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The long time behavior of DI SIR epidemic model with stochastic perturbation ☆

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ABSTRACT

In this paper, we present a DI SIR epidemic model with two categories stochastic perturbations. The long time behavior of the two stochastic systems is studied. Mainly, we show how the solution goes around the infection-free equilibrium and the endemic equilibrium of deterministic system under different conditions.

1. Introduction

Since public recognition of the acquired immunodeficiency syndrome (AIDS) epidemic in 1981, there are nearly 70 million people in worldwide infected by HIV/AIDS. Now, there are approximately 5700 people everyday infected this epidemic, which is equivalent to a new infections occur every 16 seconds. Hence the HIV/AIDS pandemic is the greatest public health disaster of modern times. In addition, the dynamics transmission of HIV is quite complex. For instance, the incubation period after infection with HIV is known to be extremely long about 10–15 years without treatment. During this period, the individuals stay healthy and can unknowingly transmit the disease to others. Except that, although the disease is known as a sexually transmitted disease, it is also passed on from contaminated needles, breast milk, and an infected mother to her baby at birth (vertical transmission). All these factors have made it more difficult to understand how this epidemic spreads in the population.

Mathematical models based on the underlying transmission of HIV can help us to understand better how the disease spreads in the community and can investigate how changes in the various assumptions and parameter values affect the course of epidemic [20]. There are many researchers to construct mathematical models, which reflect the characteristics of this epidemic to some extent [1,5–7,18,21,22,26]. Hyman et al. also paid attention on this, see [13–17].

Especially, Hyman et al. [15] proposed a differential infectivity (DI) model that accounted for differences in infectiousness between individuals during the chronic stages, and the correlation between viral loads and rates of developing AIDS. They assumed that the susceptible population was homogeneous and neglected variations in susceptibility, risk behavior,

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and many other factors associated with the dynamics of HIV spread. In addition the population they studied was a small, highrisk subset of a larger population. They divided the population as susceptible individuals S, the HIV infection population I, which was subdivided into n subgroups, $I_1, I_2, \ldots, I_n$, and the group of AIDS patients A. They gave out the DI model:

$$\begin{cases} \dfrac{dS}{dt} = \mu S^0 - \mu S - \displaystyle\sum_{j=1}^{n} \beta_j I_j S, \\ \dfrac{dI_k}{dt} = p_k \displaystyle\sum_{j=1}^{n} \beta_j I_j S - (\mu + \gamma_k) I_k, \quad k = 1, 2, \ldots, n, \\ \dfrac{dA}{dt} = \displaystyle\sum_{k=1}^{n} \gamma_k I_k - \delta A. \end{cases} \tag{1.1}$$

Here, they assumed the rate of infection, depended upon the transmission probability per partner β_k of individuals in subgroup k, the proportion of individuals in the subgroup I_k/N ($N = S + I$ and $I = \sum_{k=1}^{n} I_k$), and the number of contacts of an individual per unit of time, $r(N)$, which was supposed $r(N) = N$. The other parameters in system (1.1) were summarized as follows. S^0 presents a constant steady state of the susceptible population S, when no virus is in the population. μ is the rate of inflow and outflow, which maintains the equilibrium S^0. p_k is the probability of an individual enters subgroup k, when he is infected, where $\sum_{k=1}^{n} p_k = 1$. γ_k is the rate of leaving the high-risk population because of behavior changes that are induced by either HIV-related illnesses or a positive HIV test. Last, δ is the die rate of A which satisfies $\delta \geqslant \mu$.

Since the dynamics of group A has no effects on the transmission dynamics, they omitted the last equation of (1.1) in their analysis. Obviously, system (1.1) has only two kinds of equilibria: the infection-free equilibrium $E_0 = (S^0,\ I_1 = 0,\ I_2 = 0, \ldots, I_n = 0)$ and the endemic equilibrium $E^* = (S^*, I_1^*, I_1^*, \ldots, I_n^*)$. Hyman et al. [15] and Ma et al. [23] showed if $R_0 \leqslant 1$, the infection-free equilibrium is globally asymptotically stable in the region $G := \{(S, I_k) | 0 \leqslant N = S + \sum_{k=1}^{n} I_k \leqslant S^0\}$, while if $R_0 > 1$, the disease-free equilibrium is unstable, and the endemic equilibrium E^* is globally asymptotically stable in region G, where $R_0 = S^0 \sum_{k=1}^{n} \frac{\beta_k p_k}{\mu + \gamma_k}$.

In fact, epidemic models are inevitably affected by environmental white noise which is an important component in realism, because it can provide an additional degree of realism in compared to their deterministic counterparts. Therefore, many stochastic models for the epidemic populations have been developed. In addition, both from a biological and from a mathematical perspective, there are different possible approaches to include random effects in the model. Here, we mainly mention four approaches. The first one is through time Markov chain model to consider environment noise in HIV epidemic [27–30]. The second is with parameters perturbation, such as [10,11]. Imhof and Walcher in [19] introduced random fluctuations in the deterministic chemostat model, following the way of [3], in which they considered the environmental noise was proportional to the variables. This is the third one. The last important issue to model stochastic epidemic system is to robust the positive equilibria of deterministic models. In this situation, it is mainly to investigate whether the stochastic system preserves the asymptotic stability properties of the positive equilibria of deterministic models, see [4,8,9].

In this paper, taking into account the effect of randomly fluctuating environment in system (1.1), we incorporate white noise with the last two approaches, respectively. For the one issue, in detail we show that a reasonable stochastic analogue of system (1.1) is given by

$$\begin{cases} dS = \left(\mu S^0 - \mu S - \displaystyle\sum_{j=1}^{n} \beta_j I_j S\right) dt + \sigma_1 S dB_1(t), \\ dI_k = \left[p_k \displaystyle\sum_{j=1}^{n} \beta_j I_j S - (\mu + \gamma_k) I_k\right] dt + \sigma_{k+1} I_k \, dB_{k+1}(t), \quad k = 1, 2, \ldots, n, \\ dA = \left(\displaystyle\sum_{j=1}^{n} \gamma_j I_j - \delta A\right) dt + \sigma_{n+2} A \, dB_{n+2}(t), \end{cases} \tag{1.2}$$

where $B_1(t), B_2(t), \ldots, B_{n+2}(t)$ are independent Brownian motions, and $\sigma_1, \sigma_2, \ldots, \sigma_{n+2}$ are their intensities.

The other one, we assume $R_0 > 1$, then system (1.1) exists the positive equilibria $E^* = (S^*, I_1^*, I_2^*, \ldots, I_n^*, A^*)$. We introduce stochastic perturbations of the white noises are directly proportional to distances $S(t)$, $I_k(t)$, $A(t)$ from values of S^*, I_k^*, A^*, respectively. In detail, that is,

$$\begin{cases} dS = \left(\mu S^0 - \mu S - \sum_{j=1}^{n} \beta_j I_j S\right) dt + \sigma_1 (S - S^*)\, dB_1(t), \\ dI_k = \left[p_k \sum_{j=1}^{n} \beta_j I_j S - (\mu + \gamma_k) I_k\right] dt + \sigma_{k+1}(I_k - I_k^*)\, dB_{k+1}(t), \quad k = 1, 2, \ldots, n, \\ dA = \left(\sum_{j=1}^{n} \gamma_j I_j - \delta A\right) dt + \sigma_{n+2}(A - A^*)\, dB_{n+2}(t), \end{cases} \tag{1.3}$$

where $B_k(t)$, $k = 1, 2, \ldots, n+2$, are also independent standard Brownian motions and $\sigma_k > 0$, $k = 1, 2, \ldots, n+2$, represent the intensities of $B_k(t)$, $k = 1, 2, \ldots, n+2$, respectively.

As the deterministic system, we also omit the last equation of (1.2) and (1.3) in our analysis.

The paper is organized as follows. In Section 2, we mainly study system (1.2). First, we show there is a unique nonnegative solution of system (1.2) for any nonnegative initial value. Next, we investigate the asymptotic properties. We conclude, although the solution of system (1.2) does not converge to E_0 or E^*, under some conditions, there is a stability result like that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t E\|X(t) - E_0\|^2$$

or

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t E\|X(t) - E^*\|^2$$

is small, provided the diffusion coefficients are sufficiently small, where $X(t)$ denotes the solution of system (1.2), and $E\|X(t) - X^*\|^2 = E[\sum_{k=1}^{n}(x_k(t) - x_k^*)^2]$. In Section 3, we discuss system (1.3). When $R_0 > 1$, E^* is also the endemic equilibrium of system (1.3). We explore the solution of system (1.3) is stochastically asymptotically stable by Lyapunov's function.

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathscr{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all P-null sets). Denote

$$R_+^n = \{x \in R^n:\ x_i > 0 \text{ for all } 1 \leqslant i \leqslant n\},$$
$$\bar{R}_+^n = \{x \in R^n:\ x_i \geqslant 0 \text{ for all } 1 \leqslant i \leqslant n\},$$
$$S_h := \{x \in R^d:\ |x| \leqslant h\}.$$

In general, consider d-dimensional stochastic differential equation [24]

$$dx(t) = f(x(t), t)\, dt + g(x(t), t)\, dB(t) \quad \text{on } t \geqslant t_0 \tag{1.4}$$

with initial value $x(t_0) = x_0 \in R^d$. $B(t)$ denotes d-dimensional standard Brownian motions defined on the above probability space. Define the differential operator L associated with Eq. (1.4) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x, t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} [g^T(x, t) g(x, t)]_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on a function $V \in C^{2,1}(S_h \times \bar{R}_+; \bar{R}_+)$, then

$$LV(x, t) = V_t(x, t) + V_x(x, t) f(x, t) + \frac{1}{2} \operatorname{trace}[g^T(x, t) V_{xx}(x, t) g(x, t)],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_x = (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_d})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{d\times d}$. By Itô's formula, if $x(t) \in S_h$, then

$$dV(x(t), t) = LV(x(t), t)\, dt + V_x(x(t), t) g(x(t), t)\, dB(t).$$

2. The dynamics of system (1.2)

In this section, we discuss system (1.2). First, we show there is a unique nonnegative solution no matter how large the intensities of noises are. In the next two parts, we mainly study the long time behavior of the solution.

2.1. Existence and uniqueness of the nonnegative solution

To investigate the dynamical behavior, the first concern thing is whether the solution is global existence. Moreover, for a model of epidemic population dynamics, whether the value is nonnegative is also considered. Hence in this section we first discuss the solution of system (1.2) is global and nonnegative. In order for a stochastic differential equation to have a unique global (i.e. no explosion in a finite time) solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (cf. Arnold [2], Mao [24]). However, the coefficients of system (1.2) do not satisfy the linear growth condition (for the incidence is bilinear), though they are locally Lipschitz continuous, so the solution of system (1.2) may explode at a finite time (cf. Arnold [2], Mao [24]). In this section, using Lyapunov analysis method (mentioned in Mao [25]), we show the solution of system (1.2) is positive and global.

