

## STABLE SPLITTINGS DERIVED FROM THE STEINBERG MODULE

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IN THIS PAPER we construct a new class of stable splittings for certain classifying spaces, including  $B(\mathbb{Z}/p)^k$ . Our results involve symmetric products of the sphere spectrum and are based on the fundamental Steinberg module of modular representation theory. Splitting theorems have long played an important role in homotopy theory, see [1-4], one reason being that an equivalence  $X \xrightarrow{\cong} \vee X_i$  enables one to construct maps  $X_i \rightarrow X$  which were *a priori* inaccessible. Examples include Mahowald's maps  $\eta_j$ [5] based on Snaith's splittings and, more recently, certain maps used in Kuhn's proof of the Whitehead conjecture [6, 7]. These latter maps are based on our splitting of  $B(\mathbb{Z}/2)^k$ .

Our main result shows that the suspension spectrum of a product of lens spaces  $B(\mathbb{Z}/p)^k$  can be split using the Steinberg idempotent of  $\mathbb{F}_p[GL_k(\mathbb{F}_p)]$ . Let  $Sp^n(S^\circ)$  denote the  $n$ -fold symmetric product of the sphere spectrum. We recall  $Sp^\infty(S^\circ) = K(\mathbb{Z})$  by the Dold-Thom theorem. Let  $D(k)$  be the cofiber of the diagonal map  $d: Sp^{p^k-1}(S^\circ) \rightarrow Sp^{p^k}(S^\circ)$ . Then  $D(\infty) = K(\mathbb{Z}/p)$ . Let  $M(k) = \Sigma^{-k}D(k)/D(k-1)$ . In mod- $p$  cohomology  $H^*(M(k))$  has a basis consisting of admissible Steenrod operations of length exactly  $k$ .

**THEOREM A.** *Stably,  $B(\mathbb{Z}/p)^k$  contains  $p^{\binom{k}{2}}$  summands each equivalent to  $M(k)$ . These summands correspond to the  $p^{\binom{k}{2}}$  summands of the Steinberg module in  $\mathbb{F}_p[GL_k(\mathbb{F}_p)]$ .*

Here and throughout, all spaces are localized at  $p$ . Let  $L(k) = \Sigma^{-k}Sp^{p^k}(S^\circ)/Sp^{p^k-1}(S^\circ)$ . A simple argument shows that  $L(k)$  is also a summand of  $B(\mathbb{Z}/p)^k$ ; in fact,  $M(k) = L(k) \vee L(k-1)$ .

Let  $\int^k \mathbb{Z}/p$  denote the  $k$ -fold wreath product. Using the transfer  $t: B(\int^k \mathbb{Z}/p) \rightarrow B(\mathbb{Z}/p)^k$  and the double coset formula we prove

**THEOREM B.**  *$M(k)$  is a stable summand in  $B(\int^k \mathbb{Z}/p)$ . Let  $O(k)$  be the real orthogonal*

*Let  $O(k)$  be the real orthogonal group. Using Becker-Gottlieb transfer for the fibration  $O(k)/(\mathbb{Z}/2)^k \rightarrow B(\mathbb{Z}/2)^k \rightarrow BO(k)$  we prove*

**THEOREM C.**  *$M(k)$  is a stable summand in  $BO(k)$ .*

Let  $T^k = (S^1)^k$  be the  $k$ -torus. We construct a spectrum  $BP(k)$  such that  $H^*BP(k)$  has a basis consisting of admissible Steenrod operations in the reduced powers of length exactly  $k$ . Using a lifting of the Steinberg idempotent to  $GL_k(\mathbb{Z}_p)$  we show

**THEOREM D.** *Completed at  $p$ ,  $BT^k$  contains  $p^{\binom{k}{2}}$  stable summands each equivalent to  $BP(k)$ . Further,  $BP(k)$  is a stable summand of  $BU(k)$ .*

This paper is organized as follows: The brief §1 contains a few remarks about the length filtration of the Steenrod algebra. In §2 we give an account of those facts about  $GL_n(\mathbb{F}_p)$

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and the Steinberg module needed for our subsequent constructions. Sections 3 and 4 are devoted to the construction and properties of various spectra including Thom spectra and symmetric product spectra. The proof of Theorem A is given in §5. Section 6 contains proofs of Theorems B and C. Finally the construction of  $BP(k)$  and the proof of Theorem D is given in §7.

§1. PRELIMINARIES ON THE STEENROD ALGEBRA

Let  $A$  denote the Steenrod algebra, and let  $A_n$  denote the subalgebra generated by  $\beta, P^1, \dots, P^{p^n-1}$ . (For  $p = 2, \beta = Sq^1$  and  $P^i = Sq^{2^i}$ .) If  $I$  is a finite sequence  $(\epsilon_0, r_1, \epsilon_1, r_2, \dots)$ ,  $r_i \geq 0, \epsilon_j = 0, 1$ , then  $\theta^I = \beta^{\epsilon_0} P^{r_1} \beta^{\epsilon_1} P^{r_2} \dots$ ; if  $\epsilon_i = 0$  for all  $i$  we write  $P^I$  for  $\theta^I$ . As usual,  $I$  is admissible if  $r_i \geq pr_{i+1} + \epsilon_i$  for all  $i$ . By a classical theorem of Cartan and Serre, the admissible  $\theta^I$  are a basis for  $A$ . The length  $l(I)$  is defined by  $l(I) = n$  if  $r_i = 0$  for  $i > n$  and  $\epsilon_i = 0$  for  $i \geq n$ . Thus we obtain vector space filtrations on  $A$  defined by  $F_n = \langle \theta^I : l(I) \leq n \rangle$  and  $G_n = \langle \theta^I : I \text{ admissible, } l(I) > n \rangle$ . Finally, we recall that  $A_n$  contains an exterior algebra on primitive elements  $Q_0, \dots, Q_{n-1}$ , where  $Q_0 = \beta$  and  $Q_{k+1} = [P^{p^k}, Q_k]$ .

PROPOSITION 1.1. (a)  $F_n$  is spanned by the admissible  $\theta^I, l(I) \leq n$ ; (b)  $F_n$  is a subcoalgebra of  $A$ ; (c)  $F_n$  is a left  $A_{n-1}$  submodule of  $A$ . Moreover  $F_n$  is free over  $E[Q_0, \dots, Q_{n-1}]$  on  $\{P^I : I \text{ admissible, } l(I) \leq n\}$ ; (d)  $G_n$  is a left ideal. Moreover  $A/G_n = F_n$  as  $A_{n-1}$  modules.

Proof: (a) follows from the Adem relations, and (b) is obvious. The first part of (c) also follows from the Adem relations, using induction on  $n$ . For the second part, note that the  $E[Q_0, \dots, Q_{n-1}]$  submodule of  $F_n$  generated by  $\{P^I : I \text{ admissible, } l(I) \leq n\}$  is indeed free as claimed; this follows from Milnor[8], Theorem 4(a). Hence this module has Poincaré series  $\prod_0^{n-1} (1 + t^{2p^i-1}) / \prod_1^n (1 - t^{2(p^i-1)})$ , which is precisely the Poincaré series of  $F_n$  (by (a)). Finally, (d) also follows from the Adem relations; alternatively, it is a consequence of (3.5) below.

§2.  $GL_n \mathbb{F}_q$  AND THE STEINBERG MODULE

Let  $V^n$  be a vector space over the finite field  $\mathbb{F}_q, q = p^r$ , with basis  $y_1, \dots, y_n$ . Then  $GL_n \mathbb{F}_q$  is the automorphism group of  $V^n$ , acting on the right.  $GL_n \mathbb{F}_q$  has order  $q^{\binom{n}{2}} \prod_{i=1}^n (q^i - 1)$ , and contains the following distinguished subgroups:

- $\Sigma_n = \text{symmetric group}$  (permutation matrices).
- $D_n = \text{diagonal matrices}$ .
- $B_n = \text{Borel subgroup} = \text{upper triangular matrices}$ .
- $U_n = \text{unipotent subgroup} = \text{upper triangular matrices with all diagonal entries equal to 1}$ .  
(Note  $U_n$  is a  $p$ -Sylow subgroup.)

