Stability and Convergence of the Explicit-Implicit Conservative Domain Decomposition Procedure for Parabolic Problems

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Abstract—In this note, we present an improved stability condition of a finite difference domain decomposition procedure for the parabolic equation. This procedure is proposed in [1], in which interface fluxes are calculated from the solution at the previous time level, and then these fluxes serve as Neumann boundary conditions for implicit subdomain problems. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In solving numerical parabolic equations, the method of explicit time-stepping is easy to implement on parallel computers, but it has the severe time-step constraint that stability imposes upon them. The methods of implicit time-stepping do not have these stability restrictions of the mesh step, but instead a global linear system of equations needs to be solved at each time step. The domain decomposition allows one to divide the global problem into smaller subdomain problems. For the nonoverlapping domain decomposition method, the explicit nature of the calculation at the interface of subdomains gives rise to a constraint involving the time step and an interface discretization parameter. A lot of work (e.g., see [1–4]) has been devoted to the construction of...
finite difference domain decomposition methods, which have much less severe stability conditions of step restriction than that needed for the pure explicit scheme. In [1] a finite difference domain decomposition procedure for the parabolic equation is proposed. In this procedure, interface fluxes are calculated from the solution at the previous time level, and then these fluxes serve as Neumann boundary conditions for implicit subdomain problems. The aim of this paper is to derive an improved stability and convergence condition of this domain decomposition procedure, which yields that the ratio of the meshstep can be taken as at least $2(9 - \sqrt{21})/3$ times that given by [1] (see the following (1.4) and (1.6)).

Consider the initial-boundary value problem for the parabolic equation

$$u_t - u_{xx} = 0, \quad 0 < x < 1, \quad 0 < t \leq T,$$

$$u_x(0,t) = u_x(1,t) = 0, \quad 0 < t \leq T,$$

$$u(x, 0) = u^0(x), \quad 0 \leq x \leq 1.$$  \hfill (1.1a)

$$u_t + q_x = 0.$$  \hfill (1.1b)

$$q(x,t) = -u_x(x,t).$$

Define

$$q(0,t) = q(1,t), \quad 0 < t \leq T,$$

and

$$u_t + q_x = 0.$$  \hfill (1.1c)

Then

$$q(0,t) = q(1,t), \quad 0 < t \leq T,$$

and

$$u_t + q_x = 0.$$  \hfill (1.2)

Here and below we adopt the same notations and symbols as those in [2]. Let $(\cdot, \cdot)$ and $\| \cdot \|$ be the $L^2(0,1)$ inner product and norm, respectively.

For a positive integer $I$, let $h = 1/I$ and $x_i = ih$ for $i = 0, 1, \ldots, I$. And set $x_i = (x_{i-1/2} + x_{i+1/2})/2$, $\Omega_i = [x_{i-1/2}, x_{i+1/2}]$.

For functions $f(x)$, $g(x)$, let $f_i = f(x_i)$, $f_{i+1/2} = f(x_{i+1/2})$, and define the discrete inner product

$$\langle f, g \rangle = \sum_{i=0}^{I} f_{i+1/2} g_{i+1/2} h,$$

and the corresponding discrete norm $\| f \|^2 = \langle f, f \rangle$.

Denote by $\mathcal{U}$ the finite dimensional subspace of $L^2(0,1)$ consisting of all functions which are piecewise constant on $\Omega_i$, $i = 1, \ldots, I$. Denote this constant by $w_i$. Denote by $\mathcal{Q}$ the subspace of $C[0, 1]$ such that if $v \in \mathcal{Q}$, then $v$ is a linear function on each $\Omega_i$ and $v(0) = v(1) = 0$.

Assume that $\bar{x} = x_{k+1/2}$ for some integer $k$, $0 < k < I$. Let $H$ be an integral multiple of $h$, i.e., $H = mh$, and $0 < H \leq \min(x, 1 - x)$.

