assumption $(\mathrm{U_j})$, $p \leqslant x_i$ for some $i < 3$, a contradiction. □

5. First steps together of the identities (S), (U), and (B)

5.1. Udav–Bond partitions

The goal of this subsection is to prove the following partition result of the sets $[p]^D$ (see Notation 3.3).

Proposition 5.1. *Let L be a complete, lower continuous, dually 2-distributive lattice that satisfies* (U) *and* (B). *Then for every $p \in \mathrm{J}(L)$, there are subsets A and B of $[p]^D$ that satisfy the following properties*:

(i) $[p]^D = A \cup B$ *and* $A \cap B = \emptyset$.
(ii) *For all $x, y \in [p]^D$, $p \leqslant x \vee y$ iff $\langle x, y\rangle \in (A \times B) \cup (B \times A)$.*

Moreover, the set $\{A, B\}$ is uniquely determined by these properties.

The set $\{A, B\}$ will be called the *Udav–Bond partition* (of $[p]^D$) associated with p. We observe that every conjugate with respect to p of an element of A (respectively, B) belongs to B (respectively, A).

Proof. If $[p]^D = \emptyset$ the result is obvious, so suppose that $[p]^D \neq \emptyset$. By Lemma 3.2, there are $a, b \in [p]^D$ minimal such that $p \leqslant a \vee b$. We define A and B by the formulas

$$A = \{x \in [p]^D \mid p \leqslant x \vee b\}, \qquad B = \{y \in [p]^D \mid p \leqslant a \vee y\}.$$

Let $x \in [p]^D$. By Corollary 3.4, x has a conjugate with respect to p, denote it by y. By Lemma 3.2(ii), y is join-irreducible, thus $y \in [p]^D$. By applying $(\mathrm{B_j})$ to the inequalities $p \leqslant a \vee b, x \vee y$, we obtain that either $p \leqslant a \vee x, b \vee y$ or $p \leqslant a \vee y, b \vee x$, thus either $p \leqslant a \vee x$ or $p \leqslant b \vee x$. If both inequalities hold simultaneously, then, since $p \leqslant a \vee b$ and by $(\mathrm{U_j})$, we obtain that p lies below either a or b or x, a contradiction. Hence we have established (i).

Let $x, y \in [p]^D$, we shall establish in which case the inequality $p \leqslant x \vee y$ holds. Suppose first that $x \in A$ and $y \in B$. By applying $(\mathrm{B_j})$ to the inequalities $p \leqslant b \vee x, a \vee y$, we obtain that either $p \leqslant x \vee y$ or $p \leqslant b \vee y$. In the second case, $y \in B$, but $y \in A$, a contradiction by item (i); hence $p \leqslant x \vee y$.

Now suppose that $x, y \in A$. If $p \leqslant x \vee y$, then, by applying $(\mathrm{U_j})$ to the inequalities $p \leqslant x \vee y, b \vee x, b \vee y$, we obtain that p lies below either x or y or b, a contradiction. Hence $p \nleqslant x \vee y$. The conclusion is the same for $\langle x, y\rangle \in B \times B$. This concludes the proof of item (ii).

Finally, the uniqueness of $\{A, B\}$ follows easily from items (i) and (ii). □

5.2. Choosing orientation with Stirlitz

In this subsection we shall investigate further the configuration on which $(\mathrm{S_j})$ is based. The following lemma suggests an 'orientation' of the join-irreducible elements in such a configuration. More specifically, we are trying to embed the given lattice into $\mathbf{Co}(P)$, for some poset $\langle P, \trianglelefteq \rangle$. Attempting to define P as $\mathrm{J}(L)$, this would suggest to order the elements a, b, b_0, and b_1 by $c \lhd a \lhd b$ and $b_{1-i} \lhd b \lhd b_i$. Although the elements of P will be defined *via finite sequences* of elements of $\mathrm{J}(L)$, rather than just elements of $\mathrm{J}(L)$, this idea will be crucial in the construction of Section 7.

Lemma 5.2. *Let L be a lattice satisfying* $(\mathrm{S_j})$ *and* $(\mathrm{U_j})$. *Let $a, b, b_0, b_1, c \in \mathrm{J}(L)$ such that $a \neq b$ and satisfying the inequalities $a \leqslant b \vee c$ with b minimal such, and $b \leqslant b_0 \vee b_1$ with $b \nleqslant b_0, b_1$. Then the following assertions hold*:

(i) *The inequalities $b \leqslant a \vee b_i$ and $a \leqslant b_i \vee c$ together are equivalent to the single inequality $b \leqslant b_i \vee c$, for all $i < 2$.*
(ii) *There is exactly one $i < 2$ such that $b \leqslant b_i \vee c$.*

Proof. We first observe that $b \nleqslant c$ (otherwise $a \leqslant c$). If $b \leqslant a \vee b_i$ and $a \leqslant b_i \vee c$, then obviously $b \leqslant b_i \vee c$. Suppose, conversely, that $b \leqslant b_0 \vee c$. If $b \leqslant b_1 \vee c$, then, by observing that $b \leqslant b_0 \vee b_1$ and applying $(\mathrm{U_j})$, we obtain that either $b \leqslant b_0$ or $b \leqslant b_1$ or $b \leqslant c$, a contradiction. Hence $b \nleqslant b_1 \vee c$, the uniqueness statement of (ii) follows. Furthermore, by $(\mathrm{S_j})$, there exists $i < 2$ such that $b \leqslant a \vee b_i$ and $a \leqslant b_i \vee c$, whence $b \leqslant b_i \vee c$, thus $i = 0$. Therefore, $b \leqslant a \vee b_0$ and $a \leqslant b_0 \vee c$. □

Next, for a conjugate pair $\langle b, b' \rangle$ of elements of $\mathrm{J}(L)$ with respect to some element a of $\mathrm{J}(L)$, we define

$$C\big[b, b'\big] = \big\{x \in \mathrm{J}(L) \mid b D x \text{ and } b \leqslant b' \vee x\big\}. \tag{5.1}$$

Notation 5.3. Let **SUB** denote the class of all lattices that satisfy the identities (S), (U), and (B).

Hence **SUB** is a variety of lattices. It is *finitely based*, that is, it is defined by finitely many equations.

Lemma 5.4. *Let L be a complete, lower continuous, finitely spatial lattice in* **SUB**. *Let $a, b \in \mathrm{J}(L)$ such that $a D b$. Then the equality $C[b, b_0] = C[b, b_1]$ holds, for all conjugates b_0 and b_1 of b with respect to a.*

Proof. We prove, for example, that $C[b, b_0]$ is contained in $C[b, b_1]$. Let $x \in C[b, b_0]$ (so $b \leqslant b_0 \vee x$), and suppose that $x \notin C[b, b_1]$ (so $b \nleqslant b_1 \vee x$). By Corollary 3.4, x has a conjugate, say, y, with respect to b. Since both relations $a \leqslant b \vee b_1$ and $b \leqslant x \vee y$ are minimal nontrivial join-covers, it follows from Lemma 5.2 that either $b \leqslant b_1 \vee x$ or $b \leqslant b_1 \vee y$, but the first possibility does not hold. Hence the following inequalities hold:

$$b \leqslant b_0 \vee x, b_1 \vee y, x \vee y. \tag{5.2}$$

Furthermore, by the uniqueness statement of Lemma 5.2, $b \nleqslant b_0 \vee y$. Thus, by ($\mathrm{B_j}$) and the first two inequalities in (5.2) (observe that $b \nleqslant b_0, b_1, x, y$), we obtain that $b \leqslant b_0 \vee b_1$. Hence $a \leqslant b \vee b_0 \leqslant b_0 \vee b_1$, whence $a \leqslant b \vee b_0, b \vee b_1, b_0 \vee b_1$, a contradiction by ($\mathrm{U_j}$). □

For all $a, b \in \mathrm{J}(L)$ such that $a D b$, there exists, by Corollary 3.4, a conjugate b' of b with respect to a. By Lemma 5.4, for fixed a, the value of $C[b; b']$ does not depend of b'. This entitles us to *define*

$$C(a, b) = C\big[b, b'\big], \quad \text{for any conjugate } b' \text{ of } b \text{ with respect to } a. \tag{5.3}$$

Lemma 5.5. *Let $a, b \in \mathrm{J}(L)$ such that $a D b$. Then the set $\{C(a, b), [b]^D \setminus C(a, b)\}$ is the Udav–Bond partition of $[b]^D$ associated with b.*

Proof. It suffices to prove that the assertions (i) and (ii) of Proposition 5.1 are satisfied by the set $\{C(a, b), [b]^D \setminus C(a, b)\}$. We first observe the following immediate consequence of Lemma 5.2.

Claim. *For any $x \in [b]^D$ and any conjugate x' of x, $x \notin C(a, b)$ iff $x' \in C(a, b)$.*

From now on we fix a conjugate b' of b with respect to a. Let $x, y \in [b]^D$, let x' (respectively, y') be a conjugate of x (respectively, y) with respect to b.

Suppose first that $x \in C(a, b)$ and $y \notin C(a, b)$, we prove that $b \leqslant x \vee y$. It follows from Claim above that $y' \in C(a, b)$, whence the inequalities $b \leqslant b' \vee x, b' \vee y'$ hold, hence, by ($\mathrm{U_j}$), $b \nleqslant x \vee y'$. But $b \leqslant x \vee x', y \vee y'$, thus, since $b \nleqslant x, x', y, y'$ and by ($\mathrm{B_j}$), the inequality $b \leqslant x \vee y$ holds.

Suppose next that $x, y \in C(a, b)$. Since $b \leqslant b' \vee x, b' \vee y$, the inequality $b \leqslant x \vee y$ would yield, by ($\mathrm{U_j}$), a contradiction; whence $b \nleqslant x \vee y$.

Suppose, finally, that $x, y \notin C(a, b)$. Thus, by Claim, $y' \in C(a, b)$, whence, by the above, $b \leqslant x \vee y', y \vee y'$, whence, by ($\mathrm{U_j}$), $b \nleqslant x \vee y$. □

5.3. *Stirlitz tracks*

Throughout this subsection, we shall fix a lattice L satisfying the identities (S), (U), and (B). By Lemma 4.2, L is dually 2-distributive as well. Furthermore, it follows from Propositions 4.4, 4.6, and 4.8 that L satisfies ($\mathrm{S_j}$), ($\mathrm{U_j}$), and ($\mathrm{B_j}$).

Definition 5.6. For a natural number n, a *Stirlitz track* of length n is a pair $\sigma = \langle\langle a_i \mid 0 \leqslant i \leqslant n\rangle, \langle a'_i \mid 1 \leqslant i \leqslant n\rangle\rangle$, where the elements a_i for $0 \leqslant i \leqslant n$ and a'_i for $1 \leqslant i \leqslant n$ are join-irreducible and the following conditions are satisfied:

(i) the inequality $a_i \leqslant a_{i+1} \vee a'_{i+1}$ holds, for all $i \in \{0, \ldots, n-1\}$, and it is a minimal nontrivial join-cover;
(ii) the inequality $a_i \leqslant a'_i \vee a_{i+1}$ holds, for all $i \in \{1, \ldots, n-1\}$.

We shall call a_0 the *base* of σ. Observe that $a_i D a_{i+1}$, for all $i \in \{0, \ldots, n-1\}$.

Observe that if σ is a Stirlitz track as above, then, by Lemma 5.2, the following inequalities also hold:

$$a_{i+1} \leqslant a_i \vee a_{i+2}, \tag{5.4}$$

$$a_i \leqslant a'_{i+1} \vee a_{i+2}, \tag{5.5}$$

for all $i \in \{0, \ldots, n-2\}$.

