Maximal inequalities and Riesz transform estimates on $L^p$ spaces for magnetic Schrödinger operators I

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Abstract

The paper concerns the magnetic Schrödinger operator $H(a, V) = \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} - a_j \right)^2 + V$ on $\mathbb{R}^n$. Under certain conditions, given in terms of the reverse Hölder inequality on the magnetic field and the electric potential, we prove some $L^p$ estimates on the Riesz transforms of $H$ and we establish some related maximal inequalities.

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Keywords: Schrödinger operators; Maximal inequalities; Riesz transforms; Fefferman–Phong inequality; Reverse Hölder estimates; Magnetic field

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1. Introduction

Consider the Schrödinger operator with magnetic field

\[ H(a, V) = \sum_{j=1}^{n} \left( \frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right)^2 + V \quad \text{in } \mathbb{R}^n, \quad n \geq 2, \quad (1.1) \]

where \( a = (a_1, a_2, \ldots, a_n) : \mathbb{R}^n \to \mathbb{R}^n \) is the magnetic potential and \( V : \mathbb{R}^n \to \mathbb{R} \) is the electric potential. Let

\[ B(x) = \text{curl} \, a(x) = (b_{jk}(x))_{1 \leq j, k \leq n} \quad (1.2) \]

be the magnetic field generated by \( a \), where

\[ b_{jk} = \frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j}. \quad (1.3) \]

We will assume that \( a \in L^2_{\text{loc}}(\mathbb{R}^n)^n \) and \( V \in L^1_{\text{loc}}(\mathbb{R}^n), \quad V \geq 0 \). Let

\[ L_j = \frac{1}{i} \frac{\partial}{\partial x_j} - a_j \quad \text{for } 1 \leq j \leq n. \quad (1.4) \]

Set \( L = (L_1, \ldots, L_n) \) and \( |Lu(x)| = (\sum_{j=1}^{n} |L_ju(x)|^2)^{1/2} \).

Note that \( L_j^* = L_j \) for all \( 1 \leq j \leq n \), and let

\[ L^* = (L_1^*, \ldots, L_n^*)^T. \]

We define the form \( Q \) by

\[ Q(u, v) = \sum_{k=1}^{n} \int_{\mathbb{R}^n} L_k u \overline{L_k v} \, dx + \int_{\mathbb{R}^n} V u \overline{v} \, dx, \quad (1.5) \]

with domain \( D(Q) = \mathcal{V} \times \mathcal{V} \) where

\[ \mathcal{V} = \{ u \in L^2, \; L_k u \in L^2 \text{ for } k = 1, \ldots, n \text{ and } \sqrt{V} u \in L^2 \}. \]

We denote \( H(a, V) = H \), the self-adjoint operator on \( L^2(\mathbb{R}^n) \) associated to this symmetric and closed form.
The domain of $H$ is given by

$$\mathcal{D}(H) = \left\{ u \in \mathcal{D}(Q), \exists v \in L^2 \text{ so that } \mathcal{Q}(u, \phi) = \int v \bar{\phi} \, dx, \forall \phi \in \mathcal{D}(Q) \right\}.$$ 

The operators $L_j H(a, V)^{-1/2}$ are called the Riesz transforms associated with $H(a, V)$. We know that

$$n \sum_{j=1}^n \|L_j u\|_2^2 + \|V^{1/2} u\|_2^2 = \|H(a, V)^{1/2} u\|_2^2, \quad \forall u \in \mathcal{D}(Q) = \mathcal{D}(H(a, V)^{1/2}). \quad (1.6)$$

Hence, the operators $L_j H(a, V)^{-1/2}$ are bounded on $L^2(\mathbb{R}^n)$, for all $j = 1, \ldots, n$.

The aim of this paper is to establish the $L^p$ boundedness of the operators $L_j H(a, V)^{-1/2}$ and $V^{1/2} H(a, V)^{-1/2}$. In the presence of the magnetic field, the only known result is that these operators are of weak type $(1,1)$ and hence, by interpolation, are $L^p$ bounded for all $1 < p \leq 2$. This result was proved by Sikora using the finite speed propagation property [22]. Independently, Duong, Ouhabaz and Yan have proved the same result using another method.