For a smooth function $\psi(x)$, define

$$B(\psi) = \frac{1}{H} \int_0^1 \phi'(x) \psi(x) \, dx,$$

where

$$\phi(x) = \begin{cases} 
\frac{x - \bar{x} + H}{H}, & \bar{x} - H \leq x \leq \bar{x}, \\
\frac{\bar{x} + H - x}{H}, & \bar{x} \leq x \leq x + H, \\
0, & \text{otherwise}.
\end{cases}$$

Let $\tau > 0$ and $t^n = n\tau$ ($n = 0, 1, \ldots, M$), $t^M = T$, and for $f = f(t)$ let $f^n = f(t^n)$ and

$$\partial_t f^n = \frac{f^n - f^{n-1}}{\tau}.$$
Following [1] we shall define a block-centered finite difference domain decomposition procedure as follows. The numerical approximations $U^n_i$ to $u^n_i$ and $Q^n_{i+1/2}$ to $q^n_{i+1/2}$ are defined

$$\partial_t U^n_i + \frac{Q^n_{i+1/2} - Q^n_{i-1/2}}{h} = 0, \quad i = 1, \ldots, I, \quad (1.3a)$$

$$Q^n_{i+1/2} = -\frac{U^n_{i+1} - U^n_i}{h}, \quad 1 \leq i \leq I, \quad i \neq k, \quad (1.3b)$$

$$Q^n (\bar{x}) \equiv Q^n_{k+1/2} = B(U^{n-1}), \quad (1.3c)$$

and the boundary condition

$$Q^n_{I/2} = Q^n_{I+1/2} = 0. \quad (1.3d)$$

In [1] the following interesting error estimate is proved.

**THEOREM 1.** Let $u$ be the smooth solution of (1.1) and $\bar{u}(\cdot, t)$ be the $L^2$ projection of $u(\cdot, t)$ into $\mathcal{U}$. Assume

$$\frac{\tau}{H^2} \leq \frac{1}{4}, \quad (1.4)$$

then there exists a constant $C$, independent of $\tau$, $h$, and $H$, such that

$$\left( \sum_{n=1}^M ||q^n - Q^n||^2 \right)^{1/2} + \max_n ||\bar{u}^n - U^n|| \leq C (\tau + h^2 + H^3). \quad (1.5)$$

Our main result is the following.

**THEOREM 2.** Let $I$, $I - k$, $k - m$, and $m$ be sufficiently large. Assume

$$\frac{\tau}{H^2} < \frac{9 - \sqrt{21}}{6}. \quad (1.6)$$

Then

(i) the scheme (1.3) is $L_2$ stable, i.e.,

$$||U^n|| \leq ||U^{n-1}||; \quad (1.7)$$

(ii) the following convergence result holds:

$$\max_n ||\bar{u}^n - U^n|| \leq C (\tau + h^2 + H^3). \quad (1.8)$$

**REMARK 1.** Inequality (1.6) shows that the time step of (1.3) can be taken as at least $(9 - \sqrt{21})/3m^2$ times that for the pure explicit scheme.

**REMARK 2.** Usually the steplength $h$ is small, so $I$ is sufficiently large. Asymptotically one would expect to choose $h$ and $H$ such that $H^3 = O(h^3)$. From $H = mh$ it follows that $m = O(I^{1/3})$, and hence, $m$ should be sufficiently large. When $k \approx [I/2]$, then $I - k$ and $k - m$ are sufficiently large also.

**REMARK 3.** In practical computation we can take $m$ to be some small integers, e.g., $m = 1$ and $m = 2$. These two cases are discussed in Section 3, and the stability restrictions now become $(\tau/H^2) \leq 1$ and $(\tau/H^2) < (3 + \sqrt{23}/8)$, respectively. Note that $1 > (3 + \sqrt{23}/8) > (9 - \sqrt{21})/6 > 1/4$, which means the stability and convergence condition of scheme (1.3) is improved.
2. PROOF OF THEOREM 2

PROOF OF THEOREM 2(i). System (1.3) is equivalent to the following system of equations: find 
\((Q^n, U^n) \in Q \times U\) satisfying

\[
\begin{align*}
\langle Q^n, v \rangle - \langle U^n, v \rangle &= 0, \quad v \in \bar{Q}, \\
(\partial_t U^n, w) + \langle Q^n_x, w \rangle &= 0, \quad w \in U,
\end{align*}
\]

(2.1a)  (2.1b)

where \(\bar{Q} = Q \cap \{v \mid v(\bar{x}) = 0\}\).