The main property that we will need about Stirlitz tracks is the following:

Lemma 5.7. *For a positive integer n, let $\sigma = \langle\langle a_i \mid 0 \leqslant i \leqslant n\rangle, \langle a'_i \mid 1 \leqslant i \leqslant n\rangle\rangle$ be a Stirlitz track of length n. Then the inequalities $a_i \leqslant a_0 \vee a_n$ and $a_i \leqslant a'_1 \vee a_n$ hold, for all $i \in \{0, \ldots, n\}$. Furthermore, $0 \leqslant k < l \leqslant n$ implies that $a_k \nleqslant a_l$; in particular, the elements a_i, for $0 \leqslant i \leqslant n$, are distinct.*

Proof. We argue by induction on n. The result is trivial for $n = 1$, and it follows from (5.4) and (5.5) for $n = 2$. Suppose that the result holds for $n \geqslant 2$, and let $\sigma = \langle\langle a_i \mid 0 \leqslant i \leqslant n+1\rangle, \langle a'_i \mid 1 \leqslant i \leqslant n+1\rangle\rangle$ be a Stirlitz track of length $n+1$. We observe that $\sigma_* = \langle\langle a_i \mid 0 \leqslant i \leqslant n\rangle, \langle a'_i \mid 1 \leqslant i \leqslant n\rangle\rangle$ is a Stirlitz track of length n, whence, by the induction hypothesis, the following inequalities hold:

$$a_{n-1} \leqslant a_0 \vee a_n, \tag{5.6}$$

$$a_{n-1} \leqslant a'_1 \vee a_n. \tag{5.7}$$

We first prove that $a_{n-1} \leqslant a_0 \vee a_{n+1}$. Indeed, suppose that this does not hold. Hence, *a fortiori* $a_{n-1} \nleqslant a_0, a_{n+1}$. Hence, by applying ($\mathrm{B_j}$) to (5.5) (for $i = n-1$) and (5.6) and observing that $a_{n-1} \nleqslant a_n, a'_n$, we obtain that $a_{n-1} \leqslant a_n \vee a_{n+1}$. Therefore, $a_{n-1} \leqslant a_n \vee a_{n+1}, a_n \vee a'_n, a'_n \vee a_{n+1}$, a contradiction by ($\mathrm{U_j}$). Hence, indeed, $a_{n-1} \leqslant a_0 \vee a_{n+1}$. Consequently, by (5.4), $a_n \leqslant a_{n-1} \vee a_{n+1} \leqslant a_0 \vee a_{n+1}$. Hence, for $i \in \{0, \ldots, n\}$, it follows from the induction hypothesis (applied to σ_*) that $a_i \leqslant a_0 \vee a_n \leqslant a_0 \vee a_{n+1}$.

The proof of the inequalities $a_i \leqslant a'_1 \vee a_{n+1}$, for $i \in \{0, \ldots, n\}$, is similar, with a_0 replaced by a'_1 and (5.6) replaced by (5.7).

Finally, let $0 \leqslant k < l \leqslant n$, and suppose that $a_k \leqslant a_l$. By applying the previous result to the Stirlitz track $\langle\langle a_{k+i} \mid 0 \leqslant i \leqslant l-k\rangle, \langle a'_{k+i} \mid 1 \leqslant i \leqslant l-k\rangle\rangle$, we obtain that $a_{l-1} \leqslant a_k \vee a_l = a_l$, a contradiction. Hence $a_k \nleqslant a_l$, in particular, $a_k \neq a_l$. □

Lemma 5.8. *For positive integers $m, n > 0$, let*

$$\sigma = \big\langle \langle a_i \mid 0 \leqslant i \leqslant m\rangle, \langle a'_i \mid 1 \leqslant i \leqslant m\rangle \big\rangle, \qquad \tau = \big\langle \langle b_j \mid 0 \leqslant j \leqslant n\rangle, \langle b'_j \mid 1 \leqslant j \leqslant n\rangle \big\rangle$$

be Stirlitz tracks with the same base $p = a_0 = b_0$ and $p \leqslant a_1 \vee b_1$. Then $a_i, b_j \leqslant a_m \vee b_n$, for all $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$.

Proof. Suppose first that the inequality $p \leqslant a_1 \vee b'_1$ holds. Then $p \leqslant a_1 \vee b'_1, b'_1 \vee b_1, b_1 \vee a_1$, a contradiction by ($\mathrm{U_j}$). Hence $p \nleqslant a_1 \vee b'_1$, thus, by applying ($\mathrm{B_j}$) to the inequalities $p \leqslant a_1 \vee a'_1, b_1 \vee b'_1$, we obtain that $p \leqslant a'_1 \vee b'_1$.

Furthermore, from Lemma 5.7 it follows that $a_i \leqslant p \vee a_m$, for all $i \in \{0, \dots, m\}$, and $b_j \leqslant p \vee b_n$, for all $j \in \{0, \dots, n\}$, thus it suffices to prove that $p \leqslant a_m \vee b_n$. Again, from Lemma 5.7 it follows that $p \leqslant a'_1 \vee a_m, b'_1 \vee b_n$. Suppose that $p \nleqslant a_m \vee b_n$. Then $p \nleqslant a'_1, a_m, b'_1, b_n$, thus, by ($\mathrm{B_j}$), $p \leqslant a'_1 \vee b_n$. Furthermore, we have seen that $p \leqslant b'_1 \vee b_n$ and $p \leqslant a'_1 \vee b'_1$. Hence, by ($\mathrm{U_j}$), p lies below either a'_1 or b'_1 or b_n, a contradiction. □

6. The small poset associated with a lattice in SUB

Everywhere in this section before Theorem 6.7, we shall fix a complete, lower continuous, finitely spatial lattice L in **SUB**. For every element $p \in \mathrm{J}(L)$, we denote by $\{A_p, B_p\}$ the Udav–Bond partition of $[p]^D$ associated with p (see Section 5.1). We let $+$ and $-$ be distinct symbols, and we put $R = R_0 \cup R_- \cup R_+$, where R_0, R_-, and R_+ are the sets defined as follows:

$$R_0 = \{\langle p\rangle \mid p \in \mathrm{J}(L)\},$$
$$R_+ = \{\langle a, b, +\rangle \mid a, b \in \mathrm{J}(L),\ a D b\},$$
$$R_- = \{\langle a, b, -\rangle \mid a, b \in \mathrm{J}(L),\ a D b\}.$$

We define a map $e : R \to \mathrm{J}(L)$ by putting $e(\langle p\rangle) = p$, for all $p \in \mathrm{J}(L)$, while $e(\langle a, b, +\rangle) = e(\langle a, b, -\rangle) = b$, for all $a, b \in \mathrm{J}(L)$ with $a D b$.

Let $\prec$ be the binary relation on R that consists of the following pairs:

$$\langle p, a, -\rangle \prec \langle p\rangle \prec \langle p, b, +\rangle \quad \text{whenever } a \in A_p \text{ and } b \in B_p, \tag{6.1}$$
$$\langle b, c, -\rangle \prec \langle a, b, +\rangle \prec \langle b, d, +\rangle, \quad \text{and} \tag{6.2}$$
$$\langle b, d, -\rangle \prec \langle a, b, -\rangle \prec \langle b, c, +\rangle, \quad \text{whenever } c \in [b]^D \setminus C(a, b) \text{ and } d \in C(a, b). \tag{6.3}$$

Lemma 6.1. *Let $\varepsilon \in \{+, -\}$, let $n < \omega$, and let $a_0, \dots, a_n, b_0, \dots, b_n \in \mathrm{J}(L)$ such that $a_i D b_i$, for all $i \in \{0, \dots, n\}$ and $\langle a_0, b_0, \varepsilon\rangle \prec \cdots \prec \langle a_n, b_n, \varepsilon\rangle$. Then exactly one of the following cases occurs*:

(i) *$\varepsilon = +$ and, putting $a_{n+1} = b_n$, the equality $a_{i+1} = b_i$ holds, for all $i \in \{0, \dots, n\}$, while there are join-irreducible elements $a'_1, \dots, a'_{n+1}$ of L such that $\langle\langle a_i \mid 0 \leqslant i \leqslant n+1\rangle, \langle a'_i \mid 1 \leqslant i \leqslant n+1\rangle\rangle$ is a Stirlitz track.*

(ii) *$\varepsilon = -$ and, putting $a_{-1} = b_0$, the equality $a_{i-1} = b_i$ holds, for all $i \in \{0, \dots, n\}$, while there are join-irreducible elements $a'_{-1}, \dots, a'_{n-1}$ of L such that $\langle\langle a_{n-i} \mid 0 \leqslant i \leqslant n+1\rangle, \langle a'_{n-i} \mid 1 \leqslant i \leqslant n+1\rangle\rangle$ is a Stirlitz track.*

Proof. Suppose that $\varepsilon = +$ (the proof for $\varepsilon = -$ is similar). We argue by induction on n. If $n = 0$, then, from the assumption that $a_0 D b_0$ and by using Corollary 3.4, we obtain a conjugate a'_1 of b_0 with respect to a_0, and $\langle\langle a_0, a_1\rangle, \langle a'_1\rangle\rangle$ is obviously a Stirlitz track.

Suppose that $n > 0$. From the assumption that $\langle a_{n-1}, b_{n-1}, +\rangle \prec \langle a_n, b_n, +\rangle$ and the definition of $\prec$, we obtain that $a_n = b_{n-1}$. Furthermore, from the induction hypothesis it follows that there exists a Stirlitz track of the form

$$\big\langle\langle a_i \mid 0 \leqslant i \leqslant n\rangle, \big\langle a'_i \mid 1 \leqslant i \leqslant n\big\rangle\big\rangle.$$

Put $a_{n+1} = b_n$, and let a'_{n+1} be a conjugate of a_{n+1} with respect to a_n. Using again the assumption that $\langle a_{n-1}, b_{n-1}, +\rangle \prec \langle a_n, b_n, +\rangle$, we obtain the inequality $a_n \leqslant a'_n \vee a_{n+1}$. Therefore, $\langle\langle a_i \mid 0 \leqslant i \leqslant n+1\rangle, \langle a'_i \mid 1 \leqslant i \leqslant n+1\rangle\rangle$ is a Stirlitz track. $\square$

Let $\trianglelefteq$ denote the reflexive and transitive closure of $\prec$.

Lemma 6.2. *The relation $\trianglelefteq$ is a partial ordering on R, and $\prec$ is the predecessor relation of $\trianglelefteq$.*

Proof. We need to prove that for any $n > 0$, if $r_0 \prec \cdots \prec r_n$ in R, then $r_0 \neq r_n$. We have three cases to consider.

Case 1. $r_0 \in R_+$. In this case, $r_i = \langle a_i, b_i, +\rangle \in R_+$, for all $i \in \{1, \dots, n\}$. By Lemma 6.1, if we put $a_{n+1} = b_n$, then $a_{i+1} = b_i$, for all $i \in \{0, \dots, n\}$, and there are join-irreducible elements $a'_1, \dots, a'_{n+1}$ of L such that

$$\big\langle\langle a_i \mid 0 \leqslant i \leqslant n+1\rangle, \big\langle a'_i \mid 1 \leqslant i \leqslant n+1\big\rangle\big\rangle$$

is a Stirlitz track. In particular, by Lemma 5.7, $a_0 \neq a_n$, whence $r_0 \neq r_n$.

Case 2. $r_0 \in R_0$. Then $r_i \in R_+$, for all $i \in \{1, \dots, n\}$, thus $r_0 \neq r_n$.

Case 3. $r_0 \in R_-$. If $r_n \notin R_-$, then $r_0 \neq r_n$. Suppose that $r_n \in R_-$. Then $r_i = \langle a_i, b_i, -\rangle$ belongs to R_-, for all $i \in \{0, \dots, n\}$. By Lemma 6.1, if we put $a_{-1} = b_0$, then $a_{i-1} = b_i$, for all $i \in \{0, \dots, n\}$, and there are join-irreducible elements $a'_{-1}, \dots, a'_{n-1}$ of L such that $\langle\langle a_{n-i} \mid 0 \leqslant i \leqslant n+1\rangle, \langle a'_{n-i} \mid 1 \leqslant i \leqslant n+1\rangle\rangle$ is a Stirlitz track. In particular, by Lemma 5.7, $a_0 \neq a_n$, whence $r_0 \neq r_n$. $\square$

Definition 6.3.

(i) Two finite sequences $\mathbf{r} = \langle r_0, \dots, r_{n-1}\rangle$ and $\mathbf{s} = \langle s_0, \dots, s_{n-1}\rangle$ of same length of R are *isotype*, if either $e(r_i) = e(s_i)$, for all $i \in \{0, \dots, n-1\}$, or $e(r_i) = e(s_{n-1-i})$, for all $i \in \{0, \dots, n-1\}$.

(ii) An oriented path (see Section 2) $\mathbf{r} = \langle r_0, \dots, r_{n-1}\rangle$ of elements of R is
- *positive* (respectively, *negative*), if there are elements a_i, b_i (for $0 \leqslant i < n$) of $\mathrm{J}(L)$ such that $r_i = \langle a_i, b_i, +\rangle$ (respectively, $r_i = \langle a_i, b_i, -\rangle$), for all $i \in \{0, \dots, n-1\}$,

- *reduced*, if either it is positive or is has the form

$$\big\langle u_0, \dots, u_{k-1}, \langle p\rangle, v_0, \dots, v_{l-1}\big\rangle,$$

where $p \in \mathrm{J}(L)$, $\langle u_0, \dots, u_{k-1}\rangle$ is negative, and $\langle v_0, \dots, v_{l-1}\rangle$ is positive.