For convenience, we change the variables: $Q_k = \frac{I_k}{p_k}$, $k = 1, 2, \dots, n$, then (1.2) can be written as

$$\begin{cases} dS = \left(\mu S^0 - \mu S - \sum_{j=1}^{n} \beta_j p_j Q_j S\right) dt + \sigma_1 S\, dB_1(t), \\ dQ_k = \left[\sum_{j=1}^{n} \beta_j p_j Q_j S - (\mu + \gamma_k) Q_k\right] dt + \sigma_{k+1} Q_k\, dB_{k+1}(t), \quad k = 1, 2, \dots, n. \end{cases} \tag{2.1}$$

Hence we only need to show the solution of system (2.1) is positive and global existence. Besides, if that, from the last equation of system (1.2), we easily get

$$A(t) = e^{-(\delta + \frac{\sigma_{n+2}^2}{2})t + \sigma_{n+2} B_{n+2}(t)} \left[A(0) + \int_0^t \sum_{j=1}^{n} \gamma_j I_j(r) e^{(\delta + \frac{\sigma_{n+2}^2}{2})r - \sigma_{n+2} B_{n+2}(r)}\, dr\right],$$

which is also positive and global. This also verifies we can omit to analyze the last equation of (1.2).

Theorem 2.1. *There is a unique solution $(S(t), Q_1(t), Q_2(t), \dots, Q_n(t))$ of system (2.1) on $t \geqslant 0$ for any initial value $(S(0), Q_1(0), Q_2(0), \dots, Q_n(0)) \in R_+^{n+1}$, and the solution will remain in R_+^{n+1} with probability 1, namely $(S(t), Q_1(t), Q_2(t), \dots, Q_n(t)) \in R_+^{n+1}$ for all $t \geqslant 0$ almost surely.*

Proof. Since the coefficients of the equation are locally Lipschitz continuous for any given initial value $(S(0), Q_1(0), Q_2(0), \dots, Q_n(0)) \in R_+^{n+1}$, there is a unique local solution $(S(t), Q_1(t), Q_2(t), \dots, Q_n(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time (see Arnold [2]). To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $m_0 \geqslant 0$ be sufficiently large so that $S(0)$, $Q_k(0)$, $k = 1, 2, \dots, n$, all lie within the interval $[1/m_0, m_0]$. For each integer $m \geqslant m_0$, define the stopping time

$$\tau_m = \inf\{t \in [0, \tau_e)\colon \min\{S(t), Q_1(t), Q_2(t), \dots, Q_n(t)\} \leqslant 1/m \text{ or } \max\{S(t), Q_1(t), Q_2(t), \dots, Q_n(t)\} \geqslant m\},$$

where throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m \to \infty} \tau_m$, whence $\tau_\infty \leqslant \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(S(t), Q_1(t), Q_2(t), \dots, Q_n(t)) \in R_+^{n+1}$ a.s. for all $t \geqslant 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. For if this statement is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leqslant T\} > \epsilon.$$

Hence there is an integer $m_1 \geqslant m_0$ such that

$$P\{\tau_m \leqslant T\} \geqslant \epsilon \quad \text{for all } m \geqslant m_1. \tag{2.2}$$

Define a C^2-function $V : R_+^{n+1} \to \bar{R}_+$ by

$$V\big(S(t), Q_1(t), Q_2(t), \dots, Q_n(t)\big) = \sum_{k=1}^{n} a_k \left[\left(S - c - c \log \frac{S}{c}\right) + (Q_k - 1 - \log Q_k)\right],$$

where a_k, $k = 1, 2, \dots, n, c$, are $n + 1$ positive constants to be determined later. The nonnegativity of this function can bee seen from $u - 1 - \log u \geqslant 0$, $\forall u > 0$. Using Itô's formula, we get

$$\begin{aligned} dV &= \sum_{k=1}^{n} a_k \left[\left(1 - \frac{c}{S}\right) dS + \frac{c}{2S^2}(dS)^2 + \left(1 - \frac{1}{Q_k}\right) dQ_k + \frac{1}{2Q_k^2}(dQ_k)^2\right] \\ &:= LV\, dt + \sum_{k=1}^{n} a_k \left[\left(1 - \frac{c}{S}\right)\sigma_1 S\, dB_1(t) + \left(1 - \frac{1}{Q_k}\right)\sigma_{k+1} Q_k\, dB_{k+1}(t)\right], \end{aligned}$$

where

$$LV=\sum_{k=1}^{n}a_k\left\{\left(1-\frac{c}{S}\right)\left(\mu S^0-\mu S-\sum_{j=1}^{n}\beta_j p_j Q_j S\right)+\frac{c\sigma_1^2}{2}+\left(1-\frac{1}{Q_k}\right)\left[\sum_{j=1}^{n}\beta_j p_j Q_j S-(\mu+\gamma_k)Q_k\right]+\frac{\sigma_{k+1}^2}{2}\right\}$$
$$=\sum_{k=1}^{n}a_k\left[\mu S^0-\mu S-\frac{c\mu S^0}{S}+c\mu-\sum_{j=1}^{n}\beta_j p_j\frac{Q_j}{Q_k}S+(\mu+\gamma_k)+\frac{c\sigma_1^2}{2}+\frac{\sigma_{k+1}^2}{2}\right]$$
$$+\sum_{k=1}^{n}a_k\left[c\sum_{j=1}^{n}\beta_j p_j Q_j-(\mu+\gamma_k)Q_k\right].$$

Note that

$$\sum_{k=1}^{n}a_k\left[c\sum_{j=1}^{n}\beta_j p_j Q_j-(\mu+\gamma_k)Q_k\right]=\sum_{k=1}^{n}\sum_{j=1}^{n}a_k c\beta_j p_j Q_j-\sum_{k=1}^{n}a_k(\mu+\gamma_k)Q_k$$
$$=\sum_{k=1}^{n}\sum_{j=1}^{n}a_j c\beta_k p_k Q_k-\sum_{k=1}^{n}a_k(\mu+\gamma_k)Q_k$$
$$=\sum_{k=1}^{n}\left[\sum_{j=1}^{n}a_j c\beta_k p_k-a_k(\mu+\gamma_k)\right]Q_k,$$

choosing $a_k=\frac{\beta_k p_k}{\mu+\gamma_k}$, $k=1,2,\ldots,n$, and $c=\frac{1}{\sum_{j=1}^{n}a_j}$, then

$$\sum_{k=1}^{n}a_k\left[c\sum_{j=1}^{n}\beta_j p_j Q_j-(\mu+\gamma_k)Q_k\right]=0,$$

which implies

$$LV=\sum_{k=1}^{n}\frac{\beta_k p_k}{\mu+\gamma_k}\left[\mu S^0-\mu S-\frac{c\mu S^0}{S}+c\mu-\sum_{j=1}^{n}\beta_j p_j\frac{Q_j}{Q_i}S+(\mu+\gamma_k)+\frac{c\sigma_1^2}{2}+\frac{\sigma_{k+1}^2}{2}\right]$$
$$\leqslant\sum_{k=1}^{n}\frac{\beta_k p_k}{\mu+\gamma_k}\left[\mu S^0+c\mu+(\mu+\gamma_k)+\frac{c}{2}\sigma_1^2+\frac{\sigma_{k+1}^2}{2}\right]:=K.$$

Therefore,

$$\int_0^{\tau_m\wedge T}dV\big(S(r),Q_1(r),Q_2(r),\ldots,Q_n(r)\big)$$
$$\leqslant\int_0^{\tau_m\wedge T}K\,dr+\int_0^{\tau_m\wedge T}\sum_{k=1}^{n}a_k\left[\left(1-\frac{c}{S(r)}\right)\sigma_1 S(r)\,dB_1(r)+\left(1-\frac{1}{Q_k(r)}\right)\sigma_{k+1}Q_k(r)\,dB_{k+1}(r)\right].$$

Taking expectation, yields

$$E\big[V\big(S(\tau_m\wedge T),Q_1(\tau_m\wedge T),Q_2(\tau_m\wedge T),\ldots,Q_n(\tau_m\wedge T)\big)\big]$$
$$\leqslant V\big(S(0),Q_1(0),Q_2(0),\ldots,Q_n(0)\big)+E\int_0^{\tau_m\wedge T}K\,dr\leqslant V\big(S(0),Q_1(0),Q_2(0),\ldots,Q_n(0)\big)+KT. \tag{2.3}$$

Set $\Omega_m=\{\tau_m\leqslant T\}$ for $m\geqslant m_1$ and by (2.2), $P(\Omega_m)\geqslant\epsilon$. Note that for every $\omega\in\Omega_m$, there is at least one of $S(\tau_m,\omega)$, $Q_1(\tau_m,\omega),Q_2(\tau_m,\omega),\ldots,Q_n(\tau_m,\omega)$ equals either m or $1/m$. If $S(\tau_m,\omega)=m$ or $1/m$, then

$$V\big(S(\tau_m\wedge T),Q_1(\tau_m\wedge T),Q_2(\tau_m\wedge T),\ldots,Q_n(\tau_m\wedge T)\big)$$
$$\geqslant\sum_{k=1}^{n}a_k\left(m-c-c\log\frac{m}{c}\right)\wedge\sum_{k=1}^{n}a_k\left(\frac{1}{m}-c-c\log\frac{1}{cm}\right)=\left[\frac{m}{c}-1-\log\frac{m}{c}\right]\wedge\left[\frac{1}{cm}-1-\log\frac{1}{cm}\right],$$

while if for some $1 \leqslant k_0 \leqslant n$, $Q_{k_0}(\tau_m, \omega) = m$ or $1/m$, then

$$V\big(S(\tau_m \wedge T), Q_1(\tau_m \wedge T), Q_2(\tau_m \wedge T), \dots, Q_n(\tau_m \wedge T)\big) \geqslant \big[a_{k_0}(m - 1 - \log m)\big] \wedge \left[a_{k_0}\left(\frac{1}{m} - 1 - \log\frac{1}{m}\right)\right].$$

Consequently,

$$\begin{aligned} &V\big(S(\tau_m \wedge T), Q_1(\tau_m \wedge T), Q_2(\tau_m \wedge T), \dots, Q_n(\tau_m \wedge T)\big) \\ &\quad \geqslant \left[\frac{m}{c} - 1 - \log\frac{m}{c}\right] \wedge \left[\frac{1}{cm} - 1 - \log\frac{1}{cm}\right] \wedge \big[a_{k_0}(m - 1 - \log m)\big] \wedge \left[a_{k_0}\left(\frac{1}{m} - 1 - \log\frac{1}{m}\right)\right]. \end{aligned}$$

It then follows from (2.2) and (2.3) that

$$\begin{aligned} &V\big(S(0), Q_1(0), Q_2(0), \dots, Q_n(0)\big) + KT \\ &\quad \geqslant E\big[1_{\Omega_m(\omega)} V\big(S(\tau_m \wedge T), Q_1(\tau_m \wedge T), Q_2(\tau_m \wedge T), \dots, Q_n(\tau_m \wedge T)\big)\big] \\ &\quad \geqslant \epsilon\left[\frac{m}{c} - 1 - \log\frac{m}{c}\right] \wedge \left[\frac{1}{cm} - 1 - \log\frac{1}{cm}\right] \wedge \big[a_{k_0}(m - 1 - \log m)\big] \wedge \left[a_{k_0}\left(\frac{1}{m} - 1 - \log\frac{1}{m}\right)\right], \end{aligned}$$

where $1_{\Omega_m(\omega)}$ is the indicator function of Ω_m. Letting $m \to \infty$ leads to the contradiction $\infty > V(S(0), Q_1(0), Q_2(0), \dots, Q_n(0)) + KT = \infty$. So we must therefore have $\tau_\infty = \infty$ a.s. □

In reality, the initial value $S(0)$, $Q_k(0)$, $k = 1, 2, \dots, n$, can be zero. It is both interesting and practically important to consider what happens when this occurs, i.e. $(S(0), Q_1(0), Q_2(0), \dots, Q_n(0)) \in \bar{R}_+^{n+1}$.