Taking \(v = Q^n - Q^n(\bar{x})\phi(x)\) in (2.1a) and \(w = U^n\) in (2.1b), and noticing (1.2) and (1.3c) and summing up the resulting equalities, we obtain

\[
(\partial_t U^n, U^n) + ||Q^n||^2 + H|Q^n(\bar{x})|^2
= - (Q^n_x, U^n) + (Q^n, Q^n(\bar{x}) \phi) + (U^n, Q^n_x - Q^n(\bar{x}) \phi_x) + (U^{n-1}, Q^n(\bar{x}) \phi_x)
= (Q^n, Q^n(\bar{x}) \phi) + (U^{n-1} - U^n, Q^n(\bar{x}) \phi_x).
\]

(2.2)

Note that

\[
(\partial_t U^n, U^n) = \frac{1}{2\tau} \left( ||U^n||^2 - ||U^{n-1}||^2 + ||U^n - U^{n-1}||^2 \right).
\]

(2.3)

Let

\[
F \equiv \frac{1}{2\tau} \left( ||U^n - U^{n-1}||^2 \right) + ||Q^n||^2 + H|Q^n(\bar{x})|^2
- (Q^n, Q^n(\bar{x}) \phi) - (U^{n-1} - U^n, Q^n(\bar{x}) \phi_x).
\]

(2.4)

Obviously, if \(F \geq 0\), then from (2.2)-(2.4) we have \(||U^n|| \leq ||U^{n-1}||\), and then it follows that the conclusion of Theorem 2 is proved. Therefore, it remains to show \(F \geq 0\) under the conditions of Theorem 2.

Denote \(r_j = Q^n_{j+1/2}\) and \(\lambda = \tau/H^2\). From (1.3a) and the definition of \(\phi(x)\) it follows

\[
F = \frac{m^2\lambda}{2} \sum_{i=1}^I |r_i - r_{i-1}|^2 h + \sum_{i=1}^I r_i^2 h + mr_k^2 h - r_k \sum_{i=k-m}^k r_i \left( 1 - \frac{1}{m} |k - i| \right) h + m\lambda(r_{k+m} - 2r_k + r_{k-m})r_k h.
\]

Set \(\gamma = 2/m^2\lambda\) and \(\bar{F} = \gamma F/h\). Define

\[
\begin{align*}
c_1 &= 2 + \gamma, \quad c_i = 2 + \gamma - \frac{1}{c_{i-1}}, \quad i = 2, \ldots, k - m - 1, \\
d_{I-1} &= 2 + \gamma, \quad d_i = 2 + \gamma - \frac{1}{d_{i+1}}, \quad i = k + m + 1, \ldots, I - 2.
\end{align*}
\]

(2.5a)  (2.5b)

Then the quadratic form \(\bar{F}\) can be written as the following:

\[
\bar{F} = \sum_{i=1}^{k-m-1} c_i \left( r_i - \frac{r_{i+1}}{c_i} \right)^2 + \sum_{i=I-1}^{k+m+1} d_i \left( r_i - \frac{r_{i-1}}{d_i} \right)^2
+ \left( 2 + \gamma - \frac{1}{c_{k-m-1}} \right) r_{k-m}^2 + \left( 2 + \gamma - \frac{1}{d_{k+m+1}} \right) r_{k+m}^2
+ (2 + \gamma) \sum_{j=k-m+1}^{k+m} r_j^2 - 2 \sum_{j=k-m+1}^{k+m} r_{j-1}r_j + m\gamma r_k^2
+ \frac{2}{m} (r_{k-m} + r_{k+m} - 2r_k) r_k - \gamma r_k \sum_{j=k-m+1}^{k+m-1} r_j \left( 1 - \frac{1}{m} |k - j| \right).
\]

(2.6)
Now we estimate the last two terms of (2.6). Using the following elementary inequality:

\[ ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \quad a, b \in \mathcal{R}, \quad \varepsilon > 0, \]

and \( \sum_{j=1}^{m-1} j^2 = (m-1)m(2m-1)/6 \), we have

\[
\gamma r_k \sum_{j=k-m+1}^{k+m-1} r_j \left( 1 - \frac{1}{m} |k-j| \right) \\
\leq \gamma r_k^2 + \frac{\gamma m}{3} \left( 1 - \frac{1}{m} \right) \left( 2 - \frac{1}{m} \right) r_k^2 + \frac{\gamma}{4\varepsilon} \left( \sum_{j=k-m+1}^{k-1} r_j^2 + \sum_{j=k+1}^{k+m-1} r_j^2 \right),
\]