Lemma 6.4. *Every oriented path of R is isotype to a reduced oriented path.*

Proof. Let $\mathbf{r}$ be an oriented path of R, we prove that $\mathbf{r}$ is isotype to a reduced oriented path. If $\mathbf{r}$ is either positive or reduced there is nothing to do. Suppose that $\mathbf{r}$ is neither positive nor reduced. Then $\mathbf{r}$ has the form

$$\big\langle\langle a_{k-1}, a_k, -\rangle, \dots, \langle a_0, a_1, -\rangle, \langle b_0, b_1, +\rangle, \dots, \langle b_{l-1}, b_l, +\rangle\big\rangle$$

for some integers $k > 0$ and $l \geqslant 0$. If $l = 0$, then $\mathbf{r}$ is isotype to the positive path

$$\big\langle\langle a_0, a_1, +\rangle, \dots, \langle a_{k-1}, a_k, +\rangle\big\rangle.$$

Suppose now that $l > 0$. Since $\langle a_0, a_1, -\rangle \prec \langle b_0, b_1, +\rangle$, two cases can occur.

Case 1. $a_0 = b_1$ and $a_1 \notin C(b_0, b_1)$ (see (6.2)). Observe that $\langle a_0, a_1, -\rangle \prec \langle a_0\rangle$ if $a_1 \in A_{a_0}$ while $\langle a_0\rangle \prec \langle a_0, a_1, +\rangle$ if $a_1 \in B_{a_0}$ (see (6.1)). In the first case, it follows from Lemma 5.5 (applied to $C(a_0, a_1)$) that the sequence

$$\big\langle\langle a_{k-1}, a_k, -\rangle, \dots, \langle a_0, a_1, -\rangle, \langle a_0\rangle, \langle b_1, b_2, +\rangle, \dots, \langle b_{l-1}, b_l, +\rangle\big\rangle$$

is an oriented path, isotype to $\mathbf{r}$. Similarly, in the second case, the oriented path

$$\big\langle\langle b_{l-1}, b_l, -\rangle, \dots, \langle b_1, b_2, -\rangle, \langle a_0\rangle, \langle a_0, a_1, +\rangle, \dots, \langle a_{k-1}, a_k, +\rangle\big\rangle$$

is isotype to $\mathbf{r}$.

Case 2. $a_1 = b_0$ and $b_1 \notin C(a_0, a_1)$ (see (6.3)). Observe that $\langle b_0\rangle \prec \langle b_0, b_1, +\rangle$ if $b_1 \in B_{b_0}$ while $\langle b_0, b_1, -\rangle \prec \langle b_0\rangle$ if $b_1 \in A_{b_0}$ (see (6.1)). In the first case, the oriented path

$$\big\langle\langle a_{k-1}, a_k, -\rangle, \dots, \langle a_1, a_2, -\rangle, \langle b_0\rangle, \langle b_0, b_1, +\rangle, \dots, \langle b_{l-1}, b_l, +\rangle\big\rangle$$

is isotype to $\mathbf{r}$. Similarly, in the second case, the oriented path

$$\big\langle\langle b_{l-1}, b_l, -\rangle, \dots, \langle b_0, b_1, -\rangle, \langle b_0\rangle, \langle a_1, a_2, +\rangle, \dots, \langle a_{k-1}, a_k, +\rangle\big\rangle$$

is isotype to $\mathbf{r}$. This concludes the proof. □

We define a map φ from L into the powerset of R as follows:

$$\varphi(x) = \big\{r \in R \mid e(r) \leqslant x\big\}, \quad \text{for all } x \in L. \tag{6.4}$$

Lemma 6.5. *The set* $\varphi(x)$ *belongs to* $\mathbf{Co}(R, \trianglelefteq)$, *for all* $x \in L$.

Proof. It is sufficient to prove that if $r_0 \prec \cdots \prec r_n$ in R such that $e(r_0), e(r_n) \leqslant x$, the relation $e(r_k) \leqslant x$ holds whenever $0 < k < n$. By Lemma 6.4, it is sufficient to consider the case where the oriented path $\mathbf{r} = \langle r_0, \dots, r_n \rangle$ is reduced. If it is positive, then, by Lemma 6.1, there exists a Stirlitz track of the form

$$\big\langle \langle a_i \mid 0 \leqslant i \leqslant n+1 \rangle, \langle a'_i \mid 1 \leqslant i \leqslant n+1 \rangle \big\rangle$$

for join-irreducible elements a_i, a'_i of L with $r_i = \langle a_i, a_{i+1}, + \rangle$, for all $i \in \{0, \dots, n\}$. But then, by Lemma 5.7 applied to the Stirlitz track

$$\big\langle \langle a_{i+1} \mid 0 \leqslant i \leqslant n \rangle, \langle a'_{i+1} \mid 1 \leqslant i \leqslant n \rangle \big\rangle,$$

$e(r_k) = a_{k+1} \leqslant a_1 \vee a_{n+1} \leqslant x$. Suppose from now on that $\mathbf{r}$ is not positive. Then three cases can occur.

Case 1. $\mathbf{r} = \langle \langle a_0 \rangle, \langle a_0, a_1, + \rangle, \dots, \langle a_{n-1}, a_n, + \rangle \rangle$ for some $a_0, \dots, a_n \in \mathrm{J}(L)$. It follows from Lemma 6.1 that there exists a Stirlitz track of the form

$$\big\langle \langle a_i \mid 0 \leqslant i \leqslant n \rangle, \langle a'_i \mid 1 \leqslant i \leqslant n \rangle \big\rangle,$$

hence, by Lemma 5.7, $e(r_k) = a_k \leqslant a_0 \vee a_n \leqslant x$.

Case 2. $\mathbf{r} = \langle \langle a_{n-1}, a_n, - \rangle, \dots, \langle a_0, a_1, - \rangle, \langle a_0 \rangle \rangle$ for some $a_0, \dots, a_n \in \mathrm{J}(L)$. The argument is similar to the one for Case 1.

Case 3. $\mathbf{r} = \langle \langle a_{n'-1}, a_{n'}, - \rangle, \dots, \langle a_0, a_1, - \rangle, \langle a_0 \rangle, \langle b_0, b_1, + \rangle, \dots, \langle b_{n''-1}, b_{n''}, + \rangle \rangle$ for some positive integers n' and n'' and join-irreducible $a_0 = b_0$, $a_1, \dots, a_{n'}$, $b_1, \dots, b_{n''}$. From $\langle a_0, a_1, - \rangle \prec \langle a_0 \rangle \prec \langle b_0, b_1, + \rangle$ it follows that $a_0 = b_0 \leqslant a_1 \vee b_1$. From Lemma 6.1 it follows that there are Stirlitz tracks of the form

$$\sigma = \big\langle \langle a_i \mid 0 \leqslant i \leqslant n' \rangle, \langle a'_i \mid 1 \leqslant i \leqslant n' \rangle \big\rangle,$$
$$\tau = \big\langle \langle b_j \mid 0 \leqslant j \leqslant n'' \rangle, \langle b'_j \mid 1 \leqslant j \leqslant n'' \rangle \big\rangle,$$

with the same base $a_0 = b_0 \leqslant a_1 \vee b_1$. Since $e(r_k)$ has either the form a_i, where $0 \leqslant i < n'$, or b_j, where $0 \leqslant j < n''$, it follows from Lemma 5.8 that $e(r_k) \leqslant a_{n'} \vee b_{n''} \leqslant x$. This concludes the proof. □

Lemma 6.6. *The map* φ *is a* $\langle 0, 1 \rangle$*-lattice embedding from* L *into* $\mathbf{Co}(R)$.

Proof. It is obvious that φ is a $\langle \wedge, 0, 1 \rangle$-homomorphism. Let $x, y \in L$ such that $x \nleqslant y$. Since L is finitely spatial, there exists $p \in \mathrm{J}(L)$ such that $p \leqslant x$ and $p \nleqslant y$. Hence, $\langle p \rangle \in \varphi(x) \setminus \varphi(y)$, so $\varphi(x) \nsubseteq \varphi(y)$. Therefore, φ is a $\langle \wedge, 0, 1 \rangle$-embedding.

Now let $x, y \in L$ and let $r \in \varphi(x \vee y)$, we prove that $r \in \varphi(x) \vee \varphi(y)$. The conclusion is trivial if $r \in \varphi(x) \cup \varphi(y)$, so suppose that $r \notin \varphi(x) \cup \varphi(y)$. We need to consider two cases:

Case 1. $r = \langle p \rangle$, for some $p \in \mathrm{J}(L)$. So $p \leqslant x \vee y$ while $p \nleqslant x, y$. By Lemma 3.2, there are minimal $u \leqslant x$ and $v \leqslant y$ such that $p \leqslant u \vee v$, hence u and v are join-irreducible and they do not belong to the same side of the Udav–Bond partition of $[p]^D$ associated with p (see Proposition 5.1). Hence, by the definition of $\prec$, either $\langle p, u, - \rangle \prec \langle p \rangle \prec \langle p, v, + \rangle$ or $\langle p, v, - \rangle \prec \langle p \rangle \prec \langle p, u, + \rangle$. Since $\langle p, u, \varepsilon \rangle \in \varphi(x)$ and $\langle p, v, \varepsilon \rangle \in \varphi(y)$, for all $\varepsilon \in \{+, -\}$, it follows from this that $\langle p \rangle \in \varphi(x) \vee \varphi(y)$.

Case 2. $r = \langle a, b, + \rangle$ for some $a, b \in \mathrm{J}(L)$ such that $a D b$. So $b \leqslant x \vee y$ while $b \nleqslant x, y$. By Lemma 3.2, there are minimal $u \leqslant x$ and $v \leqslant y$ such that $b \leqslant u \vee v$, hence u and v are join-irreducible and they do not belong to the same side of the Udav–Bond partition of $[b]^D$ associated with b (see Proposition 5.1). Hence, it follows from Lemma 5.5 that either $u \notin C(a, b)$ and $v \in C(a, b)$ or $u \in C(a, b)$ and $v \notin C(a, b)$. In the first case,

$$\langle b, u, - \rangle \prec \langle a, b, + \rangle \prec \langle b, v, + \rangle,$$

while in the second case,

$$\langle b, v, - \rangle \prec \langle a, b, + \rangle \prec \langle b, u, + \rangle.$$

Since $\langle b, u, \varepsilon \rangle \in \varphi(x)$ and $\langle b, v, \varepsilon \rangle \in \varphi(y)$, for all $\varepsilon \in \{+, -\}$, it follows from this that $r \in \varphi(x) \vee \varphi(y)$.

Case 3. $r = \langle a, b, - \rangle$ for some $a, b \in \mathrm{J}(L)$ such that $a D b$. The proof is similar to the proof of Case 2. □

We can now formulate the main theorem of this paper.

Theorem 6.7. *Let L be a lattice. Then the following are equivalent*:

(i) *L embeds into a lattice of the form* $\mathbf{Co}(P)$*, for some poset P*;
(ii) *L satisfies the identities* (S), (U)*, and* (B) (*i.e., it belongs to the class* **SUB**);
(iii) *L has a lattice embedding into a lattice of the form* $\mathbf{Co}(R)$*, for some poset R, that preserves the existing bounds. Furthermore, if L is finite, then R is finite, with*

$$|R| \leqslant 2\left|\mathrm{J}(L)\right|^2 - 5\left|\mathrm{J}(L)\right| + 4.$$

Proof. (i) ⇒ (ii) follows immediately from Lemmas 4.1, 4.5, and 4.7.

(ii) ⇒ (iii). Denote by $\operatorname{Fil} L$ the lattice of all *dual ideals* (= filters) of L, ordered by reverse inclusion; if L has no unit element, then we allow the empty set in $\operatorname{Fil} L$, otherwise we require filters to be nonempty. This way, $\operatorname{Fil} L$ is complete and the canonical lattice embedding $x \mapsto {\uparrow} x$ from L into $\operatorname{Fil} L$ preserves the existing bounds. It is well known that $\operatorname{Fil} L$ is a dually algebraic lattice that extends L and that satisfies the same identities as L (see, for example, G. Grätzer [11]), in particular, it belongs to **SUB**. Furthermore, $\operatorname{Fil} L$ is dually algebraic, thus lower continuous and spatial, thus it is *a fortiori* finitely spatial.

We consider the poset $\langle R, \trianglelefteq \rangle$ constructed above from $\operatorname{Fil} L$. By Lemmas 6.5 and 6.6, the canonical map φ defines a $\langle 0, 1\rangle$-embedding from $\operatorname{Fil} L$ into $\mathbf{Co}(R)$.

(iii) $\Rightarrow$ (i) is trivial.