Many authors have been interested in the study of the Riesz transforms of $H(a, V)$ in the case when the magnetic potential $a$ is absent, i.e. $LH(a, V)^{-1} = \nabla (-\Delta + V)^{-1/2}$. We mention the works of Helffer and Nourrigat [13], Guibourg [10] and Zhong [25], in which they considered the case of polynomial potentials. A generalization of their results was given by Shen [17], he proved the $L^p$ boundedness of Riesz transforms of Schrödinger operators with electric potential contained in certain reverse Hölder classes. Auscher and I improved this result in [1], using a different approach based on local estimates. Note that this approach can be extended to more general spaces for instance some Riemannian manifolds and Lie groups (see [3]). The main purpose of this work is to find some sufficient conditions on the electric potential and the magnetic field, for which the Riesz transforms of $H(a, V)$ are $L^p$ bounded for a range $p > 2$. Many arguments follow those of [1], the contribution of the magnetic field will be controlled by introducing an auxiliary function $m(\cdot, |B|)$.

Note that, because of the gauge invariance of the operator $H(a, V)$ and the nature of the $L^p$ estimates, any quantitative condition should be imposed on magnetic field $B$, not directly on $a$.

This article also aims to establish some maximal inequalities related to the $L^p$ behaviour of $L_j L_k H(a, V)^{-1}$, $V H(a, V)^{-1}$ and other operators called the second order Riesz transforms. The only known result for a range $p > 2$ is given by Shen in [20]. He generalized the $L^2$ estimate proved by Guibourg in [11] for polynomial potentials. Estimates on these operators are of great interest in the study of spectral theory of $H(a, V)$. In this paper our assumptions on potentials will be given in terms of reverse Hölder inequality. Let recall the definition of these weight classes:

**Definition 1.1.** Let $\omega \in L^q_{\text{loc}}(\mathbb{R}^n)$, $\omega > 0$ almost everywhere, $\omega \in \mathcal{R}H_q$, $1 < q \leq \infty$, the class of the reverse Hölder weights with exponent $q$, if there exists a constant $C$ such that for any cube $Q$ of $\mathbb{R}^n$,

$$\left( \int_Q \omega^q(x) \, dx \right)^{1/q} \leq C \left( \int_Q \omega(x) \, dx \right), \quad (1.7)$$
If $q = \infty$, then the left-hand side is the essential supremum on $Q$. The smallest $C$ is called the RH$_q$ constant of $\omega$.

**A note about notations:** Throughout this paper we will use the following notation $\int_Q \omega = \frac{1}{|Q|} \int_Q \omega$. $C$ and $c$ denote constants. As usual, $\lambda Q$ is the cube co-centered with $Q$ with side-length $\lambda$ times that of $Q$.

We give the definition of an auxiliary function introduced by Shen in [17].

**Definition 1.2.** Let $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\omega \geq 0$, for $x \in \mathbb{R}^n$, the function $m(x, \omega)$ is defined by

$$1 \frac{1}{m(x, \omega)} = \sup \left\{ r > 0: \frac{r^2}{|Q(x, r)|} \int_{Q(x, r)} \omega(y) \, dy \leq 1 \right\}. \quad (1.8)$$

We now state our main result:

**Theorem 1.3.** Let $a \in L^2_{\text{loc}}(\mathbb{R}^n)^n$. Also assume the following conditions

$$\begin{cases} |B| \in \text{RH}_{n/2}, \\ |\nabla B| \leq cm(\cdot, |B|)^3, \end{cases} \quad (1.9)$$

where $|B| = \sum_{j,k} |b_{jk}|$ and $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. Then, for all $1 < p < \infty$, there exists a constant $C_p > 0$, such that

$$\|LH(a, 0)^{-1/2}(f)\|_p \leq C_p \|f\|_p, \quad (1.10)$$

for any $f \in C^\infty_0(\mathbb{R}^n)$, and

$$\left| \{x \in \mathbb{R}^n; |Lf(x)| > \alpha \} \right| \leq \frac{C_1}{\alpha} \|H(a, 0)^{1/2}f\|_1$$

for $\alpha > 0$ and all $f \in C^\infty_0(\mathbb{R}^n)$ if $p = 1$.

The conditions (1.9), which are dilation invariant, are used by Shen in [20] to study the operators $L_jL_kH(a, V)^{-1}$. Note that these conditions mean that the value of $|B|$ do not fluctuate too much on the average and $|\nabla B|$ is uniformly bounded in the scale $m(x, |B|)^{-1}$. It is clear that the hypothesis of Theorem 1.3 is satisfied if the magnetic potentials $a_j(x)$ are polynomials.