(2.7)

where the constant \( \varepsilon > 1/4 \) will be determined later. Moreover, for any \( \delta > 0 \) there holds

\[
\frac{2}{m} (r_{k-m} + r_{k+m} - 2r_k) r_k \geq -\frac{3}{m} r_k^2 - 2\delta r_k^2 - \frac{1}{\delta m^2} \left( r_{k-m}^2 + r_{k+m}^2 \right). \]

(2.8)

Combining (2.6)–(2.8) gives

\[
\bar{F} \geq \sum_{i=1}^{k-m-1} c_i \left( r_i - \frac{r_{i+1}}{c_i} \right)^2 + \sum_{i=1}^{k+m+1} d_i \left( r_i - \frac{r_{i-1}}{d_i} \right)^2 + c_{k-m} r_{k-m}^2 + c_{k+m} r_{k+m}^2 \\
+ \left( 2 + \gamma - \frac{\gamma}{4\varepsilon} \right) \left( \sum_{j=k-m+1}^{k-1} r_j^2 + \sum_{j=k+1}^{k+m-1} r_j^2 \right) - 2 \sum_{j=k-m+1}^{k+m-1} r_{j-1} r_j \\
+ \left[ m\gamma - \frac{\gamma m}{3} \left( 1 - \frac{1}{m} \right) \left( 2 - \frac{1}{m} \right) + 2 - \frac{4}{m} - 2\delta \right] r_k^2,
\]

(2.9)

where

\[
c_{k-m} = 2 + \gamma - \frac{1}{c_{k-m-1}} - \frac{1}{\delta m^2}, \quad c_{k+m} = 2 + \gamma - \frac{1}{c_{k+m+1}} - \frac{1}{\delta m^2}.
\]

Define

\[
\bar{c}_0 = c_{k-m}, \quad \bar{c}_i = 2 + \gamma - \frac{\gamma}{4\varepsilon} - \frac{1}{\bar{c}_{i-1}}, \quad i = 1, \ldots, m-1, \\
\bar{d}_0 = c_{k+m}, \quad \bar{d}_i = 2 + \gamma - \frac{\gamma}{4\varepsilon} - \frac{1}{\bar{d}_{i-1}}, \quad i = 1, \ldots, m-1,
\]

and

\[
c_k = m\gamma - \frac{\gamma m}{3} \left( 1 - \frac{1}{m} \right) \left( 2 - \frac{1}{m} \right) + 2 - \frac{4}{m} - 2\delta - \frac{1}{\bar{c}_{m-1}} - \frac{1}{\bar{d}_{m-1}}.
\]

Then we have

\[
\bar{F} \geq \sum_{i=1}^{k-m-1} c_i \left( r_i - \frac{1}{c_i} r_{i+1} \right)^2 + \sum_{i=1}^{k+m+1} d_i \left( r_i - \frac{1}{d_i} r_{i-1} \right)^2 + c_k r_k^2 \\
+ \sum_{i=0}^{m-1} \bar{c}_i \left( r_{k-m+i} - \frac{1}{\bar{c}_i} r_{k-m+i+1} \right)^2 + \sum_{i=0}^{m-1} \bar{d}_i \left( r_{k+m-i} - \frac{1}{\bar{d}_i} r_{k+m-i-1} \right)^2.
\]

(2.10)

Denote \( c_{(1)} \infty = 1 + (\gamma/2) + \sqrt{(1 + (\gamma/2)^2 - 1}. \) By induction on \( i \), it is easy to prove \( c_i > c_{i+1} \) and \( c_i > c_{(1)} \infty > 1 \), for all \( 1 \leq i \leq k - m - 1 \). And then there holds \( \lim_{(k-m) \rightarrow \infty} c_{k-m-1} = c_{(1)} \infty \).
Since $c^{(1)}_{\infty} + 1/c^{(1)}_{\infty} = 2 + \gamma$, we get

$$c_{k-m} + \frac{1}{c_{k-m}} > 2 + \left(1 - \frac{1}{4\varepsilon}\right)\gamma$$

provided that $k - m$ is large enough. It follows that $\bar{c}_1 < c_{k-m}$. By induction on $i$ we can show that $\bar{c}_i > 1$ and $\bar{c}_i < \bar{c}_{i-1}$ for all $1 \leq i \leq m - 1$. And then there holds $\lim_{m \to \infty} \bar{c}_{m-1} = c^{(2)}_{\infty}$, where

$$c^{(2)}_{\infty} = 1 + \frac{\gamma}{2} \left(1 - \frac{1}{4\varepsilon}\right) + \sqrt{\left(1 + \frac{\gamma}{2} \left(1 - \frac{1}{4\varepsilon}\right)\right)^2 - 1}.$$