In case L is finite, put $n = |\mathrm{J}(L)|$, we verify that $|R| \leqslant 2n^2 - 5n + 4$ for the poset $\langle R, \trianglelefteq \rangle$ constructed above, in the case where $n \geqslant 2$ (for $n \leqslant 1$ then one can take for P a singleton). Indeed, it follows from the join-semidistributivity of L (that itself follows from Lemma 4.3) that L has at least two D-maximal (= join-prime) elements, hence the number of pairs $\langle a, b\rangle$ of elements of $\mathrm{J}(L)$ such that $a D b$ is at most $(n-1)(n-2)$, whence

$$|R| \leqslant 2(n-1)(n-2) + n = 2n^2 - 5n + 4. \quad \square$$

Remark 6.8. The upper bound $2|\mathrm{J}(L)|^2 - 5|\mathrm{J}(L)| + 4$ of Theorem 6.7(iii), obtained for the particular poset R constructed above, is reached for L defined as the lattice of all order-convex subsets of a finite chain.

Corollary 6.9. *The class of all lattices that can be embedded into some* $\mathbf{Co}(P)$ *coincides with* **SUB**; *it is a finitely based variety. In particular, it is closed under homomorphic images.*

Of course, we proved more, for example, the class of all lattices that can be embedded into some *finite* $\mathbf{Co}(P)$ forms a *pseudovariety* (see [10]), thus it is closed under homomorphic images.

7. The tree-like poset associated with a lattice in SUB

Everywhere in this section before Theorem 7.7, we shall fix a complete, lower continuous, finitely spatial lattice L in **SUB**. The goal of the present section is to define a tree-like poset Γ and a lattice embedding from L into $\mathbf{Co}(\Gamma)$ that preserves the existing bounds, see Theorem 7.7.

The idea to use D-increasing finite sequences of join-irreducible elements is introduced in K.V. Adaricheva [1], where it is proved that every finite lattice without D-cycle can be embedded into the lattice of subsemilattices of some finite meet-semilattice; see also [2].

We denote by Γ the set of all finite, nonempty sequences $\alpha = \langle \alpha(0), \dots, \alpha(n)\rangle$ of elements of $\mathrm{J}(L)$ such that $\alpha(i) D \alpha(i+1)$, for all $i < n$. We put $|\alpha| = n$ (the *length* of α), and we extend this definition by putting $|\emptyset| = -1$. We further put $\bar{\alpha} = \langle \alpha(0), \dots, \alpha(n-1)\rangle$ (the *truncation* of α) and $e(\alpha) = \alpha(n)$ (the *extremity* of α). If $\alpha = \bar{\beta}$, we say that β is a *one-step extension* of α. Furthermore, for all $n \geqslant 0$, we put

$$\Gamma_n = \{\alpha \in \Gamma \mid |\alpha| \leqslant n\} \quad \text{and} \quad E_n = \Gamma_n \setminus \Gamma_{n-1} \quad \text{for } n > 0.$$

For $\alpha \in \Gamma \setminus \Gamma_0$, we say that a *conjugate* of α is an element β of Γ such that $\bar{\alpha} = \bar{\beta}$ and $e(\alpha)$ and $e(\beta)$ are conjugate with respect to $e(\bar{\alpha})$. It follows from Corollary 3.4 that *every element of* $\Gamma \setminus \Gamma_0$ *has a conjugate*. Furthermore, for $\alpha, \beta \in \Gamma$, we write $\alpha \sim \beta$, if either $\alpha = \bar{\beta}$ or $\beta = \bar{\alpha}$.

For all $n > 0$, we define inductively a binary relation $\prec_n$ on Γ_n, together with subsets A_α and B_α of $[e(\alpha)]^D$ for $\alpha \in \Gamma_{n-1}$.

The induction hypothesis to be satisfied consists of the following two assertions:

(S1) $\prec_n$ is acyclic.
(S2) For all $\alpha, \beta \in \Gamma_n$, $\alpha \sim \beta$ iff either $\alpha \prec_n \beta$ or $\beta \prec_n \alpha$.

For $n = 0$, let $\prec_n$ be empty.

The case $n = 1$ is the only place where we have some freedom in the choice of $\prec_n$. We suppose that we have already used this freedom for the construction of the poset $\langle R, \trianglelefteq\rangle$ of Section 6, that is, for each $p \in \mathrm{J}(L)$, let A_p, B_p such that $\{A_p, B_p\}$ is the Udav–Bond partition of $[p]^D$ associated with p (see Section 5.1), and we let R be the poset associated with this choice that we constructed in Section 6. Then we put $A_{\langle p\rangle} = A_p$ and $B_{\langle p\rangle} = B_p$, and we define

$$\prec_1 = \bigl\{\bigl\langle\langle p, a\rangle, \langle p\rangle\bigr\rangle \mid p \in \mathrm{J}(L),\ a \in A_{\langle p\rangle}\bigr\} \cup \bigl\{\bigl\langle\langle p\rangle, \langle p, b\rangle\bigr\rangle \mid p \in \mathrm{J}(L),\ b \in B_{\langle p\rangle}\bigr\}.$$

It is obvious that $\prec_1$ satisfies both (S1) and (S2).

Now suppose having defined $\prec_n$, for $n \geqslant 1$, that satisfies both (S1) and (S2). For all $\alpha \in E_n$, we define subsets A_α and B_α of $[e(\alpha)]^D$ as follows:

Case 1. $\bar{\alpha} \prec_n \alpha$. Then we put $A_\alpha = [e(\alpha)]^D \setminus C(e(\bar{\alpha}), e(\alpha))$ and $B_\alpha = C(e(\bar{\alpha}), e(\alpha))$.

Case 2. $\alpha \prec_n \bar{\alpha}$. Then we put $A_\alpha = C(e(\bar{\alpha}), e(\alpha))$ and $B_\alpha = [e(\alpha)]^D \setminus C(e(\bar{\alpha}), e(\alpha))$.

Then we define $\prec_{n+1}$ as

$$\begin{aligned} \prec_{n+1} = \prec_n &\cup \bigl\{\bigl\langle \alpha^\frown\langle x\rangle, \alpha\bigr\rangle \mid \alpha \in E_n \text{ and } x \in A_\alpha\bigr\} \\ &\cup \bigl\{\bigl\langle \alpha, \alpha^\frown\langle y\rangle\bigr\rangle \mid \alpha \in E_n \text{ and } y \in B_\alpha\bigr\}, \end{aligned} \tag{7.1}$$

where $\langle\alpha, \beta\rangle \mapsto \alpha^\frown\beta$ denotes *concatenation* of finite sequences.

Lemma 7.1. *The relation* $\prec_{n+1}$ *satisfies both* (S1) *and* (S2).

Proof. It is obvious that $\prec_{n+1}$ satisfies (S2). Now let us prove (S1), and suppose that $\prec_{n+1}$ has a cycle, say, $\alpha_0 \prec_{n+1} \alpha_1 \prec_{n+1} \cdots \prec_{n+1} \alpha_k = \alpha_0$, where $k \geqslant 2$. We pick k *minimal* with this property. As $A_\alpha \cap B_\alpha = \emptyset$, for all α, we cannot have $k = 2$ as well, so $k \geqslant 3$.

By the induction hypothesis, one of the elements of the cycle belongs to E_{n+1}, without loss of generality we may assume that it is the case for α_0. Hence, by (7.1), $\alpha_1 = \overline{\alpha_0}$ belongs to Γ_n. Let l be the smallest element of $\{1, \ldots, k-1\}$ such that $\alpha_{l+1} \notin \Gamma_n$ (it exists since $\alpha_k = \alpha_0 \notin \Gamma_n$). Suppose that $l < k-1$. By (S2) for $\prec_{n+1}$, $\alpha_{l+2} = \overline{\alpha_{l+1}} = \alpha_l$, a contradiction by the minimality of k. Hence $l = k-1$, which means that $\alpha_1, \ldots, \alpha_{k-1} \in \Gamma_n$. Hence, since $k-1 \geqslant 2$, we obtain that $\alpha_1 \prec_n \cdots \prec_n \alpha_{k-1} = \alpha_1$ is a $\prec_n$-cycle, a contradiction. $\square$

Lemma 7.1 completes the definition of $\prec_n$, for all $n > 0$. We define $\prec$ as the union over all $n < \omega$ of $\prec_n$. Hence $\prec$ is an acyclic binary relation on Γ such that $\alpha \sim \beta$ iff either $\alpha \prec \beta$ or $\beta \prec \alpha$, for all $\alpha, \beta \in \Gamma$. Since $\prec$ is acyclic, the reflexive and transitive closure $\trianglelefteq$ of $\prec$ is a partial ordering on Γ, for which $\prec$ is exactly the *predecessor* relation. For the sake of clarity, we rewrite below the inductive definition of $\prec$ and the sets A_α and B_α for $\alpha \in \Gamma$.

(a) For $|\alpha| = 0$, A_α and B_α are chosen such that $\{A_\alpha, B_\alpha\}$ is the Udav–Bond partition of $[e(\alpha)]^D$ associated with $e(\alpha)$.
(b) Suppose that $|\alpha| \geqslant 1$. Then we define A_α and B_α by

$$\langle A_\alpha, B_\alpha\rangle = \begin{cases} ([e(\alpha)]^D \setminus C(e(\bar{\alpha}), e(\alpha)), C(e(\bar{\alpha}), e(\alpha))) & \text{if } \bar{\alpha} \prec \alpha, \\ (C(e(\bar{\alpha}), e(\alpha)), [e(\alpha)]^D \setminus C(e(\bar{\alpha}), e(\alpha))) & \text{if } \alpha \prec \bar{\alpha}. \end{cases}$$

(c) $\alpha \prec \beta$ implies that $\alpha \sim \beta$.
(d) $\alpha^\frown\langle x\rangle \prec \alpha$ iff $x \in A_\alpha$ and $\alpha \prec \alpha^\frown\langle x\rangle$ iff $x \in B_\alpha$, for all $\alpha \in \Gamma$ and all $x \in [e(\alpha)]^D$.

By Lemma 5.5, the set $\{A_\alpha, B_\alpha\}$ is the Udav–Bond partition of $[e(\alpha)]^D$ associated with α, for all $\alpha \in \Gamma$. Therefore, by Proposition 5.1 and the definition of $\prec$, we obtain immediately the following consequence.

Corollary 7.2. *For all $\alpha \in \Gamma$ and all $x, y \in [e(\alpha)]^D$, $e(\alpha) \leqslant x \vee y$ iff either $\alpha^\frown\langle x\rangle \prec \alpha \prec \alpha^\frown\langle y\rangle$ or $\alpha^\frown\langle y\rangle \prec \alpha \prec \alpha^\frown\langle x\rangle$.*

For $\alpha, \beta \in \Gamma$, we denote by $\alpha * \beta$ the largest common initial segment of α and β. Observe that $\alpha * \beta$ belongs to $\Gamma \cup \{\emptyset\}$ and that $\alpha * \beta = \beta * \alpha$. Put $m = |\alpha| - |\alpha * \beta|$ and $n = |\beta| - |\alpha * \beta|$. We let $P(\alpha, \beta)$ be the finite sequence $\langle \gamma_0, \gamma_1, \dots, \gamma_{m+n}\rangle$, where the γ_i, for $0 \leqslant i \leqslant m + n$, are defined by $\gamma_0 = \alpha$, $\gamma_{i+1} = \overline{\gamma_i}$, for all $i < m$, $\gamma_{m+n} = \beta$, and $\gamma_{m+n-j-1} = \overline{\gamma_{m+n-j}}$, for all $j < n$. Hence the γ_i-s first decrease from $\gamma_0 = \alpha$ to $\gamma_m = \alpha * \beta$ by successive truncations, then they increase again from γ_m to $\gamma_{m+n} = \beta$ by successive one-step extensions.

For $\alpha, \beta \in \Gamma$, we observe that a path (see Section 2) from α to β is a finite sequence $\mathbf{c} = \langle \gamma_0, \gamma_1, \dots, \gamma_k\rangle$ of distinct elements of Γ such that $\gamma_0 = \alpha$, $\gamma_k = \beta$, and $\gamma_i \sim \gamma_{i+1}$, for all $i < k$.

Proposition 7.3. *For all $\alpha, \beta \in \Gamma$, there exists at most one path from α to β, and then this path is $P(\alpha, \beta)$. Furthermore, such a path exists iff $\alpha(0) = \beta(0)$.*

Hence, by using the terminology of Section 2: the poset $\langle \Gamma, \trianglelefteq\rangle$ is tree-like.

Proof. Put again $m = |\alpha| - |\alpha * \beta|$ and $n = |\beta| - |\alpha * \beta|$, and $P(\alpha, \beta) = \langle \gamma_0, \dots, \gamma_{m+n}\rangle$. Let $\mathbf{d} = \langle \delta_0, \dots, \delta_k\rangle$ (for $k < \omega$) be a path from α to β. We begin with the following essential observation.