Theorem 2.2. *For any initial value $(S(0), Q_1(0), Q_2(0), \dots, Q_n(0)) \in \bar{R}_+^{n+1}$, the solution of system* (2.1) *will remain in $\bar{R}_+^{n+1}$ with probability* 1, *namely $(S(t), Q_1(t), Q_2(t), \dots, Q_n(t)) \in \bar{R}_+^{n+1}$ for all $t \geqslant 0$ almost surely.*

Proof. Clearly,

$$S(t) = e^{-(\mu + \frac{\sigma_1^2}{2})t - \sum_{j=1}^n \int_0^t \beta_j p_j Q_j(u)\,du - \sigma_1 B_1(t)} \left[S_0 + \mu S^0 \int_0^t e^{(\mu + \frac{\sigma_1^2}{2})u + \sum_{j=1}^n \int_0^u \beta_j p_j Q_j(v)\,dv + \sigma_1 B_1(u)}\,du\right],$$

then $S(t) > 0$ no matter $S(0) > 0$ or $S(0) = 0$. Next we investigate $Q_k(t)$, $k = 1, 2, \dots, n$,

$$\begin{aligned} Q_k(t) &= e^{-(\mu + \gamma_k + \frac{\sigma_{k+1}^2}{2})t + \sigma_{k+1} B_{k+1}(t) + \beta_k p_k \int_0^t S(u)\,du} \\ &\quad \times \left[Q_{k0} + \sum_{j \neq k} \beta_j p_j \int_0^t Q_j(u) S(u) e^{(\mu + \gamma_k + \frac{\sigma_{k+1}^2}{2})u - \sigma_{k+1} B_{k+1}(u) - \beta_k p_k \int_0^u S(r)\,dr}\,du\right]. \end{aligned}$$

Obviously, $Q_k(t) \geqslant 0$ when $Q_k(0) \geqslant 0$, $k = 1, 2, \dots, n$. □

2.2. Asymptotic behavior around the disease-free equilibrium of the deterministic model

Obviously, $E_0 = (S^0, 0, \dots, 0)$ is the solution of system (1.1), which is called the disease-free equilibrium. If $R_0 \leqslant 1$, then E_0 is globally asymptotically stable, which means the disease will die out after some period of time. Therefore, it is interesting to study the disease-free equilibrium for controlling infectious disease. But, there is no disease-free equilibrium in system (1.2). It is natural to ask how we can consider the disease will be extinct. In this section we mainly through estimating the oscillation around E_0 to reflect whether the disease will die out.

Theorem 2.3. *Let $(S(t), I_1(t), I_2(t), \dots, I_n(t))$ be the solution of system* (1.2) *with initial value $(S(0), I_1(0), I_2(0), \dots, I_n(0)) \in R_+^{n+1}$. If $R_0 = S^0 \sum_{j=1}^n \frac{\beta_j p_j}{\mu + \gamma_j} \leqslant 1$, $\mu > \sigma_1^2$ and $\mu + \gamma_k > \frac{\sigma_{k+1}^2}{2}$ for each $k = 1, 2, \dots, n$, then*

$$\limsup_{t \to \infty} \frac{1}{t} E \int_0^t \left[\big(\mu - \sigma_1^2\big)\big(S(r) - S^0\big)^2 + \frac{\sum_{k=1}^n \frac{\beta_k}{(2\mu + \gamma_k) p_k}(\mu + \gamma_k - \frac{\sigma_{k+1}^2}{2})}{\sum_{j=1}^n \frac{\beta_j p_j}{2\mu + \gamma_j}} I_k^2(r)\right] dr \leqslant \sigma_1^2 \big(S^0\big)^2. \tag{2.4}$$

Proof. To prove (2.4), we only need to show the solution of system (2.1) has

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\left[\left(\mu-\sigma_1^2\right)\left(S(r)-S^0\right)^2+\frac{\sum_{k=1}^n\frac{\beta_k p_k}{2\mu+\gamma_k}(\mu+\gamma_k-\frac{\sigma_{k+1}^2}{2})}{\sum_{j=1}^n\frac{\beta_j p_j}{2\mu+\gamma_j}}Q_k^2(r)\right]dr\leqslant\sigma_1^2\left(S^0\right)^2.$$

First change the variables $u=S-S^0$, $v_k=Q_k$, $k=1,2,\dots,n$, then

$$\begin{cases} du=\left(-\mu u-\sum_{j=1}^n\beta_j p_j v_j u-\sum_{j=1}^n\beta_j p_j S^0 v_j\right)dt+\sigma_1\left(u+S^0\right)dB_1(t),\\ dv_k=\left[\sum_{j=1}^n\beta_j p_j v_j u+\sum_{j=1}^n\beta_j p_j S^0 v_j-(\mu+\gamma_k)v_k\right]dt+\sigma_{k+1}v_k\,dB_{k+1}(t),\quad k=1,2,\dots,n. \end{cases}\tag{2.5}$$

Define C^2 function $V:R_+^{n+1}\to R_+$ by

$$V(u,v_1,v_2,\dots,v_n)=\sum_{k=1}^n a_k(u+v_k)^2+\sum_{k=1}^n b_k v_k,$$

where a_k, b_k, $k=1,2,\dots,n$, are positive constants. Then V is positive definite, and along the trajectories of system (2.5) we have

$$\begin{aligned} dV &= \left\{\sum_{k=1}^n a_k\left[2(u+v_k)\left[-\mu u-(\mu+\gamma_k)v_k\right]+\sigma_1^2\left(u+S^0\right)^2+\sigma_{k+1}^2v_k^2\right]\right.\\ &\quad\left.+\sum_{k=1}^n b_k\left[\sum_{j=1}^n\beta_j p_j v_j u+\sum_{j=1}^n\beta_j p_j S^0 v_j-(\mu+\gamma_k)v_k\right]\right\}dt\\ &\quad+\sum_{k=1}^n a_k\left[2u\sigma_1\left(u+S^0\right)dB_1(t)+2v_k\sigma_{k+1}v_k\,dB_{k+1}\right]+\sum_{k=1}^n b_k\sigma_{k+1}v_k\,dB_{k+1}(t)\\ &:= LV\,dt+2\sum_{k=1}^n a_k\sigma_1(u+v_k)\left(u+S^0\right)dB_1(t)+\sum_{k=1}^n\sigma_{k+1}\left(2a_k(u+v_k)+b_k\right)v_k\,dB_{k+1}(t), \end{aligned}$$

where

$$\begin{aligned} LV &= -\sum_{k=1}^n a_k\left[\left(2\mu-\sigma_1^2\right)u^2+\left(2\mu+2\gamma_k-\sigma_{k+1}^2\right)v_k^2-2\sigma_1^2S^0u-\sigma_1^2\left(S^0\right)^2\right]\\ &\quad-2\sum_{k=1}^n a_k(2\mu+\gamma_k)uv_k+\sum_{k=1}^n\sum_{j=1}^n b_k\beta_j p_j u v_j+\sum_{k=1}^n\sum_{j=1}^n b_k\beta_j p_j S^0 v_j-\sum_{k=1}^n b_k(\mu+\gamma_k)v_k. \end{aligned}\tag{2.6}$$

Note that

$$\begin{aligned} \sum_{k=1}^n\sum_{j=1}^n b_k\beta_j p_j S^0 v_j-\sum_{k=1}^n b_k(\mu+\gamma_k)v_k &= \sum_{k=1}^n\sum_{j=1}^n b_j\beta_k p_k S^0 v_k-\sum_{k=1}^n b_k(\mu+\gamma_k)v_k\\ &= \sum_{k=1}^n\left[\sum_{j=1}^n b_j\beta_k p_k S^0-b_k(\mu+\gamma_k)\right]v_k, \end{aligned}\tag{2.7}$$

then we choose $b_k=\frac{\beta_k p_k}{\mu+\gamma_k}$, $k=1,2,\dots,n$, such that

$$\sum_{j=1}^n b_j\beta_k p_k S^0-b_k(\mu+\gamma_k)=\sum_{j=1}^n\left(\frac{\beta_j p_j}{\mu+\gamma_j}S^0-1\right)\beta_k p_k=(R_0-1)\beta_k p_k,$$

which together with $R_0\leqslant 1$ implies (2.7) that

$$\sum_{k=1}^n\sum_{j=1}^n b_k\beta_j p_j S^0 v_j-\sum_{k=1}^n b_k(\mu+\gamma_k)v_k\leqslant 0.\tag{2.8}$$

In the meantime, we see

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{j=1}^{n} b_k\beta_j p_j u v_j - 2\sum_{k=1}^{n} a_k(2\mu+\gamma_k)u v_k \\
&\quad = \sum_{k=1}^{n}\sum_{j=1}^{n} b_j\beta_k p_k u v_k - 2\sum_{k=1}^{n} a_k(2\mu+\gamma_k)u v_k = \sum_{k=1}^{n}\left[\sum_{j=1}^{n} b_j\beta_k p_k - 2a_k(2\mu+\gamma_k)\right]u v_k \\
&\quad = \sum_{k=1}^{n}\left[\sum_{j=1}^{n}\frac{\beta_j p_j}{\mu+\gamma_j}\beta_k p_k - 2a_k(2\mu+\gamma_k)\right]u v_k = \sum_{k=1}^{n}\left[\frac{R_0}{S^0}\beta_k p_k - 2a_k(2\mu+\gamma_k)\right]u v_k, \qquad (2.9)
\end{aligned}$$

choosing $a_k = \frac{R_0\beta_k p_k}{2(2\mu+\gamma_k)S^0} > 0$ such that for each k, $\frac{R_0}{S^0}\beta_k p_k - 2a_k(2\mu+\gamma_k) = 0$, and so

$$\sum_{k=1}^{n}\left[\frac{R_0}{S^0}\beta_k p_k - 2a_k(2\mu+\gamma_k)\right]u v_k = 0. \qquad (2.10)$$

Taking (2.8)–(2.10) together, (2.6) can be written as

$$LV \leqslant -\sum_{k=1}^{n}\frac{R_0\beta_k p_k}{2(2\mu+\gamma_k)S^0}\left[\left(2\mu-\sigma_1^2\right)u^2 + \left(2\mu+2\gamma_k-\sigma_{k+1}^2\right)v_k^2 - 2\sigma_1^2 S^0 u - \sigma_1^2\left(S^0\right)^2\right].$$