Once the estimates for the pure magnetic Schrödinger operator $H(a, 0)$ is established, we will proceed onto the second part of our work. We then add the positive electric potential $V \in \text{RH}_q$, with $q > 1$, while keeping the same conditions on $B$ and get the following theorem:

**Theorem 1.4.** Let $a \in L^2_{\text{loc}}(\mathbb{R}^n)^n$, $V \in \text{RH}_q$, $1 < q \leq \infty$. Also assume that the magnetic field $B$ satisfies the conditions (1.9).

Then, there exists an $\epsilon > 0$ depending on the reverse Hölder constant RH$_q$ of $V$, such that, for every $1 < p < \sup(2q, q^*) + \epsilon$, there exists a constant $C_p > 0$, such that

$$\|LH(a, V)^{-1/2}(f)\|_p \leq C_p \|f\|_p, \quad (1.11)$$
for any \( f \in C_0^\infty(\mathbb{R}^n) \). Here, \( q^* = qn / (n - q) \) is the Sobolev exponent of \( q \) if \( q < n \), and \( q^* = \infty \) if \( q \geq n \).

Taking into account the conditions on the electric potential, and pursuing step-by-step the proof of Theorem 1.3, we get the following result:

**Theorem 1.5.** Let \( a \in L^2_{\text{loc}}(\mathbb{R}^n) \), \( V \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( V \geq 0 \) a.e. on \( \mathbb{R}^n \). Also assume that there exist two positive constants \( c > 0 \) and \( C > 0 \) such that:

\[
\begin{align*}
|B| + V &\in RH_{n/2}, \\
V &\leq C m(\cdot, |B| + V)^2, \\
|\nabla B| &\leq cm(\cdot, |B| + V)^3.
\end{align*}
\] (1.12)

Then (1.11) is satisfied for all \( 1 < p < \infty \).

The following three results will be useful to prove Theorems 1.3 and 1.4. The first describes reverse inequalities of (1.11).

**Theorem 1.6.** Let \( V \in A_\infty \) or \( V = 0 \), \( a \in L^2_{\text{loc}}(\mathbb{R}^n) \) and \( |B| \in RH_{n/2} \).

Then, for all \( 1 \leq p < \infty \), there exists a constant \( C_p > 0 \) depending only on the RH \( q \) constant of \( |B| \), such that

\[
\|H(a, V)^{1/2}(f)\|_p \leq C_p \{ \|Lf\|_p + \| |B|^{1/2} f\|_p + \| V^{1/2} f\|_p \} 
\] (1.13)

for any \( f \in C_0^\infty(\mathbb{R}^n) \) if \( p > 1 \), and

\[
\{x \in \mathbb{R}^n; |H(a, V)^{1/2} f(x)| > \alpha\} \leq \frac{C_1}{\alpha} \int |Lf| + \| |B|^{1/2} f\| + \| V^{1/2} f\|,
\] (1.14)

for all \( \alpha > 0 \) and \( f \in C_0^\infty(\mathbb{R}^n) \) if \( p = 1 \).

**Remark 1.7.**

1. Under assumptions (1.9), we can replace \( \| |B|^{1/2} f\|_p \) by \( \|m(\cdot, |B|) f\|_p \) in (1.13) and (1.14).
2. Under the conditions (1.12), we can replace the term \( \| |B|^{1/2} f\|_p + \| V^{1/2} f\|_p \) by \( \|m(\cdot, |B| + V) f\|_p \).

Note that introducing (1.9) and (1.12) makes the proof of Theorem 1.6, using the same strategy as before, easier.

The result concerns some new inequalities:

**Theorem 1.8.** Let \( a \in L^2_{\text{loc}}(\mathbb{R}^n) \) and \( V \in RH_q \), \( 1 < q \leq +\infty \). Then, there exists \( \epsilon > 0 \), depending only on the RH \( q \) constant of \( V \), such that \( VH(a, V)^{-1} \) and \( H(a, 0)H(a, V)^{-1} \) are \( L^p \) bounded for all \( 1 \leq p < q + \epsilon \).
It follows by complex interpolation (see [1] for more details):

**Corollary 1.9.** Let $a \in L^2_{\text{loc}}(\mathbb{R}^n)^n$ and $V \in RH_q$, $1 < q \leq +\infty$. Then, there exists an $\epsilon > 0$, depending only on the $RH_q$ constant of $V$, such that, the operators $V^{1/2}H(a, V)^{-1/2}$ and $H(a, 0)^{1/2}H(a, V)^{-1/2}$ are $L^p$ bounded for all $1 < p < 2q + \epsilon$.