Denote

$$\gamma_m = \frac{m\gamma}{2} - \frac{\gamma \varepsilon m}{6} \left(1 - \frac{1}{m}\right) \left(2 - \frac{1}{m}\right) + 1 - \frac{2}{m} - \frac{1}{\bar{c}_{m-1}},$$

$$\gamma'_m = \frac{m\gamma}{2} - \frac{\gamma \varepsilon m}{6} \left(1 - \frac{1}{m}\right) \left(2 - \frac{1}{m}\right) + 1 - \frac{2}{m} - \frac{1}{\bar{c}_{m-1}}.$$

Then $c_k = \gamma_m + \gamma'_m$. Take $\delta = m^{-3/2}$. We shall choose $\varepsilon$ and $\gamma$ such that $\gamma_m \geq 0$ and $\gamma'_m \geq 0$. From $\bar{c}_{m-1} > c^{(2)}_{\infty}$ and the definitions of $\delta$ and $\gamma$, we get

$$\gamma_m \geq \frac{m\gamma}{2} - \frac{\gamma \varepsilon m}{6} \left(1 - \frac{1}{m}\right) \left(2 - \frac{1}{m}\right) + 1 - \frac{2}{m} - \frac{1}{m^{3/2}} - \frac{1}{c^{(2)}_{\infty}}$$

$$= \left[1 - \frac{\varepsilon}{3m} \left(1 - \frac{1}{m}\right) \left(2 - \frac{1}{m}\right) \right] \frac{1}{\lambda} - \frac{2}{m} - \frac{1}{m^{3/2}}$$

$$- \frac{1}{m^2\lambda} \left(1 - \frac{1}{4\varepsilon}\right) - \sqrt{\frac{2}{m^2\lambda} \left(1 - \frac{1}{4\varepsilon}\right) + \frac{1}{m^4\lambda^2} \left(1 - \frac{1}{4\varepsilon}\right)^2}.$$

Then, there holds

$$\lim\inf_{m \to \infty} m\gamma_m \geq \left(1 - \frac{2\varepsilon}{3}\right) \frac{1}{\lambda} - 2 + \sqrt{\left(1 - \frac{1}{4\varepsilon}\right) \frac{2}{\lambda}}. \quad (2.11)$$

We shall determine $\varepsilon (> 1/4)$ and $\lambda_0 = \lambda_0(\varepsilon)$ such that for $\lambda < \lambda_0$ there holds

$$\left(1 - \frac{2\varepsilon}{3}\right) \frac{1}{\lambda} - 2 + \sqrt{\left(1 - \frac{1}{4\varepsilon}\right) \frac{2}{\lambda}} > 0. \quad (2.12)$$

Let $\varepsilon = \varepsilon_0 > 1/4$ satisfy $1 - 2\varepsilon/3 = 2(1 - 1/4\varepsilon)$. We obtain $\varepsilon_0 = (-3 + \sqrt{21})/4$. Now we take $\lambda_0 = 2(1 - 1/4\varepsilon_0)$. It follows that, if $\lambda < \lambda_0$, i.e.,

$$\lambda < \frac{9 - \sqrt{21}}{6},$$

then (2.12) is true. Due to (2.11) we find $\gamma_m \geq 0$ for $m$ large sufficiently. Similarly, there is $\gamma'_m \geq 0$ for $m$ large enough. It follows that $c_k \geq 0$ for $m$ large enough. In view of (2.10), the proof of Theorem 2(i) is completed.

PROOF OF THEOREM 2(ii). Let $\tilde{Q}(\cdot, t) \in \mathbb{Q}$ be the interpolation of $q(\cdot, t)$, i.e., $\tilde{Q}(x_{i+1/2}, t) = q(x_{i+1/2}, t)$, $i = 0, 1, \ldots, I$. There holds

$$(\tilde{Q}_x(\cdot, t), w) = (q_x(\cdot, t), w) = -(u_x(\cdot, t), w), \quad w \in \mathcal{U}. \quad (2.13)$$