Besides, $2\sigma_1^2 S^0 u \leqslant \sigma_1^2 u^2 + \sigma_1^2 (S^0)^2$, then

$$LV \leqslant -\sum_{k=1}^{n}\frac{R_0\beta_k p_k}{(2\mu+\gamma_k)S^0}\left[\left(\mu-\sigma_1^2\right)u^2 + \left(\mu+\gamma_k-\frac{\sigma_{k+1}^2}{2}\right)v_k^2\right] + \sum_{k=1}^{n}\frac{R_0\beta_k p_k}{2\mu+\gamma_k}\sigma_1^2 S^0$$

and

$$\begin{aligned}
dV \leqslant{}& \left\{-\sum_{k=1}^{n}\frac{R_0\beta_k p_k}{(2\mu+\gamma_k)S^0}\left[\left(\mu-\sigma_1^2\right)u^2 + \left(\mu+\gamma_k-\frac{\sigma_{k+1}^2}{2}\right)v_k^2\right] + \sum_{k=1}^{n}\frac{R_0\beta_k p_k}{2\mu+\gamma_k}\sigma_1^2 S^0\right\}dt \\
&+ 2\sum_{k=1}^{n} a_k\sigma_1(u+v_k)\left(u+S^0\right)dB_1(t) + \sum_{k=1}^{n}\sigma_{k+1}\left(2a_k(u+v_k)+b_k\right)v_k\,dB_{k+1}(t). \qquad (2.11)
\end{aligned}$$

Integrating both sides of (2.11) from 0 to t, and taking expectation, yields

$$\begin{aligned}
E\left[V(t)\right] - V(0) &= E\left[\int_0^t LV(r)\,dr\right] \\
&\leqslant -E\sum_{k=1}^{n}\int_0^t \frac{R_0\beta_k p_k}{(2\mu+\gamma_k)S^0}\left[\left(\mu-\sigma_1^2\right)u^2(r) + \left(\mu+\gamma_k-\frac{\sigma_{k+1}^2}{2}\right)v_k^2(r)\right]dr + \sum_{k=1}^{n}\frac{R_0\beta_k p_k}{2\mu+\gamma_k}\sigma_1^2 S^0 t.
\end{aligned}$$

Hence

$$\limsup_{t\to\infty}\frac{1}{t}E\sum_{k=1}^{n}\int_0^t \frac{R_0\beta_k p_k}{(2\mu+\gamma_k)S^0}\left[\left(\mu-\sigma_1^2\right)u^2(r) + \left(\mu+\gamma_k-\frac{\sigma_{k+1}^2}{2}\right)v_k^2(r)\right]dr \leqslant \sum_{k=1}^{n}\frac{R_0\beta_k p_k}{2\mu+\gamma_k}\sigma_1^2 S^0,$$

which can be simplified to

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t \left[\left(\mu-\sigma_1^2\right)u^2(r) + \frac{\sum_{k=1}^{n}\frac{\beta_k p_k}{2\mu+\gamma_k}\left(\mu+\gamma_k-\frac{\sigma_{k+1}^2}{2}\right)}{\sum_{j=1}^{n}\frac{\beta_j p_j}{2\mu+\gamma_j}}v_k^2(r)\right]dr \leqslant \sigma_1^2\left(S^0\right)^2.$$

Consequently,

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t \left[\left(\mu-\sigma_1^2\right)\left(S(r)-S^0\right)^2 + \frac{\sum_{k=1}^{n}\frac{\beta_k}{(2\mu+\gamma_k)p_k}\left(\mu+\gamma_k-\frac{\sigma_{k+1}^2}{2}\right)}{\sum_{j=1}^{n}\frac{\beta_j p_j}{2\mu+\gamma_j}}I_k(r)^2\right]dr \leqslant \sigma_1^2\left(S^0\right)^2. \qquad \square$$

Remark 2.1. From Theorem 2.3, we can know, under some conditions, the solution of system (1.2) will oscillate around the disease-free equilibrium, and the intensity of fluctuation is only relation to the intensity of the white noise $\dot{B}_1(t)$, but do not relation to the intensities of the other white noises. In a biological interpretation, if the intensity of stochastic perturbations on S is small, the solution of system (1.2) will be close to the disease-free equilibrium of system (1.1) most of the time.

Besides, if $\sigma_1 = 0$, then E_0 is also the disease-free equilibrium of system (1.2). From the proof of Theorem 2.3, we can get

$$\begin{aligned} LV &\leqslant -\sum_{k=1}^{n} \frac{R_0\beta_k p_k}{2(2\mu+\gamma_k)S^0}\big[(2\mu-\sigma_1^2)u^2 + (2\mu+2\gamma_k-\sigma_{k+1}^2)v_k^2 - 2\sigma_1^2 S^0 u - \sigma_1^2 (S^0)^2\big] \\ &= -\sum_{k=1}^{n} \frac{R_0\beta_k p_k}{(2\mu+\gamma_k)S^0}\left[\mu u^2 + \left(\mu+\gamma_k - \frac{\sigma_{k+1}^2}{2}\right)v_k^2\right], \end{aligned}$$

which is negative-definite, if $\mu+\gamma_k > \frac{\sigma_{k+1}^2}{2}$, $k = 1, 2, \ldots, n$. Therefore the solution of system (1.2) is stochastically asymptotically stable in the large (see Mao [24]).

2.3. Asymptotic behavior around the endemic equilibrium of the deterministic model

When studying epidemic dynamical system, we are interested in two problems. One is when the disease will die out, which has been shown in the above part, another is when the disease will prevail and persist in a population. In the deterministic models, the second problem is solved by showing that the endemic equilibrium to corresponding model is a global attractor or is globally asymptotic stable. But, there is none of endemic equilibrium in system (1.2). Since system (1.2) is the perturbation system of system (1.1) which has an endemic equilibrium E^*, it seems reasonable to consider the disease will be prevail if the solution of system (1.2) is going around E^* at the most time. We get following results.

Theorem 2.4. *Let $(S(t), I_1(t), I_2(t), \ldots, I_n(t))$ be the solution of system* (1.2) *with any initial value $(S(0), I_1(0), I_2(0), \ldots, I_n(0)) \in \bar{R}_+^{n+1}$. If $R_0 > 1$, then*

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \frac{(S(r)-S^*)^2}{S(r)}\,dr \leqslant \frac{\sum_{k=1}^n \beta_k I_k^*(S^*\sigma_1^2 + \frac{I_k^*}{p_k}\sigma_{k+1}^2)}{2\sum_{k=1}^n \beta_k I_k^*(\mu + \sum_{j=1}^n \beta_j I_j^*)},$$

where $(S^*, I_1^*, I_2^*, \ldots, I_n^*)$ is the endemic equilibrium of system (1.1).

Proof. When $R_0 > 1$, there is the endemic equilibrium $E^* = (S^*, I_1^*, \ldots, I_n^*)$. Setting the right-hand sides of system (1.1) to be zero, we get

$$\mu S^0 - \mu S^* - \sum_{j=1}^n \beta_j I_j^* S^* = 0, \qquad p_k \sum_{j=1}^n \beta_j I_j^* S^* - (\mu+\gamma_k) I_k^* = 0, \quad k = 1, 2, \ldots, n,$$

which gives

$$\frac{\mu S^0}{S^*} = \mu + \sum_{j=1}^n \beta_j I_j^* = \mu + \sum_{j=1}^n \beta_j p_j Q_j^* \tag{2.12}$$

and

$$\mu+\gamma_k = \frac{p_k \sum_{j=1}^n \beta_j I_j^* S^*}{I_k^*} = \frac{\sum_{j=1}^n \beta_j p_j Q_j^* S^*}{Q_k^*}. \tag{2.13}$$

Let $\tilde{S} = \frac{S}{S^*}$, $\tilde{Q}_k = \frac{Q_k}{Q_k^*}$, then system (2.1) can be written as

$$\begin{cases} d\tilde{S} = \left(\mu\dfrac{S^0}{S^*} - \mu\tilde{S} - \displaystyle\sum_{j=1}^n \beta_j p_j Q_j^* \tilde{Q}_j \tilde{S}\right)dt + \sigma_1 \tilde{S}\,dB_1(t), \\ d\tilde{Q}_k = \left[\displaystyle\sum_{j=1}^n \beta_j p_j S^* \dfrac{Q_j^*}{Q_k^*}\tilde{Q}_j\tilde{S} - (\mu+\gamma_k)\tilde{Q}_k\right]dt + \sigma_{k+1}\tilde{Q}_k\,dB_{k+1}(t), \quad k = 1, 2, \ldots, n. \end{cases}$$

Define

$$V(S, Q_1, Q_2, \dots, Q_n) = \sum_{k=1}^{n} a_k \left[(\tilde{S} - 1 - \log \tilde{S}) + \frac{Q_k^*}{S^*}(\tilde{Q}_k - 1 - \log \tilde{Q}_k) \right],$$

where a_k, $k = 1, 2, \dots, n$, are positive constants to be determined later. Then V is positive definite, and

$$\begin{aligned} dV &= \left\{ \sum_{k=1}^{n} a_k \left[\left(1 - \frac{1}{\tilde{S}}\right) \left(\mu \frac{S^0}{S^*} - \mu \tilde{S} - \sum_{j=1}^{n} \beta_j p_j Q_j^* \tilde{Q}_j \tilde{S} \right) + \frac{\sigma_1^2}{2} \right] \right. \\ &\quad \left. + \sum_{k=1}^{n} \frac{a_k Q_k^*}{S^*} \left[\left(1 - \frac{1}{\tilde{Q}_k}\right) \left(\sum_{j=1}^{n} \beta_j p_j S^* \frac{Q_j^*}{Q_k^*} \tilde{Q}_j \tilde{S} - (\mu + \gamma_k) \tilde{Q}_k \right) + \frac{\sigma_{k+1}^2}{2} \right] \right\} dt \\ &\quad + \sum_{k=1}^{n} a_k \left[\left(1 - \frac{1}{\tilde{S}}\right) \sigma_1 \tilde{S}\, dB_1(t) + \frac{Q_k^*}{S^*} \left(1 - \frac{1}{\tilde{Q}_k}\right) \sigma_{k+1} \tilde{Q}_k\, dB_{k+1}(t) \right] \\ &:= LV\, dt + \sum_{k=1}^{n} a_k \left[\sigma_1 (\tilde{S} - 1)\, dB_1(t) + \frac{Q_k^* \sigma_{k+1}}{S^*} (\tilde{Q}_k - 1)\, dB_{k+1}(t) \right]. \end{aligned}$$

We compute

$$\begin{aligned} LV &= \sum_{k=1}^{n} a_k \left[\left(1 - \frac{1}{\tilde{S}}\right) \left(\mu \frac{S^0}{S^*} - \mu \tilde{S} - \sum_{j=1}^{n} \beta_j p_j Q_j^* \tilde{Q}_j \tilde{S} \right) + \frac{\sigma_1^2}{2} \right] \\ &\quad + \sum_{k=1}^{n} \frac{a_k Q_k^*}{S^*} \left[\left(1 - \frac{1}{\tilde{Q}_k}\right) \left(\sum_{j=1}^{n} \beta_j p_j S^* \frac{Q_j^*}{Q_k^*} \tilde{Q}_j \tilde{S} - (\mu + \gamma_k) \tilde{Q}_k \right) + \frac{\sigma_{k+1}^2}{2} \right] \\ &= \sum_{k=1}^{n} a_k \left[\frac{\mu S^0}{S^*} - \mu \tilde{S} - \frac{\mu S^0}{S^* \tilde{S}} + \mu + \frac{\sigma_1^2}{2} + \sum_{j=1}^{n} \beta_j p_j Q_j^* \tilde{Q}_j \right. \\ &\quad \left. - \frac{Q_k^*(\mu + \gamma_k)}{S^*} \tilde{Q}_k - \sum_{j=1}^{n} \beta_j p_j Q_j^* \frac{\tilde{Q}_j}{\tilde{Q}_k} \tilde{S} + \frac{Q_k^*(\mu + \gamma_k)}{S^*} + \frac{Q_k^* \sigma_{k+1}^2}{2S^*} \right]. \end{aligned} \tag{2.14}$$