In addition we will need to consider

$$A_n = \text{top row subgroup} = \{g \in B_n : y_i g = y_i, \forall i > 1\}.$$

$$T_n = \text{cyclic subgroup of } \Sigma_n, \text{ of order } n \text{ generated by } (1, 2, \dots, n).$$

Throughout this paper, we regard  $V^k$  as the subspace  $\langle y_{n-k+1}, \dots, y_n \rangle$  of  $V^n$ ; this convention determines inclusions  $GL_k \mathbb{F}_q \subseteq GL_n \mathbb{F}_q$ , etc. Note that many of our subgroups fit together as semi-direct products, e.g.  $\Sigma_n \tilde{\times} D_n, D_n \tilde{\times} U_n = B_n$ , and the ‘‘maximal parabolic subgroup’’  $GL_{n-1} \tilde{\times} A_n$ .

We digress briefly to review some general facts from representation theory (see [9]). All modules are understood to be right modules. Let  $R$  be any finite-dimensional algebra over

a field  $K$ . Then there is a unique set of indecomposable two-sided ideals  $B_1, \dots, B_r$ , called *blocks*, such that  $R = \prod B_i$  (as algebras). Each  $B_i$  corresponds to a central idempotent  $f_i$  such that  $B_i = Rf_i = f_iR$ ; the  $f_i$  are orthogonal and  $\sum f_i = 1$ . A nonzero right  $R$ -module  $M$  is said to *belong* to the block  $B_i$  (alternatively,  $B_i$  “contains  $M$ ”) if  $Mf_j = 0 \ \forall j \neq i$ . If  $M$  is indecomposable, then obviously  $M$  belongs to a unique block. Now if  $R$  is semisimple, then each block is a matrix algebra. More generally, suppose  $R$  is a “quasi-Frobenius” algebra, i.e. every projective over  $R$  is injective. (For example, group algebras are quasi-Frobenius). Then:

**PROPOSITION 2.1.** *If  $R$  is quasi-Frobenius, a block  $B$  of  $R$  is a matrix algebra if and only if  $B$  contains a projective irreducible module.*

*Proof.* First recall (see [9], p. 378) that two indecomposables  $U, V$  are *linked* if there is a finite sequence  $U = U_0, U_1, \dots, U_n = V$  of indecomposables such that  $U_i$  and  $U_{i+1}$  have a common irreducible constituent (i.e. composition factor) for each  $i$ . (Curtis and Reiner use only “principal” indecomposables, but this makes no difference.) This defines an equivalence relation on the set of indecomposable modules. Moreover it is true (over any finite-dimensional algebra) that  $U$  and  $V$  are linked if and only if they belong to the same block ([9], Theorem 55.2).

Now suppose  $B$  contains a projective irreducible  $N$ . Since  $N$  is also injective, it is a direct summand of any module in which it occurs as a composition factor. Hence the linking class of  $N$  consists solely of  $N$  itself. But this means every  $B$ -module is a direct sum of copies of  $N$ , and the classical Artin–Wedderburn theory then implies  $B$  is a matrix algebra over some  $K$ -central division algebra.

The converse is a standard fact.

Now take  $R = \mathbb{F}_p[GL_n\mathbb{F}_q]$ . If  $H$  is a subgroup of  $GL_n\mathbb{F}_q$ , we let  $\bar{H} = \sum_{h \in H} h$  (if  $H \not\subseteq \Sigma_n$ ) and  $\bar{H} = \sum_{h \in H} \epsilon(h)h$  (if  $H \subseteq \Sigma_n$ ); here  $\epsilon: \Sigma_n \rightarrow \{\pm 1\}$  is the usual map.

**Definition 2.2.** The Steinberg idempotent  $e_n$  is defined by  $e_n = \bar{B}_n \bar{\Sigma}_n / [GL_n: U_n]$ ; the corresponding module  $St = e_n R$  is called the *Steinberg module*.

**THEOREM 2.3.** (Steinberg[10]), (a)  $e_n$  is idempotent; (b)  $St$  is projective and absolutely irreducible; (c) as a  $U_n$ -module,  $St$  is the regular representation. In particular  $\dim St = q^{\binom{n}{2}}$  with basis  $\{e_n u: u \in U_n\}$ .

**Remark 2.4.** By Proposition 2.1, the block  $B_{St}$  containing  $St$  is a matrix algebra over  $\mathbb{F}_p$  of degree  $q^{\binom{n}{2}}$ .

**Remark.** Steinberg originally defined  $St$  as a certain composition factor of the permutation representation obtained from the action of  $GL_n$  on the flag complex  $F(V^n)$ . Later, Solomon and Tits showed that  $F(V^n)$  has the homotopy type of a wedge of  $q^{\binom{n}{2}}(n-2)$ -spheres, and that  $St$  is the representation of  $GL_n$  on the cohomology group  $N^{n-2}(V^n)$ . Yet another description of  $St$  is given in (5.12) below.

Now suppose  $K \subseteq H \subseteq GL_n, H \not\subseteq \Sigma_n$ , and let  $H = \cup_i h_i K$  (left coset decomposition). Then clearly  $\bar{H} = (\sum h_i) \bar{K}$ . If  $K$  is normal in  $H$ , then also  $(\sum h_i) \bar{K} = \bar{K} (\sum h_i)$ . Similar remarks apply if  $H \subseteq \Sigma_n$ . For example,  $\bar{B}_n = \bar{A}_n \bar{B}_{n-1} = \bar{B}_{n-1} \bar{A}_n, \bar{\Sigma}_n = \bar{\Sigma}_{n-1} \bar{T}_n, \bar{B}_n = \bar{D}_n \bar{U}_n = \bar{U}_n \bar{D}_n$ , and  $\bar{A}_n \bar{\Sigma}_{n-1} = \bar{\Sigma}_{n-1} \bar{A}_n$ . The following inductive formula is then immediate:

**PROPOSITION 2.5.**  $e_n = e_{n-1} \bar{A}_n \bar{T}_n / (q^n - 1)$ .

Our last proposition will be needed in Section 6. Let  $e'_n = \bar{\Sigma}_n \bar{B}_n / [GL_n; U_n]$ .

**PROPOSITION 2.6.** (a)  $e'_n$  is a primitive idempotent belonging to the Steinberg block  $B_{St}$ . For any  $M$  belonging to  $B_{St}$ ,  $Me'_n = M^{B_n}$ ; (b) for any  $GL_n$ -module  $W$ , there are vector space isomorphisms  $We_n \xrightarrow{\cong} We'_n$  and  $We'_n \xrightarrow{\cong} We_n$  given by  $\bar{B}_n, \bar{\Sigma}_n$  (respectively).

*Proof.* Since  $e'_n$  is the conjugate of  $e_n$  in the Hopf algebra  $\mathbb{F}_p[GL_n]$ ,  $e'_n$  is a primitive idempotent. Now by Theorem 2.3,  $St^{B_n}$  is equal to  $St\bar{B}_n = \langle e_n \bar{B} \rangle$  and has dimension one. Thus  $e'_n$  is the identity on  $St^{B_n}$ . In particular  $Ste'_n \neq 0$ , so  $e'_n$  belongs to  $B_{St}$ . This also shows  $Me'_n = M^{B_n}$  for  $M$  belonging to  $B_{St}$ , since such an  $M$  is just a direct sum of copies of  $St$  (by Remark 2.4). (b) is obvious.

**§3.  $B(\mathbb{Z}/p)^n$  AND ASSOCIATED SPECTRA**

Let  $L^{2n+1}$  denote the lens manifold  $S^{2n+1}/(\mathbb{Z}/p)$ . We identify  $B\mathbb{Z}/p$  with  $L^\infty = \lim L^{2n+1}$  and  $B(\mathbb{Z}/p)^n$  with  $\prod_1^n B\mathbb{Z}/p$ . The canonical complex line bundle  $\lambda$  over  $B\mathbb{Z}/p$  is  $S^\infty \times_{\mathbb{Z}/p} \mathbb{C}$ , where  $\mathbb{Z}/p$  acts on  $\mathbb{C}$  via the standard inclusion  $\mathbb{Z}/p \subset S^1$ . Let  $P_n = H^*B(\mathbb{Z}/p)^n$ . Then, at odd primes,  $P_1 = E[x] \otimes \mathbb{Z}/p[y]$ , where  $y = c_1(\lambda)$  and  $\beta x = y$ . From the Künneth theorem we then have

$$P_n = E[x_1, \dots, x_n] \otimes \mathbb{Z}/p[y_1, \dots, y_n]. \tag{3.1}$$

For  $p = 2$ ,  $P_n = \mathbb{Z}/2[x_1, \dots, x_n]$ . However, in order to avoid separating cases, we will make use of the following device: Let  $y_i = x_i^2$ , and replace  $P_n$  by the quotients of the filtration

$$0 \rightarrow P_n^2 \rightarrow P_n \rightarrow P_n/P_n^2 \rightarrow 0$$

where  $P_n^2$  denotes the subring of squares. Then (3.1) becomes valid for all primes. In particular (3.1) describes  $P_n$  as a module over the Steenrod algebra.