Claim. *The path* $\mathbf{d}$ *consists of a sequence of truncations followed by a sequence of one-step extensions.*

Proof of Claim. Suppose that there exists an index $i \in \{1, \ldots, k-1\}$ such that δ_i extends both δ_{i-1} and δ_{i+1}. Then $\delta_{i-1} = \overline{\delta_i} = \delta_{i+1}$, which contradicts the fact that all entries of $\mathbf{d}$ are distinct.

Hence, either $\mathbf{d}$ consists of a sequence of truncations, or there exists a least index $l \in \{0, \ldots, k-1\}$ such that δ_{l+1} is an extension of δ_l. If δ_{i+1} is not an extension of δ_i for some $i \in \{l, \ldots, k-1\}$, then, taking the least such i, we obtain that δ_i extends both δ_{i-1} and δ_{i+1}, a contradiction by the first paragraph of the present proof. Hence δ_{i+1} is a one-step extension of δ_i, for all $i \in \{l, \ldots, k-1\}$. □

Let l denote the least element of $\{0, \ldots, k\}$ such that $l < k$ implies that δ_{l+1} extends δ_l. In particular, δ_l is a common initial segment of both α and β, thus of $\alpha * \beta$. Furthermore,

$$|\alpha| - l = |\delta_0| - l = |\delta_l| \leqslant |\alpha * \beta| = |\alpha| - m,$$

thus $l \geqslant m$. Similarly,

$$|\beta| - (k - l) = |\delta_l| \leqslant |\alpha * \beta| = |\beta| - n,$$

thus $k - l \geqslant n$. In addition, both $\alpha * \beta$ and δ_m are initial segments of α of the same length $|\alpha| - m$, thus $\alpha * \beta = \delta_m$. Similarly, both $\alpha * \beta$ and δ_{k-n} are initial segments of β of the same length $|\beta| - n$, whence $\alpha * \beta = \delta_{k-n}$. Therefore, $\delta_m = \delta_{k-n}$, whence, since all entries of $\mathbf{d}$ are distinct, $m = k - n$, so $k = m + n$, whence $l = m$ since $m \leqslant l \leqslant k - n$. It follows then from the claim that $\mathbf{d} = P(\alpha, \beta)$.

Furthermore, from $\alpha \sim \beta$ it follows that $\alpha(0) = \beta(0)$, thus the same conclusion follows from the assumption that there exists a path from α to β. Conversely, if $\alpha(0) = \beta(0)$, then $\alpha * \beta$ is nonempty, thus so are all entries of $P(\alpha, \beta)$. Hence $P(\alpha, \beta)$ is a path from α to β. □

Now we define a map $\pi : \Gamma \to R$ by the following rule:

$$\pi(\alpha) = \begin{cases} \alpha & \text{if } |\alpha| = 0, \\ \langle e(\bar{\alpha}), e(\alpha), + \rangle & \text{if } \bar{\alpha} \prec \alpha, \\ \langle e(\bar{\alpha}), e(\alpha), - \rangle & \text{if } \alpha \prec \bar{\alpha}, \end{cases} \qquad \text{for all } \alpha \in \Gamma.$$

Lemma 7.4. *$\alpha \prec \beta$ in Γ implies that $\pi(\alpha) \prec \pi(\beta)$ in R, for all $\alpha, \beta \in \Gamma$. In particular, π is order-preserving.*

Proof. We argue by induction on the least integer n such that $\alpha, \beta \in \Gamma_n$. We need to consider first the case where $p, a, b \in \mathrm{J}(L)$, $a \in A_p$, $b \in B_p$ (so that $\langle p, a \rangle \prec \langle p \rangle \prec \langle p, b \rangle$ in Γ), and prove that $\pi(\langle p, a \rangle) \prec \pi(\langle p \rangle) \prec \pi(\langle p, b \rangle)$ in R. But by the definition of π, the following equalities hold,

$$\pi\big(\langle p, a \rangle\big) = \langle p, a, - \rangle, \quad \pi\big(\langle p \rangle\big) = \langle p \rangle, \quad \text{and} \quad \pi\big(\langle p, b \rangle\big) = \langle p, b, + \rangle,$$

while, by the definition of $\prec$ on R,

$$\langle p, a, - \rangle \prec \langle p \rangle \prec \langle p, b, + \rangle,$$

which solves the case where $n = 1$.

The remaining case to consider is where $\alpha^\frown\langle x\rangle \prec \alpha \prec \alpha^\frown\langle y\rangle$ in Γ, for $|\alpha| > 0$. Thus $x \in A_\alpha$ and $y \in B_\alpha$, whence

$$\pi\bigl(\alpha^\frown\langle x\rangle\bigr) = \bigl\langle e(\alpha), x, -\bigr\rangle,$$
$$\pi\bigl(\alpha^\frown\langle y\rangle\bigr) = \bigl\langle e(\alpha), y, +\bigr\rangle.$$

Suppose first that $\bar{\alpha} \prec \alpha$. Then

$$A_\alpha = \bigl[e(\alpha)\bigr]^D \setminus C\bigl(e(\bar{\alpha}), e(\alpha)\bigr) \quad \text{while } B_\alpha = C\bigl(e(\bar{\alpha}), e(\alpha)\bigr).$$

Furthermore, $\pi(\alpha) = \langle e(\bar{\alpha}), e(\alpha), +\rangle$, while, by the definition of $\prec$ on R,

$$\bigl\langle e(\alpha), x, -\bigr\rangle \prec \bigl\langle e(\bar{\alpha}), e(\alpha), +\bigr\rangle \prec \bigl\langle e(\alpha), y, +\bigr\rangle,$$

in other words,

$$\pi\bigl(\alpha^\frown\langle x\rangle\bigr) \prec \pi(\alpha) \prec \pi\bigl(\alpha^\frown\langle y\rangle\bigr).$$

Suppose now that $\alpha \prec \bar{\alpha}$. Then

$$A_\alpha = C\bigl(e(\bar{\alpha}), e(\alpha)\bigr) \quad \text{while } B_\alpha = \bigl[e(\alpha)\bigr]^D \setminus C\bigl(e(\bar{\alpha}), e(\alpha)\bigr).$$

Furthermore, $\pi(\alpha) = \langle e(\bar{\alpha}), e(\alpha), -\rangle$, while, by the definition of $\prec$ on R,

$$\bigl\langle e(\alpha), x, -\bigr\rangle \prec \bigl\langle e(\bar{\alpha}), e(\alpha), -\bigr\rangle \prec \bigl\langle e(\alpha), y, +\bigr\rangle,$$

in other words,

$$\pi\bigl(\alpha^\frown\langle x\rangle\bigr) \prec \pi(\alpha) \prec \pi\bigl(\alpha^\frown\langle y\rangle\bigr),$$

which completes the proof. □

We observe the following immediate consequence of Lemma 7.4.

Corollary 7.5. *One can define a zero-preserving complete meet homomorphism* $\pi^* : \mathbf{Co}(R) \to \mathbf{Co}(\Gamma)$ *by the rule*

$$\pi^*(X) = \pi^{-1}[X], \quad \textit{for all } X \in \mathbf{Co}(R).$$

We put $\psi = \pi^* \circ \varphi$, where $\varphi : L \hookrightarrow \mathbf{Co}(R)$ is the canonical map defined in Section 6. Hence ψ is a zero-preserving meet homomorphism from L into $\mathbf{Co}(\Gamma)$. For any $x \in L$, the value $\psi(x)$ is calculated by the same rule as $\varphi(x)$, see (6.4):

$$\psi(x) = \bigl\{\alpha \in \Gamma \mid e(\alpha) \leqslant x\bigr\}.$$

Lemma 7.6. *The map ψ is a lattice embedding from L into $\mathbf{Co}(\Gamma)$. Moreover, ψ preserves the existing bounds.*

Proof. The statement about preservation of bounds is obvious. We have already seen (and it is obvious) that ψ is a meet homomorphism. Let $x, y \in L$ such that $x \nleqslant y$. Since L is finitely spatial, there exists $p \in \mathrm{J}(L)$ such that $p \leqslant x$ and $p \nleqslant y$; whence $\langle p\rangle \in \psi(x) \setminus \psi(y)$. Hence ψ is a meet embedding from L into $\mathbf{Co}(\Gamma)$.

Let $x, y \in L$, let $\alpha \in \psi(x \vee y)$, we prove that $\alpha \in \psi(x) \vee \psi(y)$. This is obvious if $\alpha \in \psi(x) \cup \psi(y)$, so suppose that $\alpha \notin \psi(x) \cup \psi(y)$. Hence $e(\alpha) \leqslant x \vee y$ while $e(\alpha) \nleqslant x, y$, thus, by Lemma 3.2, there are minimal $u \leqslant x$ and $v \leqslant y$ such that $e(\alpha) \leqslant u \vee v$, and both u and v belong to $[e(\alpha)]^D$. Therefore, by Corollary 7.2, either $\alpha^\frown\langle u\rangle \prec \alpha \prec \alpha^\frown\langle v\rangle$ or $\alpha^\frown\langle v\rangle \prec \alpha \prec \alpha^\frown\langle u\rangle$. In both cases, since $\alpha^\frown\langle u\rangle \in \psi(x)$ and $\alpha^\frown\langle v\rangle \in \psi(y)$, we obtain that $\alpha \in \psi(x) \vee \psi(y)$. Therefore, ψ is a join homomorphism. □

Now we can state the main embedding theorem of the present section.

Theorem 7.7. *Let L be a lattice. Then the following assertions are equivalent*:

(i) *there exists a poset P such that L embeds into $\mathbf{Co}(P)$*;
(ii) *L satisfies the identities* (S), (U), *and* (B) (*i.e., it belongs to the class* **SUB**);
(iii) *there exists a tree-like* (*see Section* 2) *poset Γ such that L has an embedding into $\mathbf{Co}(\Gamma)$ that preserves the existing bounds. Furthermore, if L is finite without D-cycle, then Γ is finite.*

Proof. (i) ⇒ (ii) has already been established, see Theorem 6.7.

(ii) ⇒ (iii). As in the proof of Theorem 6.7, we denote by $\mathrm{Fil}\,L$ the lattice of all filters of L, ordered by reverse inclusion; if L has no unit element, then we allow the empty set in $\mathrm{Fil}\,L$, otherwise we require filters to be nonempty. We consider the poset Γ constructed from $\mathrm{Fil}\,L$ as in Section 7. By Lemma 7.6, L embeds into $\mathbf{Co}(\Gamma)$. The finiteness statement of (iii) is obvious.

(iii) ⇒ (i) is trivial. □

Even in case $L = \mathbf{Co}(P)$, for a finite totally ordered set P, the poset Γ constructed in Theorem 7.7 is not isomorphic to P as a rule. As it is constructed from finite sequences of elements of P, it does not lend itself to easy graphic representation. However, many of its properties can be seen on the simpler poset represented on Fig. 5, which is tree-like.

As we shall see in Sections 9 and 10, the assumption in Theorem 7.7(iii) that L be without D-cycle cannot be removed.

8. Non-preservation of atoms

The posets R and Γ that we constructed in Sections 6 and 7 are defined *via* sequences of join-irreducible elements of L. This is to be put in contrast with the main result of O. Frink [8] (see also [11]), that embeds any complemented modular lattice into a geometric lattice:

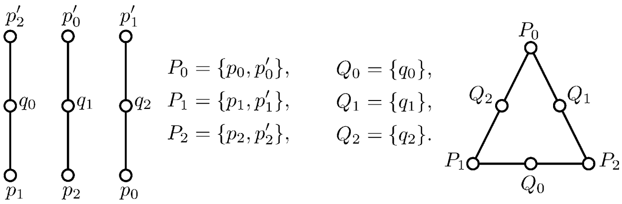


Fig. 1. The poset P and the geometry of $\boldsymbol{K}$.

namely, this construction preserves atoms. Hence the question of the necessity of the complication of the present paper, that is, using sequences of join-irreducible elements rather than just join-irreducible elements, is natural. In the present section we study two examples that show that this complication is, indeed, necessary.

Example 8.1. *A finite, atomistic lattice in* **SUB** *without D-cycle that cannot be embedded atom-preservingly into any* $\mathbf{Co}(T)$.

Proof. Let P be the nine-element poset represented on the left-hand side of Fig. 1, together with order-convex subsets P_0, P_1, P_2, Q_0, Q_1, Q_2.