Note that

$$\begin{aligned} &\sum_{k=1}^{n} \sum_{j=1}^{n} a_k \beta_j p_j Q_j^* \tilde{Q}_j - \sum_{k=1}^{n} a_k \frac{Q_k^*(\mu + \gamma_k)}{S^*} \tilde{Q}_k \\ &\quad = \sum_{k=1}^{n} \sum_{j=1}^{n} a_j \beta_k p_k Q_k^* \tilde{Q}_k - \sum_{k=1}^{n} a_k \frac{Q_k^*(\mu + \gamma_k)}{S^*} \tilde{Q}_k = \sum_{k=1}^{n} \left[\sum_{j=1}^{n} a_j \beta_k p_k Q_k^* - \sum_{j=1}^{n} a_k \beta_j p_j Q_j^* \right] \tilde{Q}_k, \end{aligned}$$

where (2.13) is used in the last equality. Choose $a_k = \beta_k p_k Q_k^*$ such that $\sum_{j=1}^{n} a_j \beta_k p_k Q_k^* = \sum_{j=1}^{n} a_k \beta_j p_j Q_j^*$. This together with (2.12) implies (2.14) that

$$\begin{aligned} LV &= \sum_{k=1}^{n} a_k \left[\frac{\mu S^0}{S^*} - \mu \tilde{S} - \frac{\mu S^0}{S^* \tilde{S}} + \mu + \frac{\sigma_1^2}{2} - \sum_{j=1}^{n} \beta_j p_j Q_j^* \frac{\tilde{Q}_j}{\tilde{Q}_k} \tilde{S} + \frac{Q_k^*(\mu + \gamma_k)}{S^*} + \frac{Q_k^* \sigma_{k+1}^2}{2S^*} \right] \\ &= \sum_{k=1}^{n} a_k \left[\mu + \sum_{j=1}^{n} \beta_j p_j Q_j^* - \mu \tilde{S} - \frac{\mu + \sum_{j=1}^{n} \beta_j p_j Q_j^*}{\tilde{S}} + \mu + \frac{\sigma_1^2}{2} \right. \\ &\quad \left. - \sum_{j=1}^{n} \beta_j p_j Q_j^* \frac{\tilde{Q}_j}{\tilde{Q}_k} \tilde{S} + \frac{Q_k^*(\mu + \gamma_k)}{S^*} + \frac{Q_k^* \sigma_{k+1}^2}{2S^*} \right] \\ &= \sum_{k=1}^{n} a_k \left[-\frac{\mu}{\tilde{S}} (1 + \tilde{S}^2 - 2\tilde{S}) + \sum_{j=1}^{n} \beta_j p_j Q_j^* - \sum_{j=1}^{n} \frac{\beta_j p_j Q_j^*}{\tilde{S}} - \sum_{j=1}^{n} \beta_j p_j Q_j^* \frac{\tilde{Q}_j}{\tilde{Q}_k} \tilde{S} \right. \\ &\quad \left. + \frac{Q_k^*(\mu + \gamma_k)}{S^*} + \frac{\sigma_1^2}{2} + \frac{Q_k^* \sigma_{k+1}^2}{2S^*} \right], \end{aligned}$$

which can be simplified to

$$LV=\sum_{k=1}^{n}a_k\left[-\frac{\mu}{\tilde S}(1-\tilde S)^2+2\sum_{j=1}^{n}\beta_j p_j Q_j^*-\sum_{j=1}^{n}\frac{\beta_j p_j Q_j^*}{\tilde S}-\sum_{j=1}^{n}\beta_j p_j Q_j^*\frac{\tilde Q_j}{\tilde Q_k}\tilde S+\frac{\sigma_1^2}{2}+\frac{Q_k^*\sigma_{k+1}^2}{2S^*}\right],$$

according to (2.13). Besides, by Cauchy inequality,

$$\begin{aligned}\sum_{k=1}^{n}a_k\sum_{j=1}^{n}\beta_j p_j Q_j^*\frac{\tilde Q_j}{\tilde Q_k}\tilde S&=\sum_{k=1}^{n}\beta_k p_k Q_k^*\sum_{j=1}^{n}\beta_j p_j Q_j^*\frac{\tilde Q_j}{\tilde Q_k}\tilde S=\sum_{k=1}^{n}\beta_k p_k\frac{Q_k^*}{\tilde Q_k}\sum_{j=1}^{n}\beta_j p_j Q_j^*\tilde Q_j\tilde S\\&=\left(\sum_{k=1}^{n}\sqrt{\beta_k p_k\frac{Q_k^*}{\tilde Q_k}}^{\,2}\right)\left(\sum_{k=1}^{n}\sqrt{\beta_k p_k Q_k^*\tilde Q_k}^{\,2}\right)\tilde S\geqslant\left(\sum_{k=1}^{n}\beta_k p_k Q_k^*\right)^2\tilde S,\end{aligned}$$

then

$$\begin{aligned}LV&\leqslant\sum_{k=1}^{n}a_k\left[-\frac{\mu}{\tilde S}(1-\tilde S)^2+2\sum_{j=1}^{n}\beta_j p_j Q_j^*-\sum_{j=1}^{n}\frac{\beta_j p_j Q_j^*}{\tilde S}-\sum_{j=1}^{n}\beta_j p_j Q_j^*\tilde S+\frac{\sigma_1^2}{2}+\frac{Q_k^*\sigma_{k+1}^2}{2S^*}\right]\\&=\sum_{k=1}^{n}a_k\left[-\frac{\mu}{\tilde S}(1-\tilde S)^2-\sum_{j=1}^{n}\beta_j p_j Q_j^*\frac{(\tilde S-1)^2}{\tilde S}\right]+\frac{1}{2}\sum_{k=1}^{n}a_k\left(\sigma_1^2+\frac{Q_k^*}{S^*}\sigma_{k+1}^2\right)\\&=-\sum_{k=1}^{n}a_k\left[\mu+\sum_{j=1}^{n}\beta_j p_j Q_j^*\right]\frac{(\tilde S-1)^2}{\tilde S}+\frac{1}{2}\sum_{k=1}^{n}a_k\left(\sigma_1^2+\frac{Q_k^*}{S^*}\sigma_{k+1}^2\right)\end{aligned}$$

and

$$\begin{aligned}dV&\leqslant\left(-\sum_{k=1}^{n}a_k\left[\mu+\sum_{j=1}^{n}\beta_j p_j Q_j^*\right]\frac{(\tilde S-1)^2}{\tilde S}+\frac{1}{2}\sum_{k=1}^{n}a_k\left(\sigma_1^2+\frac{Q_k^*}{S^*}\sigma_{k+1}^2\right)\right)dt\\&\quad+\sum_{k=1}^{n}a_k\left[\sigma_1(\tilde S-1)\,dB_1(t)+\frac{Q_k^*\sigma_{k+1}}{S^*}(\tilde Q_k-1)\,dB_{k+1}(t)\right].\end{aligned}$$

Integrating both sides of it from 0 to t, and taking expectation, yields

$$E\big[V(t)\big]-V(0)\leqslant-\sum_{k=1}^{n}a_k\left(\mu+\sum_{j=1}^{n}\beta_j p_j Q_j^*\right)E\int_0^t\frac{(\tilde S(r)-1)^2}{\tilde S(r)}\,dr+\frac{1}{2}\sum_{k=1}^{n}a_k\left(\sigma_1^2+\frac{Q_k^*}{S^*}\sigma_{k+1}^2\right)t.$$

Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\frac{(\tilde S(r)-1)^2}{\tilde S(r)}\,dr\leqslant\frac{\sum_{k=1}^{n}a_k(\sigma_1^2+\frac{Q_k^*}{S^*}\sigma_{k+1}^2)}{2\sum_{k=1}^{n}a_k(\mu+\sum_{j=1}^{n}\beta_j p_j Q_j^*)}=\frac{\sum_{k=1}^{n}\beta_k p_k Q_k^*(\sigma_1^2+\frac{Q_k^*}{S^*}\sigma_{k+1}^2)}{2\sum_{k=1}^{n}\beta_k p_k Q_k^*(\mu+\sum_{j=1}^{n}\beta_j p_j Q_j^*)},$$

i.e.

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t\frac{(S(r)-S^*)^2}{S(r)}\,dr\leqslant\frac{\sum_{k=1}^{n}\beta_k I_k^*(S^*\sigma_1^2+\frac{I_k^*}{p_k}\sigma_{k+1}^2)}{2\sum_{k=1}^{n}\beta_k I_k^*(\mu+\sum_{j=1}^{n}\beta_j I_j^*)}.\qquad\square$$

Theorem 2.5. *Let $(S(t),I_1(t),I_2(t),\ldots,I_n(t))$ be the solution of system (1.2) with any initial value $(S(0),I_1(0),I_2(0),\ldots,I_n(0))\in\bar R_+^{n+1}$. If $R_0>1$, $\mu>\sigma_1^2$ and $\mu+\gamma_k>\sigma_{k+1}^2$, $k=1,2,\ldots,n$, then*

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\sum_{k=1}^{n}a_k\left\{\left(\mu-\sigma_1^2\right)E\int_0^t\left(S(r)-S^*\right)^2dr+\frac{\mu+\gamma_k-\sigma_{k+1}^2}{p_k^2}E\int_0^t\left(I_k(r)-I_k^*\right)^2dr\right\}\\&\quad\leqslant\sum_{k=1}^{n}a_k\sigma_1^2\left(S^*\right)^2+\sum_{k=1}^{n}a_k\frac{\sigma_{k+1}^2}{p_k^2}\left(I_k^*\right)^2+\sum_{k=1}^{n}\frac{\beta_k\sigma_{k+1}^2}{2p_k}\left(I_k^*\right)^2S^*,\end{aligned}$$

where $a_k=\frac{p_k\beta_k S^*}{2\mu+\gamma_k}\sum_{j=1}^{n}\beta_j I_j^*$, $k=1,2,\ldots,n$, and $(S^*,I_1^*,I_2^*,\ldots,I_n^*)$ is the endemic equilibrium of system (1.1).