Now  $GL_n = GL_n \mathbb{Z}/p$  acts on  $(\mathbb{Z}/p)^n$  and hence on the homotopy type  $B(\mathbb{Z}/p)^n$  (on the left). The resulting right action on  $P_n$  is then the obvious one implied by (3.1) (with our usual proviso for  $p = 2$ ). As explained in [11, §1], for each idempotent  $e \in \mathbb{Z}/p[GL_n]$  we obtain a stable summand  $X$  of  $B(\mathbb{Z}/p)^n$  with cohomology  $P_n e$ . We will use the notation  $e \cdot B(\mathbb{Z}/p)^n$  for  $X$ . For example, let  $d_n = \bar{D}_n / (p - 1)^n$ , where  $D_n$  is the diagonal subgroup and  $\bar{D}_n = \sum_{g \in D_n} g$ . Then  $d_n$  is idempotent and we have the following well known fact:

**PROPOSITION 3.2.** *The map  $B(\mathbb{Z}/p)^n \rightarrow B(\Sigma_p)^n$  induced by the inclusion  $(\mathbb{Z}/p)^n \subset (\Sigma_p)^n$  restricts to an equivalence  $d_n \cdot B(\mathbb{Z}/p)^n \xrightarrow{\cong} B(\Sigma_p)^n$ .*

The transfer provides an explicit inverse. Note that

$$H^*B(\Sigma_p)^n = P_n^{D_n} = E[x_1 y_1^{p-2}, \dots, x_n y_n^{p-2}] \otimes \mathbb{Z}/p[y_1^{p-1}, \dots, y_n^{p-1}].$$

**3.3 Thom spectra**

We will need to consider various Thom spectra, and quotients of Thom spectra, over these classifying spaces. The following notation is very convenient: For any finite group  $G$  and representation  $\theta$  of  $G$ , we use the same letter  $\theta$  to denote the corresponding vector bundle over  $BG$ . In fact in place of  $\theta$ , we could take any element of the complex representation ring

$R_C(G)$ . For example, if  $\alpha$  is the reduced regular representation of  $(\mathbb{Z}/p)^n$  (i.e. the regular representation minus a trivial one-dimensional representation), then  $(B(\mathbb{Z}/p)^n)^\alpha$  is the Thom spectrum of the sum of all the nontrivial line bundles over  $B(\mathbb{Z}/p)^n$ . When  $n = 1$  and  $\lambda: \mathbb{Z}/p \rightarrow S^1$  is the standard representation mentioned above, we write  $L_{2k}^\infty$  for  $(B\mathbb{Z}/p)^{k\lambda}$  ( $k \in \mathbb{Z}$ ) and  $L_{2k+1}^\infty$  for  $L_{2k}^\infty/S^{2k}$ . When  $\beta: \Sigma_p \rightarrow U(p-1)$  is the reduced standard representation; we write  $P_{kq}^\infty$  for  $(B\Sigma_p)^{k\beta}$  ( $q = 2(p-1)$ ) and  $P_{(k+1)q-1}^\infty$  for  $P_{kq}^\infty/S^{kq}$ . (Note that  $P_{kq}^\infty$  has cells only in dimensions congruent to 0 or  $-1 \pmod{q = 2(p-1)}$ . Note also that for  $p = 2$ , this definition of  $P_k^\infty$  agrees with the usual one based on the canonical real line bundle.) In this notation, we have  $B(\mathbb{Z}/p)_+^n = \Lambda^n L_0^\infty$ ,  $B(\Sigma_p)_+^n = \Lambda^n P_0^\infty$ , etc.

The cohomology of these spectra is very easy to describe. Let  $S_n$  denote the localization of  $P_n$  obtained by inverting all nonzero linear forms in  $y_1, \dots, y_n$  (i.e. all elements of  $V^n - 0$ ). By a theorem of Wilkerson[12],  $S_n$  has a unique  $A$ -module structure extending that of  $P_n$ . Then the cohomology of virtually every spectrum considered in this paper can be regarded in a natural way as an  $A$ -submodule of  $S_n$ . For example, if  $\theta \in R_C((\mathbb{Z}/p)^n)$  then  $\theta$  has an ‘‘Thom class’’  $e(\theta) \in S_n$ , and  $H^*(B(\mathbb{Z}/p)^n)^\theta$  is the (free)  $P_n$ -submodule of  $S_n$  generated by  $e(\theta)$ . (Note this is also an  $A$ -submodule). We list here a few explicit descriptions that we will need; further examples are left to the reader.

*Examples 3.4*

- (a)  $H^*L_{2k+\epsilon}^\infty$  ( $\epsilon = 0$  or  $1$ ) is the  $P_1$ -submodule of  $S_1$  generated by  $y^k$  (or  $xy^k, y^{k+1}$  if  $\epsilon = 1$ ).
- (b)  $H^*(B(\mathbb{Z}/p)^n)^{-\alpha}$  is the  $P_n$ -submodule of  $S_n$  generated by  $l_n^{-1}$ , where  $l_n = \prod_{a \in V^n - 0} a$  is the Euler class of  $\alpha$ .
- (c)  $H^*\Lambda^n P_{-1}^\infty$  is the  $H^*B(\Sigma_p)^n$ -submodule of  $S_n$  generated by  $X_n Y_n^{-1}$ , where  $X_n = x_1 \dots x_n$  and  $Y_n = y_1 \dots y_n$ .

We emphasize that in all of these examples the  $A$ -module structure follows from the Cartan formula together with the action of  $A$  on the ‘‘Thom class’’ in the lowest dimension. This in turn is determined by the standard formulas,  $P^i y^k = \binom{k}{i} y^{k+i(p-1)}$ ,  $\beta x = y$  where  $\dim x = 1$ ,  $\dim y = 2$  and  $k$  is allowed to be negative. In fact, we make no essential use of Wilkerson’s result, since all of our  $A$ -modules will actually be submodules of the  $A$ -module of example (b).

Of particular importance for us is the  $A$ -submodule  $M_n$  of  $H^*\Lambda^n P_{-1}^\infty$  generated by  $X_n Y_n^{-1}$ .

**PROPOSITION 3.5.**  $M_n = \Sigma^{-n} A / G_n$ . Moreover  $M_n \cap P_n$  has basis  $\{\theta^l(X_n Y_n^{-1}) : I \text{ admissible, } l(I) = n\}$ .

*Proof.* Define a filtration  $\omega$  on  $H^*\Lambda^n P_{-1}^\infty$  as follows: given an  $n$ -tuple  $(a_1, \dots, a_n)$ ,  $a_i \geq -1$ , let  $z \in \omega(a_1, \dots, a_n)$  iff

(1)  $z = x_1^{\epsilon_1} \dots x_n^{\epsilon_n} y_1^{f_1} \dots y_n^{f_n}$ ,  $e_i \in \{0, 1\}$ ,  $f_i \geq -1$  and  $(f_1, \dots, f_n) \leq (a_1, \dots, a_n)$  in the lexicographical order (starting at the left), or

(2)  $z$  is a linear combination of monomials, each of which is in  $\omega(a_1, \dots, a_n)$ .

Then  $\omega(a_1, \dots, a_n) \subset \omega(a'_1, \dots, a'_n)$  if  $(a_1, \dots, a_n) \leq (a'_1, \dots, a'_n)$ .

Now for  $I = (\epsilon_0, r_1, \epsilon_1, r_2, \dots, \epsilon_{n-1}, r_n)$  define  $Y_n^I = y_1^{k_1}, \dots, y_n^{k_n}$ , where  $k_i = r_i(p-1) + \epsilon_{i-1}$ , and  $X_n^I = x_1^{1-\epsilon_0}, \dots, x_n^{1-\epsilon_{n-1}}$ .