We would give an alternative proof of the following theorem proved by Shen in [20]:

**Theorem 1.10.** Under the conditions of Theorem 1.5, for all $s = 1, \ldots, n$ and $k = 1, \ldots, n$, the operators $L_s L_k H(a, V)^{-1}$ are $L^p$ bounded for any $1 < p < \infty$.  

Note that with more general conditions on the electric potential, we have the following new result:

**Theorem 1.11.** Under the conditions of Theorem 1.4, for all $s = 1, \ldots, n$ and $k = 1, \ldots, n$, there exists an $\epsilon > 0$ depending only on the $RH_q$ constant of $V$, such that $L_s L_k H(a, V)^{-1}$ are $L^p$ bounded for all $1 < p < q + \epsilon$.

We mention without proof that our results admit local versions, replacing $V \in RH_q$ by $V \in RH_q, \text{loc}$ which is defined by the same conditions on cubes with sides less than 1. Then we get the corresponding results and estimates for $H + 1$ instead of $H$. The results on operator domains are valid under local assumptions. Our arguments are based on local estimates. Our main tools are:

1) An improved Fefferman–Phong inequality for $A_\infty$ potentials ($A_\infty$ is the Muckenhoupt weight class).
2) Criteria for proving $L^p$ boundedness of operators in absence of kernels.
3) Mean value inequalities for non-negative subharmonic functions against $A_\infty$ weights.
4) Complex interpolation, together with $L^p$ boundedness of imaginary powers of $H(a, V)$ for $1 < p < \infty$.
5) A Calderón–Zygmund decomposition adapted to level sets of the maximal function of $|Lf| + |V^{1/2}f|$.
6) A gauge transform adapted to the reverse Hölder conditions on the potentials.
7) An auxiliary global weight controlling the contribution from the magnetic field.
8) Reverse Hölder inequalities involving $Lu, m(\cdot, |B|)u, |B|^{1/2}u$ and $V^{1/2}u$ for weak solutions of $H(a, V)u = 0$.

The paper is organized as follows. In Section 2 we introduce some useful estimates. We state an improved Fefferman–Phong inequality and we establish an adapted gauge transform. Section 3 is devoted to the study of pure magnetic Schrödinger operator, first we establish some reverse estimates via a Calderón–Zygmund decomposition, then we prove the $L^p$ boundedness of Riesz transforms for all $1 < p < \infty$. In Section 4 we consider the magnetic Schrödinger operator with electric potential, we study the $L^p$ behaviour of the first and the second order Riesz transforms.

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1 Shen also proved a weak $(1, 1)$ type estimate for these operators.
2. Preliminaries

We begin by recalling some properties of the reverse Hölder classes.

Proposition 2.1. (See [1, Proposition 11.1].) Let \( \omega \) be a non-negative measurable function. Then the following are equivalent:

1. \( \omega \in A_\infty \).
2. For all \( s \in (0, 1) \), \( \omega^s \in RH_{1/s} \).
3. There exists \( s \in (0, 1) \), \( \omega^s \in RH_{1/s} \).

It is well known that if \( \omega \in RH_q \) and \( q < +\infty \), then \( \omega \in RH_p \) for all \( 1 < p < q \) and there exists an \( \varepsilon > 0 \) such that \( \omega \in RH_{q+\varepsilon} \). We also know that \( \omega \in A_\infty \) if and only if there exists \( q > 1 \) such that \( \omega \in RH_q \). Here \( A_\infty \) is the Muckenhoupt weight class, defined as the union of all \( A_p \), \( 1 \leq p < \infty \). If \( \omega \in A_\infty \), then \( \omega(x) \, dx \) is a doubling measure (see [23, Chap. V] for more details).

We will also recall some important properties of the function \( m(\cdot, \omega) \):

Lemma 2.2. Suppose \( \omega \in RH_{n/2} \), then there exist \( c > 0 \) and \( C > 0 \) such that for all \( x \) and \( y \) in \( \mathbb{R}^n \):

1. \( 0 < m(x, \omega) < \infty \) for all \( x \in \mathbb{R}^n \).
2. If \( |x - y| < \frac{C}{m(x, \omega)} \), then \( m(x, \omega) \approx m(y, \omega) \).
3. \( m(y, \omega) \leq C [1 + |x - y|m(x, \omega)]^{k_0} m(x, \omega) \).
4. \( m(y, \omega) \geq \frac{C m(x, \omega)}{[1 + |x - y|m(x, \omega)]^{k_0 + 1}} \) for some \( k_0 \) depending on \( \omega \).