Proof. Define

$$V_1(I_1, I_2, \dots, I_n) = \sum_{k=1}^{n} \frac{\beta_k I_k^* S^*}{p_k} \left(I_k - I_k^* - I_k^* \log \frac{I_k}{I_k^*} \right).$$

By Itô's formula, we get

$$dV_1 = LV_1\, dt + \sum_{k=1}^{n} \frac{\beta_k I_k^* S^*}{p_k} \left(1 - \frac{I_k^*}{I_k} \right) \sigma_{k+1} I_k\, dB_{k+1}(t),$$

where

$$\begin{aligned} LV_1 &= \sum_{k=1}^{n} \frac{\beta_k}{p_k} I_k^* S^* \left(1 - \frac{I_k^*}{I_k} \right) \left[p_k \sum_{j=1}^{n} \beta_j I_j S - (\mu + \gamma_k) I_k \right] + \sum_{k=1}^{n} \frac{\beta_k \sigma_{k+1}^2}{2p_k} \big(I_k^*\big)^2 S^* \\ &= \sum_{k=1}^{n} \frac{\beta_k}{p_k} I_k^* S^* \left[p_k \sum_{j=1}^{n} \beta_j I_j S - (\mu + \gamma_k) I_k - p_k \sum_{j=1}^{n} \beta_j \frac{I_k^*}{I_k} I_j S + (\mu + \gamma_k) I_k^* + \frac{1}{2} \sigma_{k+1}^2 I_k^* \right]. \end{aligned} \tag{2.15}$$

Substituting (2.13) into (2.15), gives

$$\begin{aligned} LV_1 &= \sum_{k=1}^{n} \frac{\beta_k}{p_k} I_k^* S^* \left[p_k \sum_{j=1}^{n} \beta_j I_j S - \frac{p_k \sum_{j=1}^{n} \beta_j I_j^* S^*}{I_k^*} I_k - p_k \sum_{j=1}^{n} \beta_j \frac{I_k^*}{I_k} I_j S + p_k \sum_{j=1}^{n} \beta_j I_j^* S^* + \frac{1}{2} \sigma_{k+1}^2 I_k^* \right] \\ &= \sum_{k=1}^{n} \beta_k I_k^* S^* \left[\sum_{j=1}^{n} \beta_j I_j S - \frac{\sum_{j=1}^{n} \beta_j I_j^* S^*}{I_k^*} I_k - \sum_{j=1}^{n} \beta_j \frac{I_k^*}{I_k} I_j S + \sum_{j=1}^{n} \beta_j I_j^* S^* + \frac{\sigma_{k+1}^2 I_k^*}{2p_k} \right]. \end{aligned}$$

By Cauchy inequality, we get

$$\begin{aligned} \sum_{k=1}^{n} \beta_k I_k^* S^* \sum_{j=1}^{n} \beta_j \frac{I_k^*}{I_k} I_j S &= \sum_{k=1}^{n} \beta_k I_k^* S^* \frac{I_k^*}{I_k} \sum_{j=1}^{n} \beta_j I_j^* S^* \frac{I_j}{I_j^*} \frac{S}{S^*} = \left(\sum_{k=1}^{n} \beta_k I_k^* S^* \frac{I_k^*}{I_k} \right) \left(\sum_{k=1}^{n} \beta_k I_k^* S^* \frac{I_k}{I_k^*} \right) \frac{S}{S^*} \\ &\geqslant \left(\sum_{k=1}^{n} \beta_k I_k^* S^* \right)^2 \frac{S}{S^*} = \left(\sum_{k=1}^{n} \beta_k I_k^* \right)^2 S^* S, \end{aligned}$$

then

$$\begin{aligned} LV_1 &\leqslant \sum_{k=1}^{n} \beta_k I_k^* S^* \left[\sum_{j=1}^{n} \beta_j I_j S - \frac{\sum_{j=1}^{n} \beta_j I_j^* S^*}{I_k^*} I_k - \sum_{j=1}^{n} \beta_j I_j^* S + \sum_{j=1}^{n} \beta_j I_j^* S^* + \frac{\sigma_{k+1}^2 I_k^*}{2p_k} \right] \\ &= \sum_{k=1}^{n} \beta_k I_k^* S^* \left[\sum_{j=1}^{n} \beta_j S \big(I_j - I_j^*\big) + \sum_{j=1}^{n} \beta_j I_j^* S^* \left(1 - \frac{I_k}{I_k^*} \right) + \frac{\sigma_{k+1}^2 I_k^*}{2p_k} \right]. \end{aligned} \tag{2.16}$$

Note that

$$\begin{aligned} \sum_{k=1}^{n} \beta_k I_k^* S^* \sum_{j=1}^{n} \beta_j I_j^* S^* \left(1 - \frac{I_k}{I_k^*} \right) &= \sum_{k=1}^{n} \beta_k \big(I_k^* - I_k\big) \sum_{j=1}^{n} \beta_j I_j^* \big(S^*\big)^2 = \sum_{j=1}^{n} \beta_j \big(I_j^* - I_j\big) \sum_{k=1}^{n} \beta_k I_k^* \big(S^*\big)^2 \\ &= -\sum_{k=1}^{n} \beta_k I_k^* \big(S^*\big)^2 \sum_{j=1}^{n} \beta_j \big(I_j - I_j^*\big), \end{aligned}$$

then (2.16) gives

$$\begin{aligned} LV_1 &\leqslant \sum_{k=1}^{n} \beta_k I_k^* S^* S \sum_{j=1}^{n} \beta_j \big(I_j - I_j^*\big) - \sum_{k=1}^{n} \beta_k I_k^* \big(S^*\big)^2 \sum_{j=1}^{n} \beta_j \big(I_j - I_j^*\big) + \sum_{k=1}^{n} \frac{\beta_k \sigma_{k+1}^2}{2p_k} \big(I_k^*\big)^2 S^* \\ &= \sum_{k=1}^{n} \beta_k I_k^* S^* \sum_{j=1}^{n} \beta_j \big(I_j - I_j^*\big)\big(S - S^*\big) + \sum_{k=1}^{n} \frac{\beta_k \sigma_{k+1}^2}{2p_k} \big(I_k^*\big)^2 S^* \end{aligned}$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{n}\beta_k I_k^* S^* \beta_j (I_j - I_j^*)(S - S^*) + \sum_{k=1}^{n}\frac{\beta_k\sigma_{k+1}^2}{2p_k}(I_k^*)^2 S^*$$
$$= \sum_{k=1}^{n}\sum_{j=1}^{n}\beta_k \beta_j I_j^* S^* (I_k - I_k^*)(S - S^*) + \sum_{k=1}^{n}\frac{\beta_k\sigma_{k+1}^2}{2p_k}(I_k^*)^2 S^*.$$

Next, define

$$V_2(S, I_1, I_2, \dots, I_n) = \frac{1}{2}\sum_{k=1}^{n} a_k\left(S - S^* + \frac{I_k - I_k^*}{p_k}\right)^2,$$

where a_k, $k = 1, 2, \dots, n$, are positive constants to be determined later. Then

$$dV_2 = LV_2\,dt + \sum_{k=1}^{n} a_k\left(S - S^* + \frac{I_k - I_k^*}{p_k}\right)\left[\sigma_1 S\,dB_1(t) + \frac{\sigma_{k+1}}{p_k} I_k\,dB_{k+1}(t)\right],$$

where

$$\begin{aligned}
LV_2 &= \sum_{k=1}^{n} a_k\left(S - S^* + \frac{I_k - I_k^*}{p_k}\right)\left(\mu S^0 - \mu S - \frac{\mu+\gamma_k}{p_k} I_k\right) + \frac{1}{2}\sum_{k=1}^{n} a_k\left(\sigma_1^2 S^2 + \frac{\sigma_{k+1}^2}{p_k^2} I_k^2\right) \\
&= \sum_{k=1}^{n} a_k\left(S - S^* + \frac{I_k - I_k^*}{p_k}\right)\left(\mu S^* + \frac{\mu+\gamma_k}{p_k} I_k^* - \mu S - \frac{\mu+\gamma_k}{p_k} I_k\right) + \frac{1}{2}\sum_{k=1}^{n} a_k\left(\sigma_1^2 S^2 + \frac{\sigma_{k+1}^2}{p_k^2} I_k^2\right) \\
&= -\sum_{k=1}^{n} a_k\left[\mu(S - S^*)^2 + \frac{\mu+\gamma_k}{p_k^2}(I_k - I_k^*)^2\right] - \sum_{k=1}^{n} a_k\frac{2\mu+\gamma_k}{p_k}(S - S^*)(I_k - I_k^*) \\
&\quad + \frac{1}{2}\sum_{k=1}^{n} a_k\left(\sigma_1^2 S^2 + \frac{\sigma_{k+1}^2}{p_k^2} I_k^2\right).
\end{aligned} \tag{2.17}$$

Besides

$$\begin{aligned}
\frac{1}{2}\sum_{k=1}^{n} a_k\left(\sigma_1^2 S^2 + \frac{\sigma_{k+1}^2}{p_k^2} I_k^2\right) &= \frac{1}{2}\sum_{k=1}^{n} a_k\left[\sigma_1^2(S - S^* + S^*)^2 + \frac{\sigma_{k+1}^2}{p_k^2}(I_k - I_k^* + I_k^*)^2\right] \leqslant \sum_{k=1}^{n} a_k\sigma_1^2(S - S^*)^2 \\
&\quad + \sum_{k=1}^{n} a_k\frac{\sigma_{k+1}^2}{p_k^2}(I_k - I_k^*)^2 + \sum_{k=1}^{n} a_k\sigma_1^2(S^*)^2 + \sum_{k=1}^{n} a_k\frac{\sigma_{k+1}^2}{p_k^2}(I_k^*)^2,
\end{aligned}$$

which implies from (2.17) that

$$\begin{aligned}
LV_2 &\leqslant -\sum_{k=1}^{n} a_k\left[(\mu - \sigma_1^2)(S - S^*)^2 + \frac{\mu+\gamma_k-\sigma_{k+1}^2}{p_k^2}(I_k - I_k^*)^2\right] \\
&\quad - \sum_{k=1}^{n} a_k\frac{2\mu+\gamma_k}{p_k}(S - S^*)(I_k - I_k^*) + \sum_{k=1}^{n} a_k\sigma_1^2(S^*)^2 + \sum_{k=1}^{n} a_k\frac{\sigma_{k+1}^2}{p_k^2}(I_k^*)^2.
\end{aligned}$$

Taking V_1 and V_2 together, define $V = V_1 + V_2$, then

$$\begin{aligned}
LV &= LV_1 + LV_2 \\
&\leqslant -\sum_{k=1}^{n} a_k\left[(\mu - \sigma_1^2)(S - S^*)^2 + \frac{\mu+\gamma_k-\sigma_{k+1}^2}{p_k^2}(I_k - I_k^*)^2\right] \\
&\quad - \sum_{k=1}^{n}\left[a_k\frac{2\mu+\gamma_k}{p_k} - \beta_k\sum_{j=1}^{n}\beta_j I_j^* S^*\right](S - S^*)(I_k - I_k^*) \\
&\quad + \sum_{k=1}^{n} a_k\sigma_1^2(S^*)^2 + \sum_{k=1}^{n} a_k\frac{\sigma_{k+1}^2}{p_k^2}(I_k^*)^2 + \sum_{k=1}^{n}\frac{\beta_k\sigma_{k+1}^2}{2p_k}(I_k^*)^2 S^*.
\end{aligned}$$