**LEMMA 3.6.** If  $I$  is admissible and  $l(I) \leq n$ ,  $\theta^l(X_n Y_n^{-1}) = \pm X_n^I Y_n^I$  modulo terms of lower filtration ( $X^I Y^I \in \omega(k_1, \dots, k_n)$ ).

*Proof of Lemma.* For  $n = 1$ , the lemma is clear; suppose inductively it is true for  $n - 1$ . By the Cartan formula  $\theta^l(X_n Y_n^{-1}) = \Sigma \pm \theta^{J_1}(x_1 y_1^{-1}), \dots, \theta^{J_n}(x_n y_n^{-1})$  where the sum is taken over all sequences  $J_1, \dots, J_n$  with  $\Sigma J_i = I$ . Those terms with  $J_1 = (\epsilon_0, r_1)$  can be

grouped as  $\theta^{(\epsilon_0, r_1)}(x_1, y_1^{-1})\theta^{I'}(X_{n-1}Y_{n-1}^{-1})$  where  $I' = (\epsilon_1, r_2, \dots, \epsilon_{n-1}, r_n)$ . By induction the sum of these terms equals  $\pm X_{n-1}^I Y_{n-1}^I \pmod{\text{elements of lower filtration}}$ . It remains to consider terms with  $l(J_1) > 1$ . For such admissible  $J_1$ ,  $\theta^{J_1}(x_1, y_1^{-1}) = 0$  for dimensional reasons. For such inadmissible  $J_1$  write  $J_1 = (\epsilon'_0, r'_1, \dots, \epsilon'_{n-1}, r'_n)$ . The Adem relations show that the only admissible summand of length 1 in  $\theta^{J_1}$  is  $c\beta^\epsilon P^r$  where  $c \in \mathbb{Z}/p$ ,  $\epsilon = 1$  if  $\sum \epsilon'_i = 1$ ,  $\epsilon = 0$  if  $\sum \epsilon'_i = 0 \pmod 2$  and  $r = \sum r'_i$  (note that  $c = 0$  if  $\sum \epsilon'_i > 1$ ). If  $\sum \epsilon'_i = 0$  then

$$c = \binom{(r'_2 + \dots + r'_n)(p-1) - 1}{r'_1} \binom{(r'_3 + \dots + r'_n)(p-1) - 1}{r'_2} \dots \binom{r'_n(p-1) - 1}{r'_{n-1}}$$

and  $c \neq 0$  implies  $r'_i < (r'_{i+1} + \dots + r'_n)(p-1)$ . Hence  $\sum r'_i < p^{n-1}r'_n \leq p^{n-1}r_n \leq r_1$  and so  $\theta^{J_1}(x_1, y_1^{-1}) \dots \theta^{J_n}(x_n, y_n^{-1})$  has filtration less than that of  $X^I Y^I$ . The case of  $\sum \epsilon'_i = 1$  is similar. This completes the proof of the lemma.

From the lemma it is immediate that the set  $\{\theta^I(X_n Y_n^{-1}), I \text{ admissible and } l(I) \leq n\}$  is independent. Moreover it is easy to see that  $l(I) = n$  iff  $\theta^I(X_n Y_n^{-1}) \in P_n$ . It then follows for dimensional reasons that the ideal  $G_n$  annihilates  $X_n Y_n^{-1}$ .

### 3.7 Transfer

We conclude this section with a discussion of the various transfer maps that we will need. Suppose  $X$  is a CW-complex,  $\eta$  is an  $n$ -dimensional complex vector bundle over  $X$  and  $\xi$  is a stable complex vector bundle over  $X$  (i.e. a map to  $BU$ ). Then the inclusion of  $\xi$  in the Whitney sum  $\eta \oplus \xi$  induces a map of Thom spectra  $X^\xi \xrightarrow{t} X^{\eta \oplus \xi}$ ; this is the transfer associated to  $\eta, \xi$ . (A quite general discussion of transfer maps can be found in [13]. We leave it to the reader to discover in what sense the construction described here is a special case of that of [13].) The following is well known:

PROPOSITION 3.8. *The following diagram commutes*

$$\begin{array}{ccc} H^* X^\xi & \xleftarrow{t^*} & H^* X^{\eta \oplus \xi} \\ \cong \uparrow & \cup e(\eta) & \uparrow \cong \\ H^* X & \xleftarrow{} & H^* X \end{array}$$

where  $\cup e(\eta)$  denotes cup product with the mod  $p$  Euler class  $e(\eta)$  and the vertical maps are Thom isomorphisms.

Remark 3.9. The proposition is in fact true for any cohomology theory  $E$  such that  $\eta$  and  $\xi$  are  $E$ -oriented.

Example 3.10. There is a transfer  $(B(\mathbb{Z}/p)^n)^{-\alpha} \xrightarrow{t} B(\mathbb{Z}/p)^n_+$ . The map  $t^*: P_n \rightarrow P_n \cdot l_n^{-1}$  is the obvious one, by (3.8).

Example 3.11.  $(B(\mathbb{Z}/p)^{-\alpha}) \xrightarrow{t} (B(\mathbb{Z}/p)^{-\lambda}) = L^{\infty}_2$ . Again the map  $t^*: P_1 \cdot y^{-1} \rightarrow P_1 \cdot y^{-(p-1)}$  is the obvious one.

Composing with the quotient map  $L^{\infty}_2 \rightarrow L^{\infty}_1$  in example (3.11), we obtain a map  $(B(\mathbb{Z}/p)^{-\alpha}) \rightarrow L^{\infty}_1$ . Maps of this type will also be referred to as ‘‘transfers’’.

Note that  $GL_1 \mathbb{Z}/p$  acts on  $(B(\mathbb{Z}/p))^{-\alpha}$ , and that  $\beta$  (of 3.3) restricted to  $\mathbb{Z}/p$  is  $\alpha$ . The final result of this section is straightforward; its proof will be left to the reader.

PROPOSITION 3.12. *The induced map of Thom spectra  $\phi: (B(\mathbb{Z}/p))^{-\alpha} \rightarrow (B(\Sigma_p))^{-\beta} = P^{\infty}_q$*

restricts to an equivalence  $d_1 \cdot (B\mathbb{Z}/p)^{-\alpha} \cong P_{-q}^\infty$ . Moreover there is a commutative diagram

$$\begin{array}{ccc} (B\mathbb{Z}/p)^{-\alpha} & \xrightarrow{\phi} & P_{-q}^\infty \\ t \downarrow & & \downarrow \psi \\ L_{-1}^\infty & \xrightarrow{\quad} & P_{-1}^\infty \\ \downarrow & & \downarrow \bar{\psi} \\ L_0^\infty & \xrightarrow{\quad} & P_0^\infty \end{array}$$

where  $t$  is the transfer and the unlabelled maps are the obvious ones. Moreover  $\psi$  and  $\bar{\psi}$  are stable retractions; in particular  $P_{-1}^\infty$  is a summand of  $L_{-1}^\infty$ .

Of course  $\bar{\psi}$  is just the retraction of (3.2).

§4. SYMMETRIC PRODUCT SPECTRA

If  $X$  is a space and  $H$  is a subgroup of  $\Sigma_n$ ,  $Sp^H X$  is the orbit space  $X^n/H$ . If  $H = \Sigma_n$ , we write  $Sp^n$  in place of  $Sp^H$ . If  $X = \{X_k, \epsilon_k\}$  is a spectrum with structure maps  $\epsilon_k: S^1 \wedge X_k \rightarrow X_{k+1}$ , then  $Sp^H X$  is the spectrum  $\{Sp^H X_k, Sp^H(\epsilon_k) \circ f_k\}$ , where  $f_k: S^1 \wedge Sp^H X_k \rightarrow Sp^H(S^1 \wedge X_{k+1})$  is defined by  $f_k(t \wedge (x_1 \dots x_n)) = (t \wedge x_1 \dots t \wedge x_n)$ . Thus  $Sp^H$  becomes a functor on the stable category; for further details the reader may consult [14].

The natural inclusions  $Sp^n X \subseteq Sp^{n+1} X$  allow us to define  $Sp^\infty X = \varinjlim Sp^n X$  for a spectrum  $X$ . By the Dold–Thom theorem,  $Sp^\infty S^0 = K\mathbb{Z}$ ; in particular  $H^* Sp^\infty S^0 = A/A\beta$ .