For each $t \in [0, T]$, define $\bar{U}(\cdot, t) \in \mathcal{U}$ by the following system:

$$\bar{U}(x_1, t) = u(0, t), \quad (2.14a)$$

$$\bar{Q}(x, t) = B\left(\bar{U}(\cdot, t)\right), \quad (2.14b)$$
Set $\mu = Q - \dot{Q}$, $\mu(x) = \mu(\dot{x}) \varphi(x)$, and $\xi = U - \dot{U}$. Since $\varphi \in Q$, there is $\bar{\mu} \in Q$. Subtract (2.14c) from (2.1a), and (2.13) from (2.1b) to obtain

$$
\langle \mu^n, v \rangle - \langle \xi^n, v \rangle = 0, \quad v \in \dot{Q},
$$

(2.15a)

$$
(\partial_t \xi^n, w) + (\mu^n, w) = (\partial_t (\bar{u}^n - \bar{U}^n), w) + (u^n_i - \partial_t u^n_i, w), \quad w \in \mathcal{U}.
$$

(2.15b)

Note that

$$
\mu^n(\bar{x}) = Q^n(\bar{x}) - \bar{Q}^n(\bar{x}) = \frac{1}{H} \left( \xi^{n-1, \phi_x} - (q^n(\bar{x}) - q^{n-1}(\bar{x})) \right).
$$

(2.16)

Set $v = \mu^n - \bar{\mu}$ in (2.15a), $w = \xi^n$ in (2.15b), multiply (2.16) by $H \mu^n(\bar{x})$, and add the resulting equalities to get

$$
\langle \partial_t \xi^n, \xi^n \rangle + ||\mu^n||^2 + H |\mu^n(\bar{x})|^2
$$

$$
= \langle \mu^n, \bar{\mu} \rangle + \langle \partial_t (\bar{u}^n - \bar{U}^n), \xi^n \rangle + (u^n_i - \partial_t u^n_i, \xi^n)
$$

$$
+ \left( \xi^{n-1} - \xi^n, \bar{\mu} \right) - H \left( q^n(\bar{x}) - q^{n-1}(\bar{x}) \right) \bar{\mu} \left( \bar{x} \right)
$$

$$
\leq \langle \mu^n, \bar{\mu} \rangle + \left( \xi^{n-1} - \xi^n, \bar{\mu} \right) + \frac{\varepsilon_1}{2} \left| \mu^n(\bar{x}) \right|^2
$$

$$
+ \frac{H}{2\varepsilon_1} \left( q^n(\bar{x}) - q^{n-1}(\bar{x}) \right)^2 + \frac{1}{4} \left| \partial_t (\bar{u}^n - \bar{U}^n) + u^n_i - \partial_t u^n_i \right|^2 + ||\xi^n||^2,
$$

(2.17)

where $\varepsilon_1$ is a small positive constant. Note that

$$
\langle \partial_t \xi^n, \xi^n \rangle = \frac{1}{2\tau} \left( ||\xi^n||^2 - ||\xi^{n-1}||^2 + ||\xi^n - \xi^{n-1}||^2 \right). \tag{2.18}
$$

From

$$
\xi^n - \xi^{n-1} = -\frac{\tau}{h} \left( \mu_i^{n+1/2} - \mu_i^{n-1/2} \right) + \tau \left[ \partial_t (\bar{u}_i^n - \bar{U}_i^n) + u^n_i - \partial_t u^n_i \right], \tag{2.19}
$$

it follows

$$
||\xi^n - \xi^{n-1}||^2 \geq (1 - \varepsilon_2)\tau \lambda \sum_{i=1}^I \left( \mu_i^{n+1/2} - \mu_i^{n-1/2} \right)^2 h
$$

$$
-(1 + \varepsilon_2)^2 \tau ^2 \left| \partial_t (\bar{u}^n - \bar{U}^n) + u^n_i - \partial_t u^n_i \right|^2,
$$

(2.20)

where $\varepsilon_2$ is a small positive constant. And there is

$$
(\xi^{n-1} - \xi^n, \bar{\mu}^2) = -\frac{\tau}{H} \left( \mu_k^{k+m+1/2} - 2\mu_{k+1/2} + \mu_{k-m+1/2} \right) \mu_k^{n+1/2}
$$

$$
+ \frac{\tau}{H} \sum_{i=k-m+1}^k \left[ \partial_t (\bar{u}_i^n - \bar{U}_i^n) + u^n_i - \partial_t u^n_i \right] \mu_k^{n+1/2} h
$$