Choosing $a_k = \frac{p_k\beta_k S^*}{2\mu+\gamma_k}\sum_{j=1}^n \beta_j I_j^*$ such that $a_k\frac{2\mu+\gamma_k}{p_k} - \beta_k\sum_{j=1}^n \beta_j I_j^* S^* = 0$, $k = 1, 2, \ldots, n$, yields

$$LV \leqslant -\sum_{k=1}^n a_k\left[\left(\mu-\sigma_1^2\right)\left(S-S^*\right)^2 + \frac{\mu+\gamma_k-\sigma_{k+1}^2}{p_k^2}\left(I_k-I_k^*\right)^2\right] + \sum_{k=1}^n a_k\sigma_1^2\left(S^*\right)^2 + \sum_{k=1}^n a_k\frac{\sigma_{k+1}^2}{p_k^2}\left(I_k^*\right)^2 + \sum_{k=1}^n \frac{\beta_k\sigma_{k+1}^2}{2p_k}\left(I_k^*\right)^2 S^*,$$

and so

$$\begin{aligned} dV \leqslant &\left\{-\sum_{k=1}^n a_k\left[\left(\mu-\sigma_1^2\right)\left(S-S^*\right)^2 + \frac{\mu+\gamma_k-\sigma_{k+1}^2}{p_k^2}\left(I_k-I_k^*\right)^2\right]\right. \\ &\left. + \sum_{k=1}^n a_k\sigma_1^2\left(S^*\right)^2 + \sum_{k=1}^n a_k\frac{\sigma_{k+1}^2}{p_k^2}\left(I_k^*\right)^2 + \sum_{k=1}^n \frac{\beta_k\sigma_{k+1}^2}{2p_k}\left(I_k^*\right)^2 S^*\right\}dt \\ &+ \sum_{k=1}^n \frac{\beta_k I_k^* S^*}{p_k}\left(1-\frac{I_k^*}{I_k}\right)\sigma_{k+1} I_k\, dB_{k+1}(t) \\ &+ \sum_{k=1}^n a_k\left(S-S^*+\frac{I_k-I_k^*}{p_k}\right)\left[\sigma_1 S\, dB_1(t) + \frac{\sigma_{k+1}}{p_k} I_k\, dB_{k+1}(t)\right]. \end{aligned} \tag{2.18}$$

Integrating both sides of (2.18) from 0 to t, and taking expectation, gives

$$\begin{aligned} E\left[V(t)\right] - V(0) \leqslant &-\sum_{k=1}^n a_k\left\{\left(\mu-\sigma_1^2\right)E\int_0^t \left(S(r)-S^*\right)^2 dr + \frac{\mu+\gamma_k-\sigma_{k+1}^2}{p_k^2}E\int_0^t \left(I_k(r)-I_k^*\right)^2 dr\right\} \\ &+ \left[\sum_{k=1}^n a_k\sigma_1^2\left(S^*\right)^2 + \sum_{k=1}^n a_k\frac{\sigma_{k+1}^2}{p_k^2}\left(I_k^*\right)^2 + \sum_{k=1}^n \frac{\beta_k\sigma_{k+1}^2}{2p_k}\left(I_k^*\right)^2 S^*\right]t. \end{aligned}$$

Consequently,

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\sum_{k=1}^n a_k\left\{\left(\mu-\sigma_1^2\right)E\int_0^t \left(S(r)-S^*\right)^2 dr + \frac{\mu+\gamma_k-\sigma_{k+1}^2}{p_k^2}E\int_0^t \left(I_k(r)-I_k^*\right)^2 dr\right\} \\ &\quad\leqslant \sum_{k=1}^n a_k\sigma_1^2\left(S^*\right)^2 + \sum_{k=1}^n a_k\frac{\sigma_{k+1}^2}{p_k^2}\left(I_k^*\right)^2 + \sum_{k=1}^n \frac{\beta_k\sigma_{k+1}^2}{2p_k}\left(I_k^*\right)^2 S^*. \qquad \square \end{aligned}$$

Remark 2.2. From the results of Theorems 2.3 and 2.4, we can see, if X^* is the equilibrium of the undisturbed system (1.1), but not of system (1.2), then under some conditions,

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t E\left[\left\|X(s)-X^*\right\|^2\right]ds < O\left(\sigma^2\right),$$

where $X(t)$ is the solution of system (1.2), $\|X(s)-X^*\|^2 = \sum_{k=1}^n (X_k(s)-X_k^*)^2$ and $\sigma^2 = \sum_{k=1}^n \sigma_k^2$. When the intensities of white noises are sufficiently small, we consider it as a stability.

3. The dynamics of system (1.3)

In this section we study the dynamics of system (1.3), which is the robustness of the endemic equilibrium E^* of the deterministic system (1.1). Thus we always assume $R_0 > 1$. Compared with system (1.2), system (1.3) does not always have nonnegative solutions, which is explained at the last of this section. Consequently, we only pay attention to the stability of solution around E^* and do not care whether the solution is positive.

If $R_0 > 1$, then $E^* = (S^*, I_1^*, I_2^*, \ldots, I_n^*)$ is the equilibrium of system (1.3). Changing the variables

$$u = S - S^*, \qquad v_k = I_k - I_k^*, \quad k = 1, 2, \ldots, n,$$

gives

$$\begin{cases} du = \left(-\mu u - \sum_{j=1}^{n} \beta_j I_j^* u - \sum_{j=1}^{n} \beta_j S^* v_j - \sum_{j=1}^{n} \beta_j u v_j\right) dt + \sigma_1 u\, dB_1(t), \\ dv_k = \left[p_k \sum_{j=1}^{n} \left(\beta_j I_j^* u + \beta_j S^* v_j\right) - (\mu + \gamma_k) v_k + p_k \sum_{j=1}^{n} \beta_j v_j u \right] dt + \sigma_{k+1} v_k\, dB_{k+1}(t), \quad k = 1, 2, \ldots, n. \end{cases} \tag{3.1}$$

It is easy to see that the stability of the endemic equilibrium of system (1.3) is equivalent to the stability of zero solution of system (3.1). In this section, following the way in [31], we show the zero solution of system (3.1) is stochastically asymptotically stable in local.

Before proving the main theorem we put forward a lemma in [24].

Assume system (1.4) that

$$f(0,t) = 0 \quad \text{and} \quad g(0,t) = 0 \quad \text{for all } t \geqslant t_0.$$

So system (1.4) has the solution $x(t) \equiv 0$ corresponding to the initial value $x(t_0) = 0$. This solution is called the trivial solution or equilibrium position.

Definition 3.1. (i) The trivial solution of system (1.4) is said to be stochastically stable or stable in probability if for every pair of $\epsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, t_0) > 0$ such that

$$P\left\{\left|x(t; t_0, x_0)\right| < r \text{ for all } t \geqslant t_0\right\} \geqslant 1 - \varepsilon,$$

whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically unstable.

(ii) The trivial solution is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\epsilon \in (0,1)$, there exists a $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ such that

$$P\left\{\lim_{t\to\infty} x(t; t_0, x_0) = 0\right\} \geqslant 1 - \varepsilon,$$

whenever $|x_0| < \delta_0$.

Lemma 3.1. *If there exists a positive-definite decrescent function $V(x,t) \in C^{2,1}(S_h \times [t_0, \infty]; \mathbb{R}_+)$ such that $LV(x,t)$ is negative-definite, then the trivial solution of system* (1.4) *is stochastically asymptotically stable.*

From the above lemma, we can obtain the stochastically asymptotically stability of equilibrium as follows.

Theorem 3.1. *Let $(S(t), I_1(t), I_2(t), \ldots, I_n(t))$ be the positive solution of system* (1.3) *with initial value $(S(0), I_1(0), I_2(0), \ldots, I_n(0)) \in \bar{R}_+^{n+1}$. If $R_0 > 1$, $\mu > \frac{\sigma_1^2}{2}$ and $\frac{\mu+\gamma_k}{(1+p_k\beta_k)} > \frac{\sigma_{k+1}^2}{2}$, $k = 1, 2, \ldots, n$, then the endemic equilibrium E^* is stochastically asymptotically stable.*

Proof. To show this result, we only need to prove the zero solution of system (3.1) is stochastically asymptotically stable. Define

$$V(u, v_1, v_2, \ldots, v_n) = \frac{1}{2} \sum_{k=1}^{n} a_k \left(u + \frac{v_k}{p_k}\right)^2 + \frac{1}{2} \sum_{k=1}^{n} b_k v_k^2, \tag{3.2}$$

where $a_k > 0$, $b_k > 0$, $k = 1, 2, \ldots, n$, are positive constants to be chosen later. Obviously, V is positive-definite and decrescent.

For sake of simplicity, divide (3.2) into two functions: $V = V_1 + V_2$, where

$$V_1(u, v_1, v_2, \ldots, v_n) = \frac{1}{2} \sum_{k=1}^{n} a_k \left(u + \frac{v_k}{p_k}\right)^2, \qquad V_2(v_1, v_2, \ldots, v_n) = \frac{1}{2} \sum_{k=1}^{n} b_k v_k^2.$$

Using Itô's formula, we compute

$$dV_1 = LV_1\, dt + \sum_{k=1}^{n} a_k \left(u + \frac{v_k}{p_k}\right) \left[\sigma_1 u\, dB_1(t) + \frac{\sigma_{k+1}}{p_k} v_k\, dB_{k+1}(t)\right],$$

$$dV_2 = LV_2\, dt + \sum_{k=1}^{n} b_k \sigma_{k+1} v_k^2\, dB_{k+1}(t),$$

where

$$
\begin{aligned}
LV_1 &= \sum_{k=1}^{n} a_k\left(u+\frac{v_k}{p_k}\right)\left[-\mu u-\sum_{j=1}^{n}\beta_j I_j^* u-\sum_{j=1}^{n}\beta_j S^* v_j+\sum_{j=1}^{n}\left(\beta_j I_j^* u+\beta_j S^* v_j\right)-\frac{\mu+\gamma_k}{p_k}v_k\right] \\
&\quad+\frac{1}{2}\sum_{k=1}^{n} a_k\sigma_1^2 u^2+\frac{1}{2}\sum_{k=1}^{n}\frac{a_k}{p_k^2}\sigma_{k+1}^2 v_k^2 \\
&= \sum_{k=1}^{n} a_k\left[-\left(\mu-\frac{\sigma_1^2}{2}\right)u^2-\left(\frac{\mu+\gamma_k}{p_k^2}-\frac{\sigma_{k+1}^2}{2p_k^2}\right)v_k^2-\frac{2\mu+\gamma_k}{p_k}u v_k\right]
\end{aligned}
$$

and

$$
\begin{aligned}
LV_2 &= \sum_{k=1}^{n} b_k v_k\left[p_k\sum_{j=1}^{n}\left(\beta_j I_j^* u+\beta_j S^* v_j\right)-(\mu+\gamma_k)v_k+p_k\sum_{j=1}^{n}\beta_j v_j u\right]+\frac{1}{2}\sum_{k=1}^{n} b_k\sigma_{k+1}^2 v_k^2 \\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j I_j^* u v_k+\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j S^* v_k v_j-\sum_{k=1}^{n} b_k(\mu+\gamma_k)v_k^2 \\
&\quad+\frac{1}{2}\sum_{k=1}^{n} b_k\sigma_{k+1}^2 v_k^2+\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j v_k v_j u.
\end{aligned}
$$

Obviously,

$$
\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j S^* v_k v_j=\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j S^* I_k^* I_j^*\frac{v_k}{I_k^*}\frac{v_j}{I_j^*}\leqslant\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j S^* I_k^* I_j^*\left[\left(\frac{v_k}{I_k^*}\right)^2+\left(\frac{v_j}{I_j^*}\right)^2\right]. \tag{3.3}
$$

Besides, from (2.13) we know

$$
\sum_{k=1}^{n} b_k(\mu+\gamma_k)v_k^2=\sum_{k=1}^{n} b_k\frac{p_k\sum_{j=1}^{n}\beta_j I_j^* S^*}{I_k^*}v_k^2=\sum_{k=1}^{n} b_k p_k I_k^*\sum_{j=1}^{n}\beta_j I_j^* S^*\left(\frac{v_k}{I_k^*}\right)^2. \tag{3.4}
$$