**THEOREM 4.1.** (Nakaoka [15]). *The inclusions  $Sp^n S^0 \rightarrow Sp^\infty S^0$  are surjective on cohomology. Moreover  $H^* Sp^\infty S^0$  has basis  $\{\theta^l: l \text{ admissible, } l(I) \leq n, \theta^l \notin A\beta\}$ .*

4.2 The spectrum  $D(n)$

If  $M\mathbb{Z}/p$  is the mod  $p$  Moore spectrum, then  $Sp^\infty M\mathbb{Z}/p = K\mathbb{Z}/p$ . In view of Theorem (4.1) it is natural to ask whether the finite symmetric products  $Sp^{p^n} M\mathbb{Z}/p$  realize the Cartan–Serre filtration  $G_n$  on  $A = H^* K\mathbb{Z}/p$ . The answer is no; it can easily be seen from Remark (4.5) that the filtration provided by the  $Sp^{p^n} M\mathbb{Z}/p$  is slightly different. Instead we use the following construction: On the space level there are obvious  $p$ -fold diagonal maps  $Sp^{p^{n-1}} S^k \rightarrow Sp^{p^n} S^k$ ; these induce maps of spectra  $Sp^{p^{n-1}} S^0 \xrightarrow{d} Sp^{p^n} S^0$ . Let  $D(n)$  denote the cofibre of  $d$ . Now clearly  $d^*$  is zero on  $H^0 Sp^{p^n} S^0$ ; hence by (4.1)  $d^*$  is zero on all of  $H^*$ . In other words, the cofibration  $Sp^{p^n} S^0 \rightarrow D(n) \rightarrow \Sigma Sp^{p^{n-1}} S^0$  has a short exact cohomology sequence. Letting  $u_n \in H^0 D(n)$  denote a generator, the following proposition is now evident:

**PROPOSITION 4.3.** *There are commutative diagrams*

$$\begin{array}{ccc} D(n-1) & \xrightarrow{j_n} & D(n) \\ i_{n-1} \searrow & & \swarrow i_n \\ & K\mathbb{Z}/p & \end{array}$$

such that  $i_n^*$  is surjective with kernel  $G_n$  for all  $n$ . In particular,  $H^* D(n)$  has basis  $\{\theta^l(u_n): l \text{ admissible, } l(I) \leq n\}$ .

Frequently, the generator  $u_n$  will be omitted from the notation. Note that  $H^* D(1) = H^* \Sigma P_{-1}^\infty$  as  $A$ -modules. In fact:

**PROPOSITION 4.4**  $\Sigma P_{-1}^\infty \cong D(1)$ .

*Proof.* Note that it is enough to exhibit a map  $\Sigma L_{-2} \xrightarrow{f} D(1)$  which is nonzero on  $H^0$ , since we can then use the following composite  $g$ :

$$\Sigma P_{-q}^\infty \xrightarrow{i} \Sigma B(\mathbb{Z}/p)^{-i} \xrightarrow{t} \Sigma L_{-1} \xrightarrow{f} D(1).$$

Here  $i$  is the inclusion of  $\Sigma P_{-q}^\infty$  as a stable summand, as in (3.12) and  $t$  is the transfer. The induced map  $\tilde{g}: \Sigma P_{-1}^\infty = \Sigma P_{-q}^\infty/S^{-q+1} \rightarrow D(1)$  is then clearly an equivalence.

Now let  $-\lambda_n$  denote the complement in  $\mathbb{C}^{n+1}$  of the canonical complex line bundle  $\lambda_n$  over  $L^{2n+1}$ . Thus  $-\lambda_n$  has total space  $\{([x], v) : \langle x, v \rangle = 0\}$  where  $x \in S^{2n+1}$ ,  $[ \ ]$  denotes equivalence class in  $L^{2n+1}$ , and  $\langle, \rangle$  is the usual Hermitian inner product on  $\mathbb{C}^{n+1}$ . Now if  $L_x$  is the complex line spanned by  $x$ , and  $|v| \leq 1$ , then  $L_x + v$  intersects  $S^{2n+1}$  in a circle of radius  $\sqrt{(1 - |v|^2)}$ . Hence we may define a map  $\tilde{f}_n$  from the unit disc bundle  $D(-\lambda_n)$  to  $Sp^p S^{2n+1}$  by  $\tilde{f}_n([x], v) = (\sqrt{(1 - |v|^2)}x + v, \sqrt{(1 - |v|^2)}ax + v, \dots, \sqrt{(1 - |v|^2)}a^{p-1}x + v)$ , where  $a = \exp(2\pi i/p)$ . (In fact  $\tilde{f}_n$  is well defined as a map into the cyclic product  $Sp^{z/p} S^{2n+1}$ ). Moreover, if  $S(-\lambda_n)$  is the unit sphere bundle, we have a commutative diagram of cofibrations:

$$\begin{array}{ccc} S(-\lambda_n) & \rightarrow & S^{2n+1} \\ \downarrow & & \downarrow d \\ D(-\lambda_n) & \xrightarrow{\tilde{f}_n} & Sp^p S^{2n+1} \\ \downarrow & & \downarrow \\ (L^{2n+1})^{-i_n} & \xrightarrow{f_n} & Sp^p S^{2n+1}/d(S^{2n+1}). \end{array}$$

The maps  $f_n$  fit together to yield a map of spectra  $\Sigma(B\mathbb{Z}/p)^{-i} = \Sigma L_{-2} \xrightarrow{f} M(1)$ . To show  $f^*$  is an isomorphism on  $H^0$ , it is enough to show  $(f_n)_*$  is an isomorphism on  $H_{2n+1}$ . Consider the restriction of  $\tilde{f}_n$  to the zero section  $L^{2n+1}$ :  $\tilde{f}_n([x]) = (x, ax, \dots, a^{p-1}x)$ . There is a commutative diagram

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{F} & Sp^p S^{2n+1} \\ \pi \searrow & \nearrow \tilde{f}_n & \\ L^{2n+1} & & \end{array}$$

where  $F$  is the composite

$$S^{2n+1} \xrightarrow{\Delta} (S^{2n+1})^p \xrightarrow{1 \times a \times \dots \times a^{p-1}} (S^{2n+1})^p \rightarrow Sp^p S^{2n+1}.$$

Now  $\pi_*$  is multiplication by  $p$  on  $H_{2n+1}(\cdot; \mathbb{Z})$ . Since  $a^k: S^{2n+1} \rightarrow S^{2n+1}$  has degree one,  $F_*$  is also multiplication by  $p$ . Hence  $(\tilde{f}_n)_*$  is an isomorphism on  $H_{2n+1}$ , and the proposition follows.

*Remark 4.5.* By a theorem of Kan and Whitehead ([16], see also [14]) the functors  $Sp^H$  preserve cofibrations in the category of spectra. An equivalent statement is that the natural map  $Sp^H S^0 \wedge X \rightarrow Sp^H X$  is an equivalence. Now if  $H$  is a wreath product  $K \int L$ , it is easy to see that  $Sp^H X \cong Sp^K(Sp^L X)$  (on the space level, this is actually a homeomorphism). Combining these remarks, we see that if  $H_n = \int^n \Sigma_p$ , then  $Sp^{H_n} S^0 \cong \Lambda^n Sp^p S^0$ .

If  $D'(n)$  is the cofibre of the diagonal  $Sp^{H_{n-1}}S^0 \rightarrow Sp^{H_n}S^0$ , as in the definition (4.2) of  $D(n)$ , there is an analogous equivalence  $D'(n) \cong \Lambda^n D'(1) \cong \Lambda^n \Sigma P_{-1}^x$  (by 4.4). Although we make no essential use of these facts, they are very helpful for understanding symmetric product spectra.

§5. PROOF OF THEOREM A

Let  $M(n) = \Sigma^{-n}(D(n)/D(n-1))$ . It follows from Proposition 4.3 that  $H^*(M(n))$  has basis  $\{\theta^I: I \text{ admissible, } l(I) = n\}$ .