We let $\boldsymbol{K}$ be the set of all elements X of $\mathbf{Co}(P)$ such that $p_i \in X \Leftrightarrow p_i' \in X$, for all $i < 3$. It is obvious that $\boldsymbol{K}$ is a meet-subsemilattice of $\mathbf{Co}(P)$ which contains $\{\emptyset, P\} \cup \Omega$, where $\Omega = \{P_0, P_1, P_2, Q_0, Q_1, Q_2\}$. We prove that $\boldsymbol{K}$ is a join-subsemilattice of $\mathbf{Co}(P)$. Indeed, for all $i < 3$, both p_i and p_i' are either maximal or minimal in P, hence, for all X, $Y \in \mathbf{Co}(P)$, $p_i \in X \vee Y$ iff $p_i \in X \cup Y$, and, similarly, $p_i' \in X \vee Y$ iff $p_i' \in X \cup Y$. Hence $X, Y \in \boldsymbol{K}$ implies that $X \vee Y \in \boldsymbol{K}$.

Therefore, $\boldsymbol{K}$ is a sublattice of $\mathbf{Co}(P)$. It follows immediately that the atoms of $\boldsymbol{K}$ are the elements of Ω, that $\boldsymbol{K}$ is atomistic, and the atoms of $\boldsymbol{K}$ satisfy the following relations (see the right half of Fig. 1):

$$\begin{array}{lll} Q_0 \leqslant P_1 \vee P_2, & Q_1 \leqslant P_0 \vee P_2, & Q_2 \leqslant P_0 \vee P_1, \\ P_0 \nleqslant P_1 \vee P_2, & P_1 \nleqslant P_0 \vee P_2, & P_2 \nleqslant P_0 \vee P_1. \end{array}$$

Hence, the sequence $P_0 P_1 P_2 P_0 P_1$ is a zigzag of length 5 (in the sense of [3]). It follows from this and the easy direction of the main theorem of [3] that $\boldsymbol{K}$ cannot be embedded atom-preservingly into any $\mathbf{Co}(T)$. □

By contrast, our second example is subdirectly irreducible, but it has D-cycles. We shall see in a subsequent paper [15] that the latter condition is unavoidable, that is, any finite, subdirectly irreducible atomistic lattice without D-cycle that can be embedded into some $\mathbf{Co}(P)$ can be embedded atom-preservingly into some finite $\mathbf{Co}(P)$ without D-cycle.

Example 8.2. *A finite, atomistic, subdirectly irreducible lattice in* **SUB** *that cannot be embedded into* $\mathbf{Co}(T)$*, for any poset T, in an atom-preserving way.*

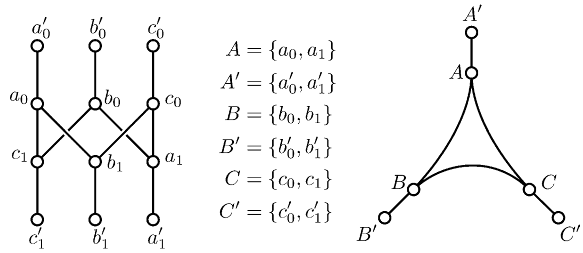


Fig. 2. The poset Q and the geometry of $\boldsymbol{L}$.

Proof. Let Q be the 12-element poset represented on the left hand side of Fig. 2, together with order-convex subsets A, B, C, A', B', C'.

We let σ be the anti-automorphism of Q defined by $\sigma(a_i) = a_{1-i}$, $\sigma(a'_i) = a'_{1-i}$, $\sigma(b_i) = b_{1-i}$, $\sigma(b'_i) = b'_{1-i}$, $\sigma(c_i) = c_{1-i}$, $\sigma(c'_i) = c'_{1-i}$, for all $i < 2$, and we let $\boldsymbol{L}$ be the set of all elements X of $\mathbf{Co}(Q)$ such that $\sigma X = X$. It is obvious that $\boldsymbol{L}$ is a meet-subsemilattice of $\mathbf{Co}(Q)$ which contains $\{\emptyset, Q\} \cup \Omega$, where $\Omega = \{A, B, C, A', B', C'\}$. We prove that $\boldsymbol{L}$ is a join-subsemilattice of $\mathbf{Co}(Q)$. Let $X, Y \in \boldsymbol{L}$, we prove that $X \vee Y \in \boldsymbol{L}$.

Since both a'_0 and a'_1 are either maximal or minimal in Q, the equivalence $a'_i \in X \vee Y \Leftrightarrow a'_i \in X \cup Y$ holds, for all $i < 2$, whence $a'_0 \in X \vee Y \Leftrightarrow a'_1 \in X \vee Y$. Similarly, $b'_0 \in X \vee Y \Leftrightarrow b'_1 \in X \vee Y$ and $c'_0 \in X \vee Y \Leftrightarrow c'_1 \in X \vee Y$.

Suppose now that $a_0 \in X \vee Y$, we prove that $a_1 \in X \vee Y$. If $a_0 \in X \cup Y$ this is obvious, so suppose that $a_0 \notin X \cup Y$. Without loss of generality, there are $x \in X$ and $y \in Y$ such that $x \lhd a_0 \lhd y$, whence $x \in \{b'_1, b_1, c'_1, c_1\}$ and $y = a'_0$. From $Y \in \boldsymbol{L}$ it follows that $a'_1 \in Y$, thus $A' \subseteq Y$. Similarly, from $X \in \boldsymbol{L}$ it follows that either $B \subseteq X$ or $C \subseteq X$ or $B' \subseteq X$ or $C' \subseteq X$. If $B \subseteq X$, then $b_0 \in X$, thus, since $a'_1 \lhd a_1 \lhd b_0$ and $a'_1 \in Y$, we obtain that $a_1 \in X \vee Y$. If $B' \subseteq X$, then $b'_0 \in X$, thus, since $a'_1 \lhd a_1 \lhd b'_0$ and $a'_1 \in Y$, we obtain again that $a_1 \in X \vee Y$. Similar results hold for either $C \subseteq X$ or $C' \subseteq X$. Therefore, $a_0 \in X \vee Y$ implies that $a_1 \in X \vee Y$. By symmetry, we obtain the converse. Similarly, $b_0 \in X \vee Y \Leftrightarrow b_1 \in X \vee Y$ and $c_0 \in X \vee Y \Leftrightarrow c_1 \in X \vee Y$. Therefore, $X \vee Y$ belongs to $\boldsymbol{L}$, which completes the proof that $\boldsymbol{L}$ is a sublattice of $\mathbf{Co}(Q)$.

It follows immediately that the atoms of $\boldsymbol{L}$ are the elements of Ω, that $\boldsymbol{L}$ is atomistic, and the atoms of $\boldsymbol{L}$ satisfy the following relations:

$$\begin{array}{lll} A, B \leqslant A' \vee B', & A \leqslant A' \vee B, & B \leqslant A \vee B', \\ B, C \leqslant B' \vee C', & B \leqslant B' \vee C, & C \leqslant B \vee C', \\ A, C \leqslant A' \vee C', & A \leqslant A' \vee C, & C \leqslant A \vee C'. \end{array}$$

Hence, $\boldsymbol{L}$ is subdirectly irreducible, with monolith (i.e., smallest nonzero congruence) the smallest congruence $\Theta(\emptyset, A)$ identifying $\emptyset$ and A, also equal to $\Theta(\emptyset, B)$ and to $\Theta(\emptyset, C)$. Furthermore, the sequence $A'B'C'A'B'$ is a zigzag of length 5 (in the sense of [3]). It follows from this and the easy direction of the main theorem of [3] that $\boldsymbol{L}$ cannot be embedded atom-preservingly into any $\mathbf{Co}(T)$. $\square$

9. Crowns in posets

We first recall the following classical definition.

Definition 9.1. For an integer $n \geqslant 2$, we denote by $\mathbb{Z}/n\mathbb{Z}$ the set of integers modulo n. The *n-crown C_n* is the poset with underlying set $(\mathbb{Z}/n\mathbb{Z}) \times \{0, 1\}$ and ordering defined by $(i, 0), (i + 1, 0) < (i, 1)$, for all $i \in \mathbb{Z}/n\mathbb{Z}$.

The crown C_n is illustrated on Fig. 3.

We shall mostly deal with sub-crowns of posets.

Definition 9.2. For $n \geqslant 2$ and a poset $(T, \trianglelefteq)$, a *n-crown* of T is a finite sequence $\langle\langle a_i, b_i\rangle \mid i \in \mathbb{Z}/n\mathbb{Z}\rangle$ of elements of $T \times T$ such that there exists an order-embedding $f : C_n \hookrightarrow T$ with $f(i, 0) = a_i$ and $f(i, 1) = b_i$, for all $i \in \mathbb{Z}/n\mathbb{Z}$.

We shall sometimes identify an integer modulo n with its unique representative in $\{0, 1, \dots, n - 1\}$ and a n-crown $\langle\langle a_i, b_i\rangle \mid i \in \mathbb{Z}/n\mathbb{Z}\rangle$ with the finite sequence

$$\big\langle\langle a_0, b_0\rangle, \langle a_1, b_1\rangle, \dots, \langle a_{n-1}, b_{n-1}\rangle\big\rangle.$$

The following lemma makes it possible to identify crowns within posets.

Lemma 9.3. *Let $(T, \trianglelefteq)$ be a poset, let $n \geqslant 3$, and let a_i, b_i $(i \in \mathbb{Z}/n\mathbb{Z})$ be elements of T. Then the following are equivalent*:

(i) $\langle\langle a_i, b_i\rangle \mid i \in \mathbb{Z}/n\mathbb{Z}\rangle$ *is a n-crown.*
(ii) $a_i \trianglelefteq b_j$ *iff* $i \in \{j, j + 1\}$, *for all* $i, j \in \mathbb{Z}/n\mathbb{Z}$.

Proof. (i) ⇒ (ii) is trivial. Conversely, suppose (ii) satisfied, we prove that $f : C_n \hookrightarrow T$ defined by $f(i, 0) = a_i$ and $f(i, 1) = b_i$, for all $i \in \mathbb{Z}/n\mathbb{Z}$, is an order-embedding. We need to prove the following assertions:

(i) $a_i \trianglelefteq a_j$ implies that $i = j$, for all $i, j \in \mathbb{Z}/n\mathbb{Z}$. Indeed, if $a_i \trianglelefteq a_j$, then $a_i \trianglelefteq b_j, b_{j-1}$ (because $a_j \trianglelefteq b_j, b_{j-1}$), thus, by assumption, $i \in \{j, j + 1\} \cap \{j, j - 1\} = \{j\}$ (we use here the inequality $n \geqslant 3$), that is, $i = j$.
(ii) $b_i \trianglelefteq b_j$ implies that $i = j$, for all $i, j \in \mathbb{Z}/n\mathbb{Z}$. The proof is similar to the one of (i).
(iii) $b_j \trianglelefteq a_i$ occurs for no $i, j \in \mathbb{Z}/n\mathbb{Z}$. Indeed, suppose that $b_j \trianglelefteq a_i$. Then $b_j \trianglelefteq b_i, b_{i-1}$ (because $a_i \trianglelefteq b_i, b_{i-1}$), thus, by (ii), $j = i = i - 1$, a contradiction.

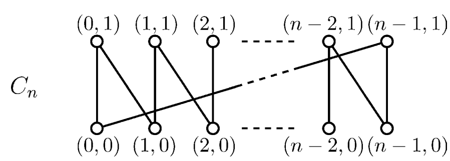


Fig. 3. The crown C_n.

This concludes the proof. $\square$

Definition 9.4. A poset T is *crown-free*, if it has no n-crown for any $n \geqslant 3$.

Strictly speaking, the 2-crown C_2 is crown-free since we are requiring $n \geqslant 3$ in the definition above. The motivation why we are putting this slight restriction on n lies in the following observation. First, the poset of Fig. 4(i) is tree-like, but it contains the 2-crown represented on Fig. 4(ii); observe also that the n-crown, for any $n \geqslant 2$, is never tree-like.

On the other hand, we shall now prove the following result.

Proposition 9.5. *Every tree-like poset is crown-free.*

As witnessed by the square $\mathbf{2}^2$, the converse of Proposition 9.5 does not hold.

Proof. Let $(T, \trianglelefteq)$ be a tree-like poset. For $x, y \in T$, we denote by $d(x, y)$ the length of the unique path from x to y if there is such a path, ∞ otherwise. Observe that $x \trianglelefteq y$ implies that $d(x, y) < \infty$ (but the converse does not hold as a rule), and then the unique path from x to y is oriented (see Section 2).