$$
- \frac{\tau}{H} \sum_{i=k+1}^{k+m} \left[ \partial_t (\bar{u}_i^n - \bar{U}_i^n) + u^n_i - \partial_t u^n_i \right] \mu_k^{n+1/2} h
$$

$$
\leq -\frac{\tau}{H} \left( \mu_k^{k+m+1/2} - 2\mu_{k+1/2} + \mu_{k-m+1/2} \right) \mu_k^{n+1/2}
$$

$$
+ \frac{\varepsilon_1 H}{2} \left( \mu_k^{n+1/2} \right)^2 + \frac{\tau \lambda}{\varepsilon_1} \left| \partial_t (\bar{u}^n - \bar{U}^n) + u^n_i - \partial_t u^n_i \right|^2.
$$

(2.21)

Moreover,

$$
\langle \mu^n, \bar{\mu} \rangle = \mu_k^{n+1/2} \sum_{i=k-m}^{k+m} \mu_i^{n+1/2} \left( 1 - \frac{1}{m} \left| k - i \right| \right) h.
$$

(2.22)
Set \( s_i = \mu_{i+1/2}^n, i = 0, 1, \ldots, I \). Let

\[
G \equiv (1 - \varepsilon_2) \frac{m^2 \lambda}{2} \sum_{i=1}^{I} |s_i - s_{i-1}|^2 h + \sum_{i=1}^{I} s_i^2 h + (1 - \varepsilon_1) m^2 s_i^2 h
- s_k \sum_{i=k-m}^{k+m} s_i \left( 1 - \frac{1}{m} |k - i| \right) h + m \lambda (s_{k+m} - 2s_k + s_{k-m}) s_k h.
\]

Combining (2.17)-(2.22) yields

\[
\frac{1}{2\tau} \left( \|\xi^n\|^2 - \|\xi^{n-1}\|^2 \right) + G \leq \|\xi^n\|^2 + \frac{H}{2\varepsilon_1} (q^n(\bar{x}) - q^{n-1}(\bar{x}))^2
+ \left( 1 + \varepsilon_2 \right) (\bar{u}^n - \bar{U}^n) + u^a - \partial_t u^n \|^2.
\]

Following the same arguments of Theorem 2(i) given as above, and taking \( \varepsilon_1 \) and \( \varepsilon_2 \) small we have \( G > 0 \) provided that (1.6) holds, then it follows

\[
\frac{1}{2\tau} \left( \|\xi^n\|^2 - \|\xi^{n-1}\|^2 \right) \leq \|\xi^n\|^2 + CH (q^n(\bar{x}) - q^{n-1}(\bar{x}))^2
+ C \left( \|\partial_t (\bar{u}^n - \bar{U}^n) + u^a - \partial_t u^n \|^2 \right).
\]

Since \( q \) and \( u \) are smooth functions,

\[
H (q^n(\bar{x}) - q^{n-1}(\bar{x}))^2 \leq CH \tau^2,
\]

\[
\|u^a - \partial_t u^n\|^2 \leq C \tau^2.
\]

Furthermore, there holds (see [1])

\[
\max_{i} \left| \bar{u}^n(x_i) - \bar{U}^n(x_i) \right| + \max_{i} \left| \partial_t (\bar{u}^n(x_i) - \bar{U}^n(x_i)) \right| \leq C \left( h^2 + H^3 \right).
\]

Hence, by substituting these inequalities above into (2.23)

\[
\frac{1}{2\tau} \left( \|\xi^n\|^2 - \|\xi^{n-1}\|^2 \right) \leq \|\xi^n\|^2 + C \left( \tau^2 + H \tau^2 + h^4 + H^6 \right).
\]

Applying Gronwall’s lemma gives

\[
\max_{n} \|\xi^n\| \leq C \left( \tau + h^2 + H^3 \right).
\]

By (2.24) and (2.25) the proof of Theorem 2(ii) is completed.