Theorem A is a consequence of the following:

**THEOREM 5.1.** *There is a map  $g: (B(\mathbb{Z}/p)^n \rightarrow M(n))$  such that on mod  $p$  cohomology,  $g^*$  is an isomorphism onto  $[H^*B(\mathbb{Z}/p)^n]e_n$ .*

For it follows that  $g$  restricts to an equivalence  $e_n \cdot B(\mathbb{Z}/p)^n \xrightarrow{\cong} M(n)$ .

Since the Steinberg block  $B_{S_i}$  is a matrix algebra of degree  $p^{(i)}$  over  $\mathbb{F}_p$ , the corresponding central idempotent decomposes into the sum of  $p^{(i)}$  primitive orthogonal idempotents one of which is  $e$ . The corresponding summands of  $B(\mathbb{Z}/p)^n$  are equivalent [11, 1.6]. Thus Theorem A follows from Theorem 5.1.

In fact the map is a very natural one, as we proceed to explain (see [11]). There are maps (of spaces)  $Sp^i S^m \wedge Sp^j S^n \rightarrow Sp^i S^{m+n}$  defined by  $(x_1 \cdot x_2 \dots x_i) \wedge (y_1 \cdot y_2 \dots y_j) \rightarrow (x_1 \wedge y_1 \cdot x_1 \wedge y_2 \dots x_i \wedge y_j)$ . These yield a map of spectra  $Sp^i S^0 \wedge Sp^j S^0 \rightarrow Sp^i S^0$  and by iteration a map  $\mu_0: \Lambda^n Sp^n S^0 \rightarrow Sp^n S^0$ . As noted in [11], by factoring out the appropriate subspectra we obtain a commutative diagram

$$\begin{array}{ccc} \Lambda^n Sp^n S^0 & \xrightarrow{\mu_0} & Sp^n S^0 \\ \downarrow & & \downarrow \\ \Lambda^n \overline{Sp}^n S^0 & \xrightarrow{\bar{\mu}_0} & \overline{Sp}^n S^0 \end{array} \tag{5.2}$$

where  $\overline{Sp}^n S^0 = Sp^n S^0 / Sp^{n-1} S^0$ .

From the definition of  $M(n)$ , it is clear on inspection that (5.2) yields a further commutative diagram

$$\begin{array}{ccc} \Lambda^n D(1) & \xrightarrow{\mu} & D(n) \\ \downarrow & & \downarrow \\ \Lambda^n(\Sigma M(1)) & \xrightarrow{\bar{\mu}} & \Sigma^n M(n). \end{array} \tag{5.3}$$

*Remark 5.4.* In view of the Dold–Thom theorem, the maps  $\mu_0, \mu$  can be viewed as filtrations of the ring spectrum multiplication on  $K\mathbb{Z}, K\mathbb{Z}/p$ . For another interpretation, see Remark (5.7) below.

Finally, from the results of §3 we obtain our main commutative diagram

$$\begin{array}{ccccc} \Lambda^n L_{-1}^x & \xrightarrow{\Lambda^n \psi} & \Lambda^n P_{-1}^x & \xrightarrow{\mu} & \Sigma^{-n} D(n) \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda^n L_0^x & \xrightarrow{\Lambda^n \bar{\psi}} & \Lambda^n P_0^x & \xrightarrow{\bar{\mu}} & M(n). \end{array} \tag{5.5}$$

Here we recall that

$$\Lambda^n L_0^x = B(\mathbb{Z}/p)_-, \quad P_{-1}^x \cong \Sigma^{-1} D(1), \quad \Lambda^n P_0^x = B(\Sigma_p)_-, \quad \text{and } P_0^x \cong M(1).$$

Let  $f = \mu(\Lambda^n \psi)$ ,  $g = \bar{\mu}(\Lambda^n \bar{\psi})$ ; we will show that  $g$  is the required map of Theorem 5.1. Let  $R_n = H^* \Lambda^n L_{-1}^x \subseteq S_n$ ,  $P_n = H^* \Lambda^n L_0^x \subseteq R_n$ , and  $M_n = A$ -submodule of  $R_n$  generated by the bottom class  $X_n Y_n^{-1}$ .

LEMMA 5.6. (a)  $f^*$  is an isomorphism  $H^* \Sigma^{-n} D(n) \rightarrow M_n$ . (b)  $g^*$  is an isomorphism  $H^* M(n) \rightarrow M_n \cap P_n$ .

*Proof.* Since  $f^*(u_n) = X_n Y_n^{-1}$ , (a) is immediate from (3.5) and (4.3). Moreover we have seen in (3.5) that  $M_n \cap P_n$  is precisely  $\langle \theta^l(X_n Y_n^{-1}) : l \text{ admissible and } l(J) = n \rangle$ . (b) then follows from (a), using (5.5).

Remark 5.7. Since our proof of (5.6) relies on Nakaoka's calculation of  $H^* Sp^p S^0$ , in a sense it puts the cart before the horse. In fact one can show directly that  $\mu_0: \Lambda^n Sp^p S^0 \rightarrow Sp^{pn} S^0$  is injective in cohomology, and indeed this is essentially equivalent to a key step in Nakaoka's original proof: As remarked in Section 4,  $\Lambda^n Sp^p S^0 \cong Sp^p(Sp^p(\dots Sp^p S^0)) \dots \cong Sp^H S^0$ , where  $H = \int^n \Sigma_p$ . Moreover, it is easy to see that the resulting map  $Sp^H S^0 \xrightarrow{\pi} Sp^{pn} S^0$  corresponding to  $\mu_0$  is the obvious "projection" associated to the inclusion  $H \subseteq \Sigma_p$ . Now algebraically one can define a transfer  $t^*: H^* Sp^H S^0 \rightarrow H^* Sp^{pn} S^0$  enjoying the usual properties, e.g.  $t^* \pi^* =$  multiplication by the index  $[\Sigma_p n: H]$ . But  $[\Sigma_p n: H]$  is prime to  $p$ , which shows  $\mu_0^*$  is injective.

Lemma (5.6) reduces Theorem (5.1) to the following purely algebraic result:

THEOREM 5.8.  $R_n e_n = M_n$ .

For then  $P_n e_n = P_n \cap R_n e_n = P_n \cap M_n = \text{Img}^*$  by 5.6b. (As usual, we are regarding  $R_n$  as embedded in  $S_n$ ). The proof of Theorem (5.8) is based on the following curious lemma, which relates the action of the Steenrod algebra on  $R_n$  to the action of  $GL_n \mathbb{F}_p$ .

LEMMA 5.9. Let  $J = (j_0, \dots, j_{n-1})$ ,  $j = \sum j_i$ ,  $j_i = 0$  or  $1$ , and let  $I$  be any multiindex of length  $\leq n - 1$ . Then

$$(x_1 y_1^{-1} Q^J P^I (X_{n-1} Y_{n-1}^{-1})) e_n = (-1)^J Q^J P^I (X_n Y_n^{-1}).$$

*Proof of Theorem 5.8.* Taking  $Q^J P^I = 1$  in the lemma we have  $(X_n Y_n^{-1}) e_n = X_n Y_n^{-1}$ , so  $M_n \subseteq R_n e_n$ . To show  $R_n e_n = M_n$  we use induction on  $n$ . For  $n = 1$  this is clear (see 3.4). Now suppose we have shown  $R_{n-1} e_{n-1} = M_{n-1}$ . From (2.5) we have  $e_n = -e_{n-1} \bar{A}_n \bar{T}_n$  and hence

$$R_n e_n = (R_1 \otimes R_{n-1} e_{n-1}) \bar{A}_n \bar{T}_n = (R_1 \otimes M_{n-1}) \bar{A}_n \bar{T}_n = (R_1 \otimes M_{n-1}) e_n.$$

Let  $R'_1 = H^* P_{-1}^\infty = H^*(\Sigma^{-1} D(1))$  (see Prop. 4.3, Ex. 3.4 (iii)). Since  $A_n$  contains the diagonal matrices  $F_p^* \times I_{n-1}$  we have  $(R_1 \otimes M_{n-1}) e_n = (R'_1 \otimes M_{n-1}) e_n$ . Further for any  $A$ -module  $N$ ,  $R'_1 \otimes N$  is generated by  $x_1 y_1^{-1} \otimes N$ . Hence it is enough to show  $(x_1 y_1^{-1} \otimes M_{n-1}) e_n \subseteq M_n$ . But this is immediate from the lemma together with (1.1(c)).