For a n-crown $\gamma = \langle \langle a_i, b_i \rangle \mid i \in \mathbb{Z}/n\mathbb{Z} \rangle$ in T, we put

$$\ell(\gamma) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} d(a_i, b_i).$$

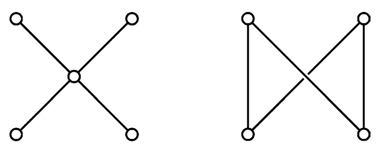
Suppose that T has a n-crown, for some integer $n \geqslant 3$. We pick such a crown $\gamma = \langle \langle a_i, b_i \rangle \mid i \in \mathbb{Z}/n\mathbb{Z} \rangle$ with $\ell(\gamma)$ *minimum*. For all $i \in \mathbb{Z}/n\mathbb{Z}$, we let

$$a_i = x_{i,0} \prec x_{i,1} \prec \cdots \prec x_{i,p_i} = b_i,$$
$$a_{i+1} = y_{i,0} \prec y_{i,1} \prec \cdots \prec y_{i,q_i} = b_i$$

be the paths from a_i (respectively, a_{i+1}) to b_i, where $\prec$ denotes the predecessor relation of T.

Claim 1. $\{x_{i,p} \mid 0 \leqslant p < p_i\} \cap \{y_{i,q} \mid 0 \leqslant q < q_i\} = \emptyset$, *for all* $i \in \mathbb{Z}/n\mathbb{Z}$.

Proof of Claim. Suppose, to the contrary, that $x_{i,p} = y_{i,q}$ for some $p \in \{0, \ldots, p_i - 1\}$ and $q \in \{0, \ldots, q_i - 1\}$. We put $b'_j = b_j$, for all $j \neq i$ in $\mathbb{Z}/n\mathbb{Z}$, while $b'_i = x_{i,p}$. Since



(i) (ii)

Fig. 4. A tree-like poset which contains the crown C_2.

$a_i, a_{i+1} \trianglelefteq b'_i$, the condition $k \in \{l, l+1\}$ implies that $a_k \trianglelefteq b'_l$, for all $k, l \in \mathbb{Z}/n\mathbb{Z}$. Conversely, let $k, l \in \mathbb{Z}/n\mathbb{Z}$ such that $a_k \trianglelefteq b'_l$. From $b'_l \trianglelefteq b_l$ it follows that $a_k \trianglelefteq b_l$, whence $k \in \{l, l+1\}$. By Lemma 9.3, the family $\gamma' = \langle \langle a_k, b'_k \rangle \mid k \in \mathbb{Z}/n\mathbb{Z} \rangle$ is a n-crown. However,

$$\ell(\gamma') \leqslant \ell(\gamma) - (p_i - p) < \ell(\gamma),$$

which contradicts the minimality of $\ell(\gamma)$. □

The proof of the following claim is symmetric.

Claim 2. *$\{y_{i,q} \mid 0 < q \leqslant q_i\} \cap \{x_{i+1,p} \mid 0 < p \leqslant p_{i+1}\} = \varnothing$, for all $i \in \mathbb{Z}/n\mathbb{Z}$.*

We define a *walk* of T to be a finite sequence $\mathbf{c} = \langle c_0, c_1, \dots, c_m \rangle$ of elements of T such that either $c_i \prec c_{i+1}$ or $c_{i+1} \prec c_i$, for all $i < m$, we say then that $\mathbf{c}$ is a walk from c_0 to c_m. Hence, a nonempty path of T is a walk with all distinct entries.

Now we let $\mathbf{d}$ be the finite sequence defined by

$$\mathbf{d} = \langle x_{0,k} \mid 0 \leqslant k \leqslant p_0 \rangle ^\frown \langle y_{0,q_0 - l} \mid 0 < l < q_0 \rangle ^\frown \langle x_{1,k} \mid 0 \leqslant k \leqslant p_1 \rangle \\ \cdots ^\frown \langle x_{n-1,k} \mid 0 \leqslant k \leqslant p_{n-1} \rangle.$$

It is obvious that $\mathbf{d}$ is a walk from $x_{0,0} = a_0$ to $x_{n-1,p_{n-1}} = b_{n-1}$. We shall now prove that $\mathbf{d}$ is a path.

Suppose, indeed, that $\mathbf{d}$ is not a path. Then one of the following cases occurs:

Case 1. There are distinct $i, j \in \mathbb{Z}/n\mathbb{Z}$, together with $k \in \{0, \dots, p_i\}$ and $l \in \{0, \dots, p_j\}$, such that $x_{i,k} = x_{j,l}$. Then $a_i \trianglelefteq x_{i,k} = x_{j,l} \trianglelefteq b_j$, thus $i \in \{j, j+1\}$, while $a_j \trianglelefteq x_{j,l} = x_{i,k} \trianglelefteq b_i$, thus $j \in \{i, i+1\}$. Since $n \geqslant 3$, we obtain that $i = j$, a contradiction.

Case 2. There are distinct $i, j \in (\mathbb{Z}/n\mathbb{Z}) \setminus \{n-1\}$, together with $k \in \{1, \dots, q_i - 1\}$ and $l \in \{1, \dots, q_j - 1\}$, such that $y_{i,k} = y_{j,l}$. Then $a_{i+1} \trianglelefteq y_{i,k} = y_{j,l} \trianglelefteq b_j$, thus $i \in \{j, j-1\}$, while $a_{j+1} \trianglelefteq y_{j,l} = y_{i,k} \trianglelefteq b_i$, thus $j \in \{i, i-1\}$, whence, since $n \geqslant 3$, $i = j$, a contradiction.

Case 3. There are $i \in \mathbb{Z}/n\mathbb{Z}$ and $j \in (\mathbb{Z}/n\mathbb{Z}) \setminus \{n-1\}$, together with $k \in \{0, \dots, p_i\}$ and $l \in \{1, \dots, q_j - 1\}$, such that $x_{i,k} = y_{j,l}$. Then from Claim 1 it follows that $i \neq j$, while from Claim 2 it follows that $i \neq j+1$. On the other hand, $a_i \trianglelefteq x_{i,k} = y_{j,l} \trianglelefteq b_j$, thus $i \in \{j, j+1\}$, a contradiction.

Therefore, we have proved that $\mathbf{d}$ is, indeed, a path from a_0 to b_{n-1}. However, the finite sequence

$$\mathbf{d}' = \langle y_{n-1,l} \mid 0 \leqslant l \leqslant q_{n-1} \rangle$$

is a path from $y_{n-1,0} = a_n = a_0$ (the indices are modulo n) to $y_{n-1,q_{n-1}} = b_{n-1}$, thus, by the uniqueness of the path from a_0 to b_{n-1}, $\mathbf{d} = \mathbf{d}'$. Thus every entry x of $\mathbf{d}$ satisfies that $x \trianglelefteq b_{n-1}$, in particular, $b_0 = x_{0,p_0} \trianglelefteq b_{n-1}$, a contradiction since $n \neq 1$. □

10. A quasi-identity for Co(*T*), for finite and crown-free *T*

Let (θ) be the following lattice-theoretical quasi-identity:

$$\begin{aligned}&\bigl[a \leqslant \bigl(a' \vee b\bigr) \wedge \bigl(a' \vee c\bigr) \;\&\; b \leqslant \bigl(b' \vee a\bigr) \wedge \bigl(b' \vee c\bigr) \;\&\; c \leqslant \bigl(c' \vee a\bigr) \wedge \bigl(c' \vee b\bigr) \\ &\quad \& \; \bigl(a' \wedge a\bigr) \vee \bigl(b' \wedge b\bigr) \vee \bigl(c' \wedge c\bigr) \vee (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \leqslant a' \wedge b' \wedge c'\bigr] \\ &\quad \Rightarrow \quad a \leqslant a'.\end{aligned}$$

It is inspired by Example 8.2 (see Corollary 10.6). The main result of Section 10 is the following.

Theorem 10.1. *Let $(T, \trianglelefteq)$ be a finite crown-free poset. Then $\mathbf{Co}(T)$ satisfies (θ).*

Let us begin with an arbitrary (not necessarily finite, not necessarily crown-free) poset $(T, \trianglelefteq)$ and convex subsets A, B, C, A', B', C' of T that satisfy the premise of (θ), that is,

$$\begin{gathered} A \subseteq A' \vee B, \qquad A \subseteq A' \vee C, \\ B \subseteq B' \vee A, \qquad B \subseteq B' \vee C, \\ C \subseteq C' \vee A, \qquad C \subseteq C' \vee B, \\ A \cap A' \subseteq B' \cap C', \qquad B \cap B' \subseteq A' \cap C', \qquad C \cap C' \subseteq A' \cap B', \\ A \cap B \subseteq A' \cap B', \qquad B \cap C \subseteq B' \cap C', \qquad A \cap C \subseteq A' \cap C'. \end{gathered}$$

We shall put $\widehat{A} = A \setminus A'$, $\widehat{B} = B \setminus B'$, and $\widehat{C} = C \setminus C'$. Observe that

$$\begin{gathered} \widehat{A} \cap (B \cup C) = \widehat{B} \cap (A \cup C) = \widehat{C} \cap (A \cup B) = \emptyset, \\ \widehat{A} \cap \widehat{B} = \widehat{A} \cap \widehat{C} = \widehat{B} \cap \widehat{C} = \emptyset. \end{gathered}$$

We shall later perform a construction whose key argument is provided by the following lemma.

Lemma 10.2. *Let $a \in \widehat{A}$ and let $a' \in A'$ with $a \lhd a'$. Then there exists $\langle b, b' \rangle \in \widehat{B} \times B'$ such that $b' \lhd b \lhd a$.*

Proof. Observe first that $a \in A \subseteq A' \vee B$. Since $a \notin A' \cup B$, there exists $(\bar{a}', b) \in A' \times B$ such that either $\bar{a}' \lhd a \lhd b$ or $b \lhd a \lhd \bar{a}'$. In the first case, $\bar{a}' \lhd a \lhd a'$, thus, by the convexity of A', $a \in A'$, a contradiction; whence $b \lhd a$. If $b \in B'$, then $b \in B \cap B' \subseteq A'$, but

$b \lhd a \lhd a'$, thus $a \in A'$, a contradiction; whence $b \in \widehat{B}$. If there exists $x \in A$ with $x \trianglelefteq b$, then, since $b \lhd a$, we obtain that $b \in A \cap B \subseteq A'$, a contradiction again. But $b \in B \subseteq A \vee B'$ and $b \notin B'$, thus there exists $b' \in B'$ such that $b' \lhd b$. □

In particular, we observe the following corollary.

Corollary 10.3. *The sets $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ are either simultaneously empty or simultaneously nonempty.*

Proof. If $\widehat{A}$ is nonempty, we pick $a \in \widehat{A}$. So $a \in A' \vee B$ while $a \notin A' \cup B$, thus there is $(a', b) \in A' \times B$ such that either $b \lhd a \lhd a'$ or $a' \lhd a \lhd b$. In the first case, we apply Lemma 10.2 to deduce that $\widehat{B} \neq \emptyset$. In the second case, we apply the dual of Lemma 10.2 to reach the same conclusion. □

Now we suppose that $\widehat{A}$ is nonempty, and we pick $a_0 \in \widehat{A}$. As in the proof of Corollary 10.3, there exists $a'_0 \in A'$ such that either $a_0 \lhd a'_0$ or $a'_0 \lhd a_0$; by replacing $\trianglelefteq$ with its dual if needed, we may assume without loss of generality that $a_0 \lhd a'_0$.

By Lemma 10.2, there are $\langle b_0, b'_0 \rangle \in \widehat{B} \times B'$ and $\langle c_1, c'_1 \rangle \in \widehat{C} \times C'$ such that $b'_0 \lhd b_0 \lhd a_0$ and $c'_1 \lhd c_1 \lhd a_0$. By applying the dual of Lemma 10.2 to $c'_1 \lhd c_1$, we obtain $\langle b_1, b'_1 \rangle \in \widehat{B} \times B'$ such that $c_1 \lhd b_1 \lhd b'_1$. By applying Lemma 10.2 to $b_1 \lhd b'_1$, we obtain $\langle a_2, a'_2 \rangle \in \widehat{A} \times A'$ such that $a'_2 \lhd a_2 \lhd b_1$. By applying in the same fashion Lemma 10.2 and its dual, we obtain $\langle c_2, c'_2 \rangle \in \widehat{C} \times C'$, $\langle b_3, b'_3 \rangle \in \widehat{B} \times B'$, and $\langle a_3, a'_3 \rangle \in \widehat{A} \times A'$ such that $a_2 \lhd c_2 \lhd c'_2$, $b'_3 \lhd b_3 \lhd c_2$, and $b_3 \lhd a_3 \lhd a'_3$.