### 3. CASES: \( m = 1 \) AND \( m = 2 \)

In the case of \( m = 1 \), there is

\[
\hat{F} = \sum_{i=1}^{k-2} c_i \left( r_i - \frac{1}{c_i} r_{i+1} \right)^2 + \sum_{i=1}^{k+2} c_i \left( r_i - \frac{1}{c_i} r_{i-1} \right)^2 + c_{k-1} r_{k-1}^2 + c_{k+1} r_{k+1}^2 + (\gamma - 2) r_k^2,
\]

where

\[
c_i = 2 + \gamma > 0, \quad c_i = 2 = \gamma - \frac{1}{c_i} \geq 0, \quad i = 2, 3, \ldots, k - 1, \quad (3.1a)
\]

\[
c_{k-1} = 2 + \gamma > 0, \quad c_i = 2 = \gamma - \frac{1}{c_i} \geq 0, \quad i = k + 1, k + 2, \ldots, I - 2. \quad (3.1b)
\]
So we have $\bar{F} \geq 0$ if and only if $\gamma \geq 2$, i.e.,

$$\lambda = \frac{\tau}{H^2} \leq 1.$$  

(3.2)

In the case of $m = 2$, the quadratic form $\bar{F}$ can be written as the sum of squares

$$\bar{F} = \sum_{i=1}^{k-3} c_i \left( r_i - \frac{1}{c_i} r_{i+1} \right)^2 + \sum_{i=1}^{k+3} c_i \left( r_i - \frac{1}{c_i} r_{i-1} \right)^2$$

$$+ c_{k-2} \left( r_{k-2} - \frac{1}{c_{k-2}} r_{k-1} + \frac{1}{2c_{k-2}} r_k \right)^2 + c_{k+2} \left( r_{k+2} - \frac{1}{c_{k+2}} r_{k+1} + \frac{1}{2c_{k+2}} r_k \right)^2$$

$$+ c_{k-1} \left( r_{k-1} + \frac{1}{c_{k-1}} \left( \frac{1}{2c_{k-2}} - 1 - \frac{\gamma}{4} \right) r_k \right)^2 + c_{k+1} \left( r_{k+1} + \frac{1}{c_{k+1}} \left( \frac{1}{2c_{k+2}} - 1 - \frac{\gamma}{4} \right) r_k \right)^2$$

$$+ \left[ 2\gamma - \frac{1}{4c_{k-2}} - \frac{1}{4c_{k+2}} - \frac{1}{c_{k-1}} \left( \frac{1}{2c_{k-2}} - 1 - \frac{\gamma}{4} \right)^2 - \frac{1}{c_{k+1}} \left( \frac{1}{2c_{k+2}} - 1 - \frac{\gamma}{4} \right)^2 \right] r_k^2,$$

where $c_i \ (i \neq k)$ are defined as those in (3.1). To assure $\bar{F} \geq 0$, it should require that

$$2\gamma \geq \frac{1}{4c_{k-2}} + \frac{1}{4c_{k+2}} + \frac{1}{c_{k-1}} \left( \frac{1}{2c_{k-2}} - 1 - \frac{\gamma}{4} \right)^2 + \frac{1}{c_{k+1}} \left( \frac{1}{2c_{k+2}} - 1 - \frac{\gamma}{4} \right)^2,$$

which can be true if

$$\gamma \geq \frac{1}{4c_{k-2}} + \frac{1}{c_{k-1}} \left( \frac{1}{2c_{k-2}} - 1 - \frac{\gamma}{4} \right)^2,$$

and

$$\gamma \geq \frac{1}{4c_{k+2}} + \frac{1}{c_{k+1}} \left( \frac{1}{2c_{k+2}} - 1 - \frac{\gamma}{4} \right)^2,$$

(3.3)

(3.4)

We might as well assume $(1/2) \leq \gamma \leq 2(\sqrt{2} - 1)$. Since $c_i > c^{(1)}_\infty$ for all $i \neq k$, we deduce that (3.3) and (3.4) are true, if

$$\gamma \geq \frac{1}{4c^{(1)}_{\infty}} + \frac{1}{c^{(1)}_{\infty}} \left( \frac{1}{2c^{(1)}_{\infty}} - 1 - \frac{\gamma}{4} \right)^2,$$

i.e.,

$$\left( \gamma - \frac{1}{2} \right) \sqrt{\left( 1 + \frac{\gamma}{2} \right)^2} - 1 + \frac{3}{4} \gamma + \frac{7}{16} \gamma^2 \geq \frac{1}{2}.$$  

It follows that $\bar{F} \geq 0$, if $\gamma \geq (\sqrt{92} - 6)/7$, i.e.,

$$\lambda \leq \frac{\sqrt{23} + 3}{8}.$$  

(3.6)

So scheme (1.3) is $L^2$ stable in the cases of $m = 1$ and $m = 2$ under the stability conditions (3.2) and (3.6), respectively.

REFERENCES