Recall  $V^n$  is the vector space  $\langle y_1, \dots, y_n \rangle$ . To prove the lemma we will need:

PROPOSITION 5.10. Let  $\alpha_{n,k} = \sum_{a \in V^n} a^k$ . Then if  $k = ip^r$  with  $0 \leq i < p^n - p^{n-1}$ ,  $\alpha_{n,k} = 0$ .

*Proof.* It is enough to prove the case  $r = 0$ . Clearly  $\alpha_{n,k}$  is a  $GL_n$  invariant. But by a classical theorem of Dickson [17], the smallest nonzero dimension in which such an invariant occurs is  $2(p^n - p^{n-1})$  for  $p$  odd and  $2^{n-1}$  for  $p = 2$ .

*Proof of Lemma 5.9.* Fix  $p > 2$ . For  $n = 1$  the lemma merely states that  $(x_1 y_1^{-1}) e_1 = x_1 y_1^{-1}$ ; this is clear since  $e_1 = d_1$ . From now on we assume  $n > 1$ . We consider first the case  $Q^J = 1$ . Suppose inductively we have shown the special case  $(X_{n-1} Y_{n-1}^{-1}) e_{n-1} = X_{n-1} Y_{n-1}^{-1}$ . Let  $\pi_i: V^n \rightarrow F_p$  denote the coordinate projections. Then

$$\begin{aligned}
 x_1 y_1^{-1} P^l(X_{n-1} Y_{n-1}^{-1}) e_n &= -x_1 y_1^{-1} P^l(X_{n-1} Y_{n-1}^{-1}) \bar{A}_n \bar{T}_n \text{ (by (2.5) and inductive hypothesis)} \\
 &= -X_n y_1^{-1} P^l(Y_{n-1}^{-1}) \bar{A}_n \bar{T}_n \\
 &= -X_n \sum_{i=1}^n \sum_{a \in V^{n-0}} \pi_i(a) a^{-1} P^l(y_1^{-1} \dots \hat{y}_i^{-1} \dots y_n^{-1}) \\
 &= -X_n \sum_{a \in V^{n-0}} a^{-1} \sum_{i=1}^n \pi_i(a) P^l(y_1^{-1} \dots \hat{y}_i^{-1} \dots y_n^{-1}) \\
 &= -X_n \sum_{a \in V^{n-0}} a^{-1} P^l(a Y_n^{-1}).
 \end{aligned}$$

Now  $\Delta P^l = \sum \theta_j' \otimes \theta_j''$  with  $l(\theta_j')$ ,  $l(\theta_j'') \leq n - 1$ . Hence  $P^l(a Y_n^{-1}) = \sum_{k=0}^{n-1} a^{pk} \theta_k(Y_n^{-1})$  for certain  $\theta_k$  independent of  $a$ , with  $\theta_0 = P^l$ . We then have

$$\begin{aligned}
 -X_n \sum_{a \in V^{n-0}} a^{-1} P^l(a Y_n^{-1}) &= -X_n \sum_{a \in V^{n-0}} a^{-1} \sum_{k=0}^{n-1} a^{pk} \theta_k(Y_n^{-1}) \\
 &= -X_n P^l Y_n^{-1} \left( \sum_{a \in V^{n-0}} a^0 \right) - X_n \sum_{k=1}^{n-1} \theta_k Y_n^{-1} (\alpha_{n,p} k_{-1}) \\
 &= P^l(X_n Y_n^{-1}). \text{ (Using 5.10).}
 \end{aligned}$$

For the general case consider the equation

$$(x_1 y_1^{-1} \theta(X_{n-1} Y_{n-1}^{-1})) \bar{A}_n \bar{T}_n = \pm \theta(X_n Y_n^{-1}), \quad \theta \in A. \tag{5.11}$$

Then it is enough to show that if (5.11) holds for  $\theta$ , then it holds for  $Q_i \theta$  if  $0 \leq i \leq n - 2$  (but with opposite sign). Since  $Q_i$  is primitive, by applying  $Q_i$  to both sides of 5.11 we are reduced to showing  $((Q_i x_1 y_1^{-1}) \theta(X_{n-1} Y_{n-1}^{-1})) \bar{A}_n \bar{T}_n = 0$ . But in fact

$$\begin{aligned}
 ((Q_i x_1 y_1^{-1}) \theta(X_{n-1} Y_{n-1}^{-1})) \bar{A}_n &= (y_1^{p^i-1} \theta(X_{n-1} Y_{n-1}^{-1})) \bar{A}_n = (\alpha_{n,p^i-1} - \alpha_{n-1,p^i-1}) \theta(X_{n-1} Y_{n-1}^{-1}) \\
 &= 0.
 \end{aligned}$$

By (5.10). This completes the proof if  $p > 2$ . The proof for  $p = 2$  is similar but easier if we use the elements  $Sq^l$ . Then

$$\begin{aligned}
 x_1^{-1} Sq^l(X_{n-1}^{-1}) e_n &= x_1^{-1} Sq^l(X_{n-1}^{-1}) \bar{A}_n \bar{T}_n = \sum_{i=1}^n \sum_{a \in W^{n-0}} \pi_i(a) a^{-1} Sq^l(x_1 \dots \hat{x}_i^{-1} \dots x_n) \\
 &= \sum_{a \in W^{n-0}} a^{-1} Sq^l(a X_n^{-1}) = Sq^l(X_n^{-1})
 \end{aligned}$$

where  $W^n$  is the vector space  $\langle x_1, \dots, x_n \rangle$ . This finishes the proof of the lemma, and the proof of Theorem 5.8.

*Remark 5.12.* Lemma (5.9) shows  $(X_n Y_n^{-1})$  is fixed by  $e_n$  (over any finite field  $\mathbb{F}_q$ ). It follows that the Steinberg module can be described as the  $GL_n \mathbb{F}_q$  submodule of  $E[x_1, \dots, x_n] \otimes \mathbb{Z}/p[y_1, \dots, y_n]$  generated by  $X_n Y_n^{-1}$ .

*Remark 5.13.* Theorem 5.8 determines the multiplicity of the Steinberg module in  $E[x_1, \dots, x_n] \otimes \mathbb{Z}/p[y_1, \dots, y_n]$ . Let  $f(s, t) = \sum a_{ij} s^i t^j$  where  $a_{ij}$  is the multiplicity of  $St$  in

$E[x_1, \dots, x_n]_{n-1} \otimes \mathbb{Z}/p[y_1, \dots, y_n]_p$ . Then using (5.8) we obtain

$$f(s, t) = t^{-n} \frac{\prod_{i=0}^{n-2} (1 + st^{2^{p^i}-2})}{\prod_{i=1}^n (1 - t^{2^{(p^i-1)}})} (st^{2^{p^n-1}-2} + t^{2^{(p^n-1)}}).$$

*Remark 5.14.* Since  $B(\Sigma_p)^n \cong d_n \cdot B(\mathbb{Z}/p)^n$ , and  $d_n$  commutes with  $\bar{U}_n$  and  $\bar{\Sigma}_n$ ,  $e_n$  restricts naturally to a self-map of  $B(\Sigma_p)^n$ . Hence  $M(n)$  is a stable summand of  $B(\Sigma_p)^n$ .

Let  $L(n) = \Sigma^{-n} S p^{p^n}(S^0)$ . We conclude this section by proving

**PROPOSITION 5.15.**  $M(n) \cong L(n) \vee L(n - 1)$ .

*Proof.* By definition, there is a cofibration  $L(n) \rightarrow M(n) \rightarrow L(n - 1)$ , with the resulting cohomology sequence short exact (§4). Hence it will be enough to produce a map  $h: M(n) \rightarrow L(n)$  such that  $h^*$  is an isomorphism onto  $\langle \theta^i: \epsilon_{n-1} = 0 \rangle$ . Let  $H$  be the composite  $\Lambda^n \Sigma P_0^\infty \rightarrow \Lambda^n \Sigma P_1^\infty \xrightarrow{\bar{\mu}_0} \overline{S p^{p^n} S^0}$ , where  $\bar{\mu}_0$  is as in (5.2) (recall  $\Sigma P_1^\infty \cong \overline{S p^p S^0}$ ). By Theorem (5.1) and Remark 5.14,  $M(n)$  is a stable summand of  $\Lambda^n \Sigma P_0^\infty$ . From diagram (5.2), it is clear that a map  $h$  with the desired property is obtained by restricting  $H$  to  $M(n)$ .