We will see that if \( u \) is a weak solution of \( H(a, V)u = 0 \), it is easier to obtain reverse Hölder inequalities using terms \( m(\cdot, |B|)u \) and \( Lu \) than is the case when we work with estimates of \( |B|^{1/2} u \).

Fix an open set \( \Omega \) and \( f \in L^\infty_{\text{comp}}(\mathbb{R}^n) \), the space of compactly supported bounded functions on \( \mathbb{R}^n \). By a weak solution of

\[
H(a, V)u = f \quad \text{in an open set } \Omega, \tag{2.1}
\]

we mean \( u \in W(\Omega) \), with

\[
W(\Omega) = \{ u \in L^1_{\text{loc}}(\Omega); \ V^{1/2} u \text{ and } L_k u \in L^2_{\text{loc}}(\Omega) \forall k = 1, \ldots, n \}
\]

and Eq. (2.1) holds in the sense of distribution on \( \Omega \). We note that if \( u \in W(\Omega) \), then by Poincaré and the diamagnetic inequalities, \( u \in L^2_{\text{loc}}(\Omega) \).

We will need the following tools:

Lemma 2.3 (Caccioppoli type inequality). Let \( u \) be a weak solution of \( H(a, V)u = f \) in \( 2Q \), where \( Q \) is a cube of \( \mathbb{R}^n \) and \( f \in L^\infty_{\text{comp}}(\mathbb{R}^n) \). Then

\[
\int_Q |Lu|^2 + V|u|^2 \leq C \left\{ \int_{2Q} |f||u| + \frac{1}{R^2} \int_{2Q} |u|^2 \right\}. \tag{2.2}
\]
Proposition 2.4 (Diamagnetic inequality). (See [15].) For all $u \in W^{1,2}_a(\mathbb{R}^n)$, with
\[ W^{1,2}_a(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n), \ L_ku \in L^2(\mathbb{R}^n), \ k = 1, \ldots, n \}, \]
we have
\[ |\nabla(|u|)| \leq |L(u)|. \quad (2.3) \]

Proposition 2.5 (Kato–Simon inequality).
\[ |(H(a, V) + \lambda)^{-1}f| \leq (-\Delta + \lambda)^{-1}|f|; \ \forall f \in L^2(\mathbb{R}^n), \ \forall \lambda > 0. \quad (2.4) \]

Fefferman–Phong inequalities. The usual Fefferman–Phong inequalities are of the form:
\[ \int_Q |u|^p \min \left\{ \int_Q \omega, \frac{1}{R^p} \right\} \leq C \left\{ \int_Q |Lu|^p + \omega|u|^p \right\}. \quad (2.5) \]
Shen proved in [19] the following global version introducing the auxiliary weight function $m(\cdot, \omega)$:

Lemma 2.6. Suppose $a \in L^2_{\text{loc}}(\mathbb{R}^n)^n$. We also assume:
\[ \begin{cases} |B| + V \in RH_{n/2}, \\ 0 \leq V \leq cm(\cdot, |B| + V)^2, \\ |\nabla B| \leq c'm(\cdot, |B| + V)^3. \end{cases} \quad (2.6) \]
Then, for all $u \in C^1(\mathbb{R}^n)$,
\[ \|m(\cdot, |B| + V)u\|_2 \leq C(\|Lu\|_2 + \|V_{1/2}u\|_2). \quad (2.7) \]

In [1] we established an improved version for these inequalities in absence of the magnetic potential. We can extend this improvement to the magnetic Schrödinger operators:

Lemma 2.7 (An improved Fefferman–Phong inequality). Let $w \in A_\infty$ and $1 \leq p < \infty$. Then there are constants $C > 0$ and $\beta \in (0, 1)$ depending only on $p$, $n$ and the $A_\infty$ constant of $w$ such that for all cubes $Q$ (with sidelength $R$) and $u \in C^1(\mathbb{R}^n)$, one has
\[ \int_Q |Lu|^p + \omega|u|^p \geq \frac{Cm_\beta(R^p \int_Q \omega)}{R^p} \int_Q |u|^p \]
where $m_\beta(x) = x$ for $x \leq 1$ and $m_\beta(x) = x^\beta$ for $x \geq 1$.