Combing (3.3) and (3.4), yields

$$
\begin{aligned}
&\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j S^* v_k v_j-\sum_{k=1}^{n} b_k(\mu+\gamma_k)v_k^2 \\
&\quad\leqslant\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j S^* I_k^* I_j^*\left[\left(\frac{v_j}{I_j^*}\right)^2-\left(\frac{v_k}{I_k^*}\right)^2\right]=\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j S^*\frac{I_k^*}{I_j^*}v_j^2-\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j S^*\frac{I_j^*}{I_k^*}v_k^2 \\
&\quad=\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} b_j p_j\beta_k S^*\frac{I_j^*}{I_k^*}v_k^2-\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j S^*\frac{I_j^*}{I_k^*}v_k^2=\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} S^*\frac{I_j^*}{I_k^*}(b_j p_j\beta_k-b_k p_k\beta_j)v_k^2.
\end{aligned}
$$

Choose $b_k=\frac{\beta_k}{p_k}$, $k=1,2,\dots,n$, such that

$$
\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j S^* v_k v_j-\sum_{k=1}^{n} b_k(\mu+\gamma_k)v_k^2=0.
$$

Therefore

$$
LV_2\leqslant\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j I_j^* u v_k+\frac{1}{2}\sum_{k=1}^{n} b_k\sigma_{k+1}^2 v_k^2+\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j v_k v_j u
$$

and

$$
\begin{aligned}
LV_1+LV_2 &\leqslant\sum_{k=1}^{n} a_k\left[-\left(\mu-\frac{\sigma_1^2}{2}\right)u^2-\left(\frac{\mu+\gamma_k}{p_k^2}-\frac{\sigma_{k+1}^2}{2p_k^2}\right)v_k^2-\frac{2\mu+\gamma_k}{p_k}u v_k\right] \\
&\quad+\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j I_j^* u v_k+\frac{1}{2}\sum_{k=1}^{n} b_k\sigma_{k+1}^2 v_k^2+\sum_{k=1}^{n}\sum_{j=1}^{n} b_k p_k\beta_j v_k v_j u.
\end{aligned}
$$

Choose $a_k = \frac{b_k p_k^2}{2\mu+\gamma_k} \sum_{j=1}^n \beta_j I_j^*$, $k = 1, 2, \ldots, n$, such that

$$\sum_{k=1}^n a_k \frac{2\mu+\gamma_k}{p_k} u v_k = \sum_{k=1}^n \sum_{j=1}^n b_k p_k \beta_j I_j^* u v_k.$$

Consequently

$$\begin{aligned} LV_1 + LV_2 &\leqslant \sum_{k=1}^n a_k \left[-\left(\mu - \frac{\sigma_1^2}{2}\right) u^2 - \left(\frac{\mu+\gamma_k}{p_k^2} - \frac{\sigma_{k+1}^2}{2p_k^2}\right) v_k^2 \right] + \frac{1}{2} \sum_{k=1}^n b_k \sigma_{k+1}^2 v_k^2 + \sum_{k=1}^n \sum_{j=1}^n b_k p_k \beta_j v_k v_j u \\ &= -\sum_{k=1}^n \left\{ a_k \left(\mu - \frac{\sigma_1^2}{2}\right) u^2 + \left[\frac{\mu+\gamma_k}{p_k^2} - \frac{1}{2}\left(\frac{1}{p_k^2} + b_k\right) \sigma_{k+1}^2 \right] v_k^2 \right\} + \sum_{k=1}^n \sum_{j=1}^n b_k p_k \beta_j v_k v_j u \\ &\leqslant -\lambda \left| y(t) \right|^2 + o\left(\left| y(t) \right|^2 \right), \end{aligned}$$

where

$$y = (u, v_1 v_2, \ldots, v_n) \quad \text{and} \quad \lambda = \min\left\{ \sum_{k=1}^n a_k \left(\mu - \frac{\sigma_1^2}{2}\right), \frac{\mu+\gamma_k}{p_k^2} - \frac{1}{2}\left(\frac{1}{p_k^2} + b_k\right)\sigma_{k+1}^2,\ k = 1, 2, \ldots, n \right\}.$$

Hence $LV(y,t)$ is a negative-definite in a sufficiently small neighborhood of $y = 0$ for $t \geqslant 0$. By Lemma 3.1 we therefore conclude that under conditions in the theorem, the trivial solution of system (3.1) is stochastically asymptotically stable. □

At the end of this section, we explain system (1.3) does not always have a nonnegative solution. In fact, the solution may be negative at some time. By Itô's formula, we can get the expression of $S(t)$,

$$S(t) = S(0)e^{A(t)} + \mu S^0 \int_0^t e^{A(t)-A(s)}\, ds - \sigma_1 S^* \int_0^t e^{A(t)-A(s)}\, dB_1(s),$$

where $A(t) = -(\mu + \frac{\sigma_1^2}{2})t - \sum_{j=1}^n \beta_j \int_0^t I_j(s)\, ds + \sigma_1 B_1(t)$. Clearly, the first two terms are positive, but the last term may be positive or negative. Moreover the properties of Brownian motion can arose the abstract of the last term very large at some time, which may lead to $S(t) < 0$. From the expressions of I_k, $k = 1, 2, \ldots, n$,

$$I_k(t) = I_k(0)e^{A_k(t)} + p_k \sum_{j \neq k} \beta_j \int_0^t S(s) I_j(s) e^{A_k(t)-A_k(s)}\, ds - \sigma_{k+1} I_k^* \int_0^t e^{A_k(t)-A_k(s)}\, dB_{k+1}(s),$$

where $A_k(t) = -(\mu + \gamma_k + \frac{\sigma_{k+1}^2}{2})t + p_k \beta_k \int_0^t S(s)\, ds + \sigma_{k+1} B_{k+1}(t)$, we can see the last term also may make the value of I_k be negative at some time. Thus, the solution of system (1.3) is not always nonnegative.

Next, we explain it from illustrations. We use Milstein's higher order method in [12] to find the strong solution of system (1.3) with given initial value and the values of parameters for $k = 2$. The corresponding discretization equation is

$$\begin{cases} x_{k+1} = x_k + \left(\mu S^0 - \mu x_k - \sum_{j=1}^2 \beta_j x_k y_{j,k} \right) \Delta t + \sigma_1 (x_k - S^*) \sqrt{\Delta t} \xi_{1,k} + \frac{\sigma_1^2}{2} (x_k - S^*) (\Delta t \xi_{1,k}^2 - \Delta t), \\ y_{1,k+1} = y_{1,k} + \left[p_1 \sum_{j=1}^2 \beta_j x_k y_{j,k} - (\mu + \gamma_1) y_{1,k} \right] \Delta t + \sigma_2 (y_{1,k} - I_1^*) \sqrt{\Delta t} \xi_{2,k} + \frac{\sigma_2^2}{2} (y_{1,k} - I_1^*) (\Delta t \xi_{2,k}^2 - \Delta t), \\ y_{2,k+1} = y_{2,k} + \left[p_2 \sum_{j=1}^2 \beta_j x_k y_{j,k} - (\mu + \gamma_2) y_{2,k} \right] \Delta t + \sigma_3 (y_{2,k} - I_2^*) \sqrt{\Delta t} \xi_{3,k} + \frac{\sigma_3^2}{2} (y_{2,k} - I_2^*) (\Delta t \xi_{3,k}^2 - \Delta t), \end{cases}$$

where $\xi_{1,k}$, $\xi_{2,k}$ and $\xi_{3,k}$, $k = 1, 2, \ldots, n$, are independent Gaussian random variables $N(0,1)$, and $\sigma_1, \sigma_2, \sigma_3$ are intensities of white noises. In Fig. 1, we choose the initial value $(S(0), I_1(0), I_2(0)) = (0.2, 1, 0.8)$ and the parameters $S^0 = 2$, $\mu = 0.3$, $\beta_1 = 0.2$, $\beta_2 = 0.4$, $\gamma_1 = 0.1$, $\gamma_2 = 0.2$, $p_1 = 0.4$, $p_2 = 0.6$ such that $R_0 > 1$. In (a)–(d), we give out the solution of system (1.3) with different values of σ_k, $k = 1, 2, 3$. Specifically, in (a), $\sigma_1 = 0.7$, $\sigma_2 = 0.2$, $\sigma_3 = 0.2$ such that $\mu > \sigma_1^2/2$, $(\mu + \gamma_k)/(1 + p_k \beta_k) > \sigma_{k+1}^2/2$, $k = 1, 2$; in (b), $\sigma_1 = 1$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$ such that $\mu > \sigma_1^2/2$ does not satisfied; in (c) and (d), $\sigma_1 = 0.1$, $\sigma_2 = 2$, $\sigma_3 = 0.1$ and $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $\sigma_3 = 2$ such that $(\mu + \gamma_1)/(1 + p_1 \beta_1) > \sigma_2^2/2$ and $(\mu + \gamma_2)/(1 + p_2 \beta_2) > \sigma_3^2/2$ do not hold, respectively. Obviously, Fig. 1 shows the solution of system (1.3) may have negative values at some time.

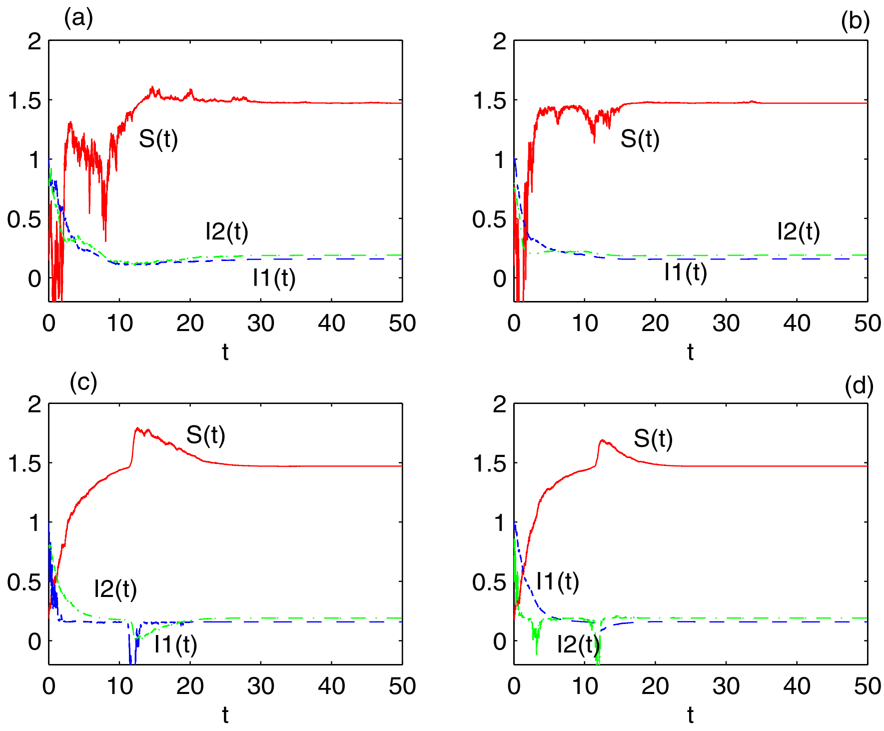

Fig. 1. The solution of system (1.3) with $k = 2$. The parameters of (a)–(d) are the same except with different values of σ_k, $k = 1, 2, 3$.

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