Now we observe that $b'_0 \lhd b_0 \lhd a_0 \lhd a'_0$ and $b'_3 \lhd b_3 \lhd a_3 \lhd a'_3$, that is, we can start the process again. Arguing by induction, we obtain elements $\langle a_i, a'_i \rangle \in \widehat{A} \times A'$ for $i \not\equiv 1 \pmod 3$, elements $\langle b_i, b'_i \rangle \in \widehat{B} \times B'$ for $i \not\equiv 2 \pmod 3$, and elements $\langle c_i, c'_i \rangle \in \widehat{C} \times C'$ for $i \not\equiv 0 \pmod 3$ such that the following relations hold, for all $i < \omega$:

$$b'_{3i} \lhd b_{3i} \lhd a_{3i} \lhd a'_{3i}, \tag{10.1}$$

$$c'_{3i+1} \lhd c_{3i+1} \lhd b_{3i+1} \lhd b'_{3i+1}, \tag{10.2}$$

$$a'_{3i+2} \lhd a_{3i+2} \lhd c_{3i+2} \lhd c'_{3i+2}. \tag{10.3}$$

This can be illustrated by Fig. 5.

Now we define subsets of T as follows:

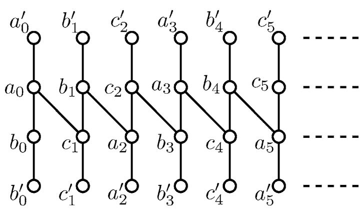


Fig. 5. A pattern in T.

$$\Omega^+ = \{a_{3i} \mid i < \omega\} \cup \{b_{3i+1} \mid i < \omega\} \cup \{c_{3i+2} \mid i < \omega\};$$
$$\Omega^- = \{a_{3i+2} \mid i < \omega\} \cup \{b_{3i} \mid i < \omega\} \cup \{c_{3i+1} \mid i < \omega\};$$
$$\Omega = \Omega^+ \cup \Omega^-.$$

Since $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ are mutually disjoint and their union contains Ω, we can define a map $\chi : \Omega \to 3$ by the rule

$$\chi(x) = \begin{cases} 0 & (x \in \widehat{A}), \\ 1 & (x \in \widehat{B}), \\ 2 & (x \in \widehat{C}), \end{cases} \quad \text{for all } x \in \Omega.$$

Lemma 10.4. *For all $\langle x, y\rangle \in \Omega^- \times \Omega^+$, $\chi(x) = \chi(y)$ implies that $x \ntrianglelefteq y$. In particular, $\Omega^- \cap \Omega^+ = \emptyset$.*

Proof. We need to prove that for all natural numbers i and j, the following inequalities hold:

- $a_{3i+2} \ntrianglelefteq a_{3j}$. Otherwise, by (10.1) and (10.3), $a'_{3i+2} \lhd a_{3i+2} \lhd a'_{3j}$, thus $a_{3i+2} \in A'$, a contradiction.
- $b_{3i} \ntrianglelefteq b_{3j+1}$. Otherwise, by (10.1) and (10.2), $b'_{3i} \lhd b_{3i} \lhd b'_{3j+1}$, thus $b_{3i} \in B'$, a contradiction.
- $c_{3i+1} \ntrianglelefteq c_{3j+2}$. Otherwise, by (10.2) and (10.3), $c'_{3i+1} \lhd c_{3i+1} \lhd c'_{3j+2}$, thus $c_{3i+1} \in C'$, a contradiction.

This concludes the proof. □

For an integer $m \geqslant 2$, we define a *m-pre-crown* to be a finite sequence $\langle\langle x_i, y_i\rangle \mid i \in \mathbb{Z}/m\mathbb{Z}\rangle$ of elements of $\Omega^- \times \Omega^+$ such that the following conditions hold, for all $i \in \mathbb{Z}/m\mathbb{Z}$:

(C1) $x_i, x_{i+1} \lhd y_i$;
(C2) $\chi(x_i) \neq \chi(x_{i+1})$ and $\chi(y_i) \neq \chi(y_{i+1})$ if $i \neq m-1$.

If $m = 2$, then, by (C1), $x_0, x_1 \lhd y_0, y_1$. Furthermore, by (C2), $\chi(x_0) \neq \chi(x_1)$, thus it follows from $x_0, x_1 \lhd y_0$ and Lemma 10.4 that $\chi(y_0)$ is the unique element of $3 \setminus \{\chi(x_0), \chi(x_1)\}$. The same holds for $\chi(y_1)$, whence $\chi(y_0) = \chi(y_1)$, which contradicts (C2). Therefore, if there exists a m-pre-crown, then $m \geqslant 3$.

We can now prove the main lemma of this section.

Lemma 10.5. *Suppose that T is crown-free. Then there are no pre-crowns in T.*

Proof. Otherwise, let m be the least positive integer such that there exists a m-pre-crown, and let $\mathbf{c} = \langle\langle x_i, y_i\rangle \mid i \in \mathbb{Z}/m\mathbb{Z}\rangle$ be such a pre-crown. As observed before, $m \geqslant 3$. By assumption on T, in order to get a contradiction, it suffices to prove that $\mathbf{c}$ is a crown of T. By (C1) and Lemma 9.3, it suffices to prove that for all $i, j \in \mathbb{Z}/m\mathbb{Z}$ such that $i \notin \{j, j+1\}$,

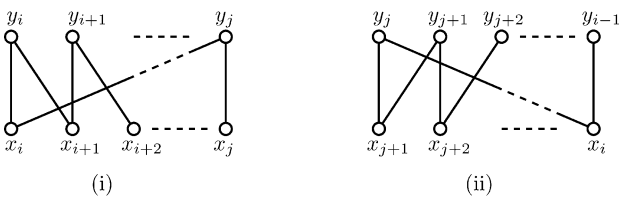


Fig. 6. Shorter pre-crowns.

the inequality $x_i \trianglelefteq y_j$ does not hold. Suppose otherwise; by Lemma 10.4, $x_i \vartriangleleft y_j$. Two cases can occur.

Case 1. $i < j$. Then the finite sequence

$$\langle\langle x_i, y_i\rangle, \langle x_{i+1}, y_{i+1}\rangle, \ldots, \langle x_j, y_j\rangle\rangle$$

is a $(j - i + 1)$-pre-crown (see Fig. 6(i)), with $1 \leqslant j - i \leqslant m - 1$. By the minimality assumption on m, this cannot happen unless $i = 0$ and $j = m - 1$, in which case $i = j + 1$ (modulo m as usual), a contradiction.

Case 2. $j < i$. Then the finite sequence

$$\langle\langle x_i, y_{i-1}\rangle, \ldots, \langle x_{j+2}, y_{j+1}\rangle, \langle x_{j+1}, y_j\rangle\rangle$$

is a $(i - j)$-pre-crown (see Fig. 6(ii)), with $2 \leqslant i - j < m$, which contradicts again the minimality of m.

Hence **c** is a m-crown of T, a contradiction. □

Now we have all the necessary tools to conclude the proof of Theorem 10.1.

Proof of Theorem 10.1. Suppose that T is finite and crown-free. There are $i < j$ such that $b_{3i} = b_{3j}$. Then the finite sequence

$$\langle\langle b_{3i}, a_{3i}\rangle, \langle c_{3i+1}, b_{3i+1}\rangle, \ldots, \langle a_{3j-1}, c_{3j-1}\rangle\rangle$$

is a $(3j - 3i)$-pre-crown in T (see Fig. 7), a contradiction.

Hence we have proved that $\widehat{A} = \emptyset$, that is, $A \subseteq A'$. Therefore, $\mathbf{Co}(T)$ satisfies (θ). □

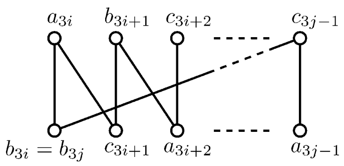


Fig. 7. A pre-crown in T.

Corollary 10.6. *Let Q be the finite poset and $\boldsymbol{L}$ the finite lattice of Example* 8.2*. Then, although $\boldsymbol{L}$ embeds into $\mathbf{Co}(Q)$, there is no finite, tree-like poset R such that $\boldsymbol{L}$ embeds into $\mathbf{Co}(R)$.*

Proof. It follows from Proposition 9.5 that R is crown-free, thus, by Theorem 10.1, $\mathbf{Co}(R)$ satisfies (θ). On the other hand, the lattice $\boldsymbol{L}$ of Example 8.2 does not satisfy (θ) (consider the atoms A, B, C, A', B', C' of $\boldsymbol{L}$), therefore it cannot be embedded into $\mathbf{Co}(R)$. □

On the other hand, it follows from Theorem 7.7(iii) that if a finite lattice L without D-cycle embeds into some $\mathbf{Co}(P)$, then it embeds into $\mathbf{Co}(R)$ for some finite, tree-like poset R. In the presence of D-cycles anything can happen, for example, take $L = \mathbf{Co}(\mathbf{4})$, the lattice of all order-convex subsets of a four-element chain; it embeds into $\mathbf{Co}(\mathbf{4})$ for the finite, tree-like poset $\mathbf{4}$, however it has D-cycles.

11. Finite generation and word problem in SUB

For a lattice term $\mathbf{s}(\mathsf{x}_1, \ldots, \mathsf{x}_n)$, a poset P, and convex subsets $X_1, \ldots, X_n$ of P, we denote by $\mathbf{s}^P(X_1, \ldots, X_n)$ the evaluation of the term $\mathbf{s}(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ at $\langle X_1, \ldots, X_n \rangle$ in the lattice $\mathbf{Co}(P)$.

The present section rests on the following lemma. Its proof is an easy induction argument on the length of $\mathbf{s}$, that we leave to the reader.

Lemma 11.1. *Let n be a positive integer, let $\mathbf{s}(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ be a lattice term, and let $X_1, \ldots, X_n$ be convex subsets of a poset P. Then $\mathbf{s}^P(X_1, \ldots, X_n)$ is the directed union of all subsets of the form $\mathbf{s}^Q(X_1 \cap Q, \ldots, X_n \cap Q)$, for $Q \subseteq P$ finite.*

As immediate corollaries, we get the following:

Corollary 11.2. *Let P be a poset. Any lattice-theoretical identity valid in all $\mathbf{Co}(Q)$, for Q a finite subset of P, is also valid in $\mathbf{Co}(P)$.*

Corollary 11.3. *A lattice-theoretical identity is valid in* **SUB** *iff it holds in $\mathbf{Co}(P)$ for every* finite *poset P.*

Consequently, the variety **SUB** is generated by its finite members. By using the results of J.C.C. McKinsey [14], we obtain the following consequence.

Corollary 11.4. *The word problem in the variety* **SUB** *is decidable.*

This means that it is decidable whether a given lattice identity $\mathbf{s}(\mathsf{x}_1, \ldots, \mathsf{x}_m) = \mathbf{t}(\mathsf{x}_1, \ldots, \mathsf{x}_m)$ holds in all lattices of the form $\mathbf{Co}(P)$. A closer look at the proof of Lemma 11.1 shows that it is sufficient to verify whether the given identity holds in all $\mathbf{Co}(P)$ for $|P| \leqslant n$, where n is the supremum of the lengths of the terms $\mathbf{s}$ and $\mathbf{t}$.

12. Open problems

We know that the class **SUB** is generated, as a variety, by its finite members (see Corollary 11.3). We also know that any finite lattice in **SUB** can be embedded into some finite $\mathbf{Co}(P)$ (see Theorem 6.7). Nevertheless we do not know whether the latter generate the whole *quasi*variety.

Problem 1. Is the class **SUB** generated, as a quasivariety, by its finite members?

Equivalently, does there exist a lattice quasi-identity that holds in all finite $\mathbf{Co}(P)$-s but not in all $\mathbf{Co}(P)$-s?

Problem 2. Is the universal theory of all lattices of the form $\mathbf{Co}(P)$ decidable?

A positive answer to Problem 1 would yield a positive answer to Problem 2.

Problem 3. Is the class **C** of all lattices that can be embedded into a product of the form $\prod_{i\in I}\mathbf{Co}(C_i)$, where the C_i are *chains*, a variety?

Problem 3 is solved by the authors in [16].

Problem 4. Can the embedding problem of a lattice in **SUB** into some $\mathbf{Co}(P)$ be solved by a *functor* (that, say, sends any L to some $\mathbf{Co}(P)$)? Can such a functor be idempotent?

Our next problem has a more computational nature.

Problem 5. For each positive integer n, denote by $\xi(n)$ the least positive integer such that every finite lattice L in **SUB** with n join-irreducible elements embeds into some $\mathbf{Co}(P)$, where $|P| \leqslant \xi(n)$. Compute $\xi(n)$, for all $n > 0$. Does $\xi(n) = O(n)$ as n goes to infinity?

For a sublattice K of a finite lattice L, the inequality $|\mathrm{J}(K)| \leqslant |\mathrm{J}(L)|$ holds, see [1, Lemma 2]. In particular, if a finite lattice L embeds into $\mathbf{Co}(P)$ for some finite poset P, then $|\mathrm{J}(L)| \leqslant |P|$. By combining this with the result of Theorem 6.7, we obtain the inequalities

$$n \leqslant \xi(n) \leqslant 2n^2 - 5n + 4.$$

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