The proof is the same as that of Lemma 2.1 in [1], combined with the diamagnetic inequality.
Remark 2.8. For more details about Fefferman–Phong inequalities and some other related problems, see [24].

Lemma 2.9 (Iwatsuka gauge transform). Let $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)^n$ and $Q$ be a cube of $\mathbb{R}^n$. Suppose $B \in C^1(\mathbb{R}^n, M_n(\mathbb{R}))$. Then there exist $\mathbf{h} \in C^1(Q, \mathbb{R}^n)$ and a real function $\phi \in C^2(Q)$, such that $\text{curl} \mathbf{h} = B$ in $Q$ and

$$h = a - \nabla \phi, \quad \text{in } Q, \quad (2.9)$$

with

$$\left( \frac{1}{|Q|} \int_{Q} |h|^n \right)^{1/n} \leq c R \left( \frac{1}{|Q|} \int_{Q} |B|^2 \right)^{\frac{2}{n}}, \quad (2.10)$$

here $c$ depends only on $n$.

Proof. We follow the proof of Lemma 2.4 in [21], which uses the construction of Iwatsuka [14].

For $x, y \in Q$, let

$$g_j(x, y) = \sum_{k=1}^{n} (x_k - y_k) \int_{0}^{1} b_{jk}(y + t(x - y)) t \, dt,$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$.

Let

$$h_j(x) = \int_{Q} g_j(x, y) \, dy, \quad j = 1, 2, \ldots, n.$$  

Then

$$|h(x)| = \left( \sum_{j} |h_j(x)|^2 \right)^{1/2} \leq n^{n-1} \int_{Q} \frac{|B(y)|}{|x-y|^{n-1}} \, dy.$$  

Now, we apply the Hardy–Littlewood–Sobolev inequality [23, p. 119] to get (2.10). Hence (2.9) holds with

$$\phi(x) = \int_{Q} \left\{ \sum_{k=1}^{n} (x_k - y_k) \int_{0}^{1} a_k(y + t(x - y)) \, dt \right\} \, dy. \quad \Box$$

Corollary 2.10. Let $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^n)^n$ and $Q$ be a cube in $\mathbb{R}^n$. We assume that $\text{curl} \mathbf{a} = B \in L^{n/2}_{\text{loc}}(\mathbb{R}^n, M_n(\mathbb{R}))$. Then, there exist $\mathbf{h} \in L^n(Q, \mathbb{R}^n)$ and a real function $\phi \in H^1(Q)$, such that $\text{curl} \mathbf{h} = B$ a.e. in $Q$ and

$$h = a - \nabla \phi \quad \text{a.e. in } Q, \quad (2.11)$$
with

\[
\left( \frac{\int_Q |h|^n}{1/n} \right)^{1/n} \leq c_R \left( \frac{\int_Q |B|^2}{n} \right)^{2/n}.
\]

**Proof.** Let \((a_m)_m \geq 0\) be the sequence of \(C^1\) functions obtained by convolution with \(a\) and converge in \(L^2_{\text{loc}}\) to \(a\). Set \((B_m)_m \geq 0\), \((\phi_m)_m \geq 0\) and \((h_m)_m \geq 0\) as the corresponding sequences of Lemma 2.9. Note that \((h_m)_m \geq 0\) converges in \(L^n(Q, \mathbb{R}^n)\). Let \(h\) be this limit, it satisfies (2.11).

Note also that \((B_m)_m \geq 0\) converges to \(B\) in \(L^{n/2}_{\text{loc}}(Q, M_n(\mathbb{R}))\) and \(\text{curl} h = B\) holds always everywhere in \(Q\), where curl is defined in the sense of distribution.

We know that for all \(m \geq 0\),

\[
\left( \frac{\int_Q |h_m|^n}{1/n} \right)^{1/n} \leq c_R \left( \frac{\int_Q |B_m|^2}{n} \right)^{2/n},
\]

uniformly in \(m\). Then applying the limit, we obtain

\[
\left( \frac{\int_Q |h|^n}{1/n} \right)^{1/n} \leq c_R \left( \frac{\int_Q |B|^2}{n} \right)^{2/n}.
\]

Hence inequality (2.11) follows easily. \(\square\)

### 3. Pure magnetic Schrödinger operator

This section is devoted to establish \(L^p\) estimates on Riesz transforms of \(H(a, 0)\) as well as its converse. Since the electric potential is absent, we cannot follow the methods of [1]. An analogous approach based on local estimates requires different localization techniques. We also use a Calderón–Zygmund decomposition adapted to the presence of magnetic field via the gauge transform previously established.

#### 3.1. Reverse estimates

In the absence of electric potential, Theorem 1.6 is of the form:

**Theorem 3.1.** Suppose \(a \in L^2_{\text{loc}}(\mathbb{R}^n)^n\) and \(|B| \in RH_{n/2}\).

Then, for all \(1 < p < \infty\), there exists \(C_p > 0\), such that

\[
\|H(a, 0)^{1/2} f\|_p \leq C_p (\|L f\|_p + \|B|^{1/2} f\|_p)
\]

for all \(f \in C_0^\infty(\mathbb{R}^n)\). There is a constant \(C > 0\) such that

\[
\left| \{ x \in \mathbb{R}^n : |H(a, 0)^{1/2} f(x)| > \alpha \} \right| \leq \frac{C}{\alpha} \int |L f| + |B|^{1/2} |f|,
\]

for \(\alpha > 0\) and for all \(f \in C_0^\infty(\mathbb{R}^n)\).
Proof. We follow step-by-step the proof of Theorem 1.2 of [1] once the appropriate Calderón–Zygmund decomposition 3.2 is established. We also use the fact that the time derivatives of the kernel of semigroup $e^{-tH}$ satisfy Gaussian estimates (see [5,6,9] and [16]).

Let’s introduce the main technical lemma of this work, interesting in its own right:

**Lemma 3.2.** Let $1 \leq p < n$ and $\alpha > 0$. Suppose $a \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $|B| \in RH_{n/2}$. Let $f \in C_0^\infty(\mathbb{R}^n)$ hence

$$\|Lf\|_p + \|B|^{1/2}f\|_p < \infty.$$  

Then, one can find a collection of cubes $(Q_k)$ and functions $g$ and $b_k$ such that

$$f = g + \sum_k b_k, \quad (3.3)$$

and the following properties hold:

$$\|Lg\|_n + \|B|^{1/2}g\|_n \leq C\alpha^{1-\frac{p}{n}}\left(\|Lf\|_p + \|B|^{1/2}f\|_p\right)^{p/n}, \quad (3.4)$$

$$\int_{Q_k} |Lb_k|^p + R_k^{-p}|b_k|^p \leq C\alpha^p|Q_k|, \quad (3.5)$$

$$\sum_k |Q_k| \leq C\alpha^{-p}\left(\int_{\mathbb{R}^n} |Lf|^p + \|B|^{1/2}f|^p\right), \quad (3.6)$$

$$\sum_k 1_{Q_k} \leq N, \quad (3.7)$$

where $N$ depends only on the dimension and $C$ on the dimension, $p$ and the $RH_{n/2}$ constant of $|B|$. Here, $R_k$ denotes the sidelength of $Q_k$ and gradients are taken in the sense of distributions in $\mathbb{R}^n$.

**Remark 3.3.** Note that by (3.4) for $p < 2$, we obtain:

$$\|Lg\|_2 + \|B|^{1/2}g\|_2 \leq C\alpha^{1-\frac{p}{2}}\left(\|Lf\|_p + \|B|^{1/2}f\|_p\right)^{p/2}. \quad (3.8)$$

We will use this inequality to prove Theorem 3.1.

The rest of the section is devoted to the demonstration of Lemma 3.2.

**Proof.** Let $\Omega$ be the open set $\{x \in \mathbb{R}^n; \ M(|Lf|^p + \|B|^{1/2}f|^p)(x) > \alpha^p\}$, where $M$ is the uncentered maximal operator over the cubes of $\mathbb{R}^n$. If $\Omega$ is empty, then set $g = f$ and $b_i = 0$. Otherwise, our argument is subdivided into six steps.
a) Construction of the cubes:
The maximal theorem gives us
\[ |\Omega| \leq C a^{-p} \int_{\mathbb{R}^n} |Lf|^p + |B|^{1/2} f|^p < \infty. \]

Let \((Q_k)\) be a Whitney decomposition of \(\Omega\) by dyadic cubes so to say \(\Omega\) is the disjoint union