**§6. SPLITTING  $B(\int^n \mathbb{Z}/p)$  AND  $B0(n)$**

Regarding  $\Sigma_{p^n}$  as the permutation group of the set  $\mathbb{F}_p^n$ , one obtains an embedding of the affine group  $Aff_n(\mathbb{F}_p) = GL_n \mathbb{F}_p \tilde{\times} \mathbb{F}_p^n$  in  $\Sigma_{p^n}$ . In particular this defines an inclusion  $j: \mathbb{F}_p^n \rightarrow \Sigma_{p^n}$  (as the group of translations) with Weyl group  $W_{\Sigma_{p^n}}(\mathbb{F}_p^n) = GL_n$ . Now the wreath product embeds  $\int^n \mathbb{Z}/p \subset \Sigma_{p^n}$  as a  $p$ -Sylow subgroup and factors  $j$

$$\begin{array}{ccc} \mathbb{F}_p^n & \xrightarrow{j} & \Sigma_{p^n} \\ & \searrow j & \cup \\ & & \int^n \mathbb{Z}/p. \end{array}$$

This embedding can be chosen so that  $Aff_n \cap \int^n \mathbb{Z}/p = U_n \tilde{\times} \mathbb{F}_p^n$ , and  $W_{\int^n \mathbb{Z}/p}(\mathbb{F}_p^n) = U_n$ . Similarly,  $\int^n \Sigma_p \subset \Sigma_{p^n}$  and  $Aff_n \cap \int^n \Sigma_p = B_n \tilde{\times} \mathbb{F}_p^n$ ; then  $W_{\int^n \Sigma_p}(\mathbb{F}_p^n) = B_n$ . Letting  $t: B \int^n \mathbb{Z}/p \rightarrow B(\mathbb{Z}/p)^n$  denote the transfer associated to  $j'$  we then have the following easy consequence of the double coset formula (see [11], Proposition 1.4).

**LEMMA 6.1.**  $j'^* t^* = \bar{U}_n$ .

*Proof of Theorem B.* From the lemma and (2.6(b)), we see that  $tj'$  restricts to an equivalence  $e'_n \cdot B(\mathbb{Z}/p)^n \xrightarrow{\cong} e_n \cdot B(\mathbb{Z}/p)^n$ . Hence Theorem B follows from Theorem A.

*Remark.* Since  $e_n$  and  $e'_n$  commute with  $d_n$ , it follows that the summand  $M(n)$  of  $B \int^n \mathbb{Z}/p$  actually is a summand of  $B \int^n \Sigma_p$ .

*Proof of Theorem C.* The inclusion of  $(\mathbb{Z}/2)^n$  in  $0(n)$  as the diagonal matrices yields a map  $B(\mathbb{Z}/2)^n \xrightarrow{i} B0(n)$  with fibre the flag manifold  $0(n)/(\mathbb{Z}/2)^n$ . Let  $t: B0(n) \rightarrow B(\mathbb{Z}/2)^n$  be the associated Becker–Gottlieb transfer.

**LEMMA 6.2.**  $i^* t^* = \bar{\Sigma}_n[18]$ .

As in the proof of Theorem B, it follows that  $i$  restricts to an equivalence  $e_n \cdot B(\mathbb{Z}/2)^n \xrightarrow{\cong} e'_n \cdot B(\mathbb{Z}/2)^n$ . Hence Theorem C follows from Theorem A.

**§7. SPLITTING  $BT^n$  AND  $BU(n)$**

In this section all spectra are completed at  $p$ . We begin by observing that  $GL_n(\mathbb{Z}_p)$  acts on  $T^n = B(\mathbb{Z}_p)^n$ ,  $BT^n$ , and hence diagonally on  $T^+_+ \wedge BT^+_+ = (T^n \times BT^n)_+$ . In mod- $p$  cohomology

$$H^*(T^n \times BT^n)_+ = E[x_1, \dots, x_n] \otimes \mathbb{Z}/p[y_1, \dots, y_n]$$

where  $x_i = 1 \otimes \dots \otimes x \otimes \dots \otimes 1 \in H^1 T^n$  and  $y_i = c_1(\pi_i)$  where  $\pi_i: T^n \rightarrow S^1$  is the  $i$ -th projection map. This notation is chosen to agree with that of (3.1) since  $H^*(T^n \times BT^n)_+ \approx H^*(B\mathbb{Z}/p)^n_+$  as  $GL_n(\mathbb{F}_p)$  modules. Here  $GL_n(\mathbb{Z}_p)$  acts via mod  $p$  reduction  $GL_n(\mathbb{Z}_p) \rightarrow GL_n(\mathbb{F}_p)$ .

Since mod  $p$  reduction is surjective, we can choose  $\hat{e}_n \in \mathbb{Z}_p[GL_n(\mathbb{Z}_p)]$  which projects to the Steinberg idempotent  $e_n \in \mathbb{F}_p[GL_n(\mathbb{F}_p)]$ ; hence  $\hat{e}_n$  defines a map

$$\hat{e}_n: (T^n + BT^n)_+ \rightarrow (T^n \times BT^n)_+$$

which induces action by  $e_n$  on  $H^*(T^n \times BT^n)_+$ . As explained in §3,  $\hat{e}_n$  splits  $(T^n \times BT^n)_+$ ; however, we wish to split  $BT^n$  at least up to suspension. Hence we define

$$\tilde{e}_n: S^n \wedge BT^+_+ \xrightarrow{i \wedge 1} T^+_+ \wedge BT^+_+ \xrightarrow{\hat{e}_n} T^+_+ \wedge BT^+_+ \xrightarrow{p \wedge 1} S^n \wedge BT^+_+$$

where  $i$  and  $p$  are inclusion and projection on the top cell. We shall see that  $\tilde{e}_n$  is an idempotent in cohomology and hence splits  $S^n \wedge BT^+_+$ .

*Definition.*  $BP(n) = \Sigma^n \tilde{e}_n(S^n \wedge BT^+_+)$ .

*Proof of Theorem D.* First we show that  $BP(n)$  has the correct cohomology. We proceed to consider some complex analogues of our previous constructions. Let  $\eta$  be the canonical line bundle over  $BS^1$  and write  $\mathbb{C}P^\infty_k$ ,  $k \in \mathbb{Z}$  for the Thom spectrum  $(BS^1)^{k\eta}$ . Then  $\Lambda^n \mathbb{C}P_0 = BT^+_+$  and we let  $P_n = H^*(T^n \wedge BT^n)_+$ ,  $S_n = P_n[l_n^{-1}]$  where  $l_n$  is the product of all non-zero linear forms in  $y_1, y_2, \dots, y_n$ . Let  $R_n = H^*(T^+_+ \wedge \Lambda^n \mathbb{C}P^\infty_{-2}) \subset S_n$  and let  $M$  be the  $P = A/(\beta)$  module generated by  $X_n Y_n^{-1}$  where  $X_n = x_1 \dots x_n$ ,  $Y_n = y_1 \dots y_n$ . Then  $M_n \cong \Sigma^n(P/P \cap G_n)$  as in Prop. 3.5. Further,  $R_n e_n = M_n$  as in Theorem 5.8. Thus  $P_n e_n = P_n \cap R_n e_n = P_n \cap M_n$  which has the required basis  $\Sigma^n\{P^l(X_n Y_n^{-1}); l(I) = n\}$  as in Prop. 3.5.

It is now clear that  $\tilde{e}_n$  is an idempotent in cohomology since  $X_n$  represents the top cell in  $S^n$ .

To see that  $BT^n$  contains  $p^{\binom{n}{2}}$  copies of  $BP(n)$  we note that lifting orthogonal idempotents of  $\mathbb{F}_p[GL_n(\mathbb{F}_p)]$  to  $\mathbb{Z}_p[GL_n(\mathbb{Z}_p)]$  results in self maps of  $(T^n \wedge BT^n)_+$  which give orthogonal idempotents in cohomology.

The proof that  $BU(n)$  splits is analogous to that of  $B0(n)$ , Theorem C (6.1). One uses the fibration  $U(n)/T^n \rightarrow BT^n \rightarrow BU(n)$  and Becker–Gottlieb transfer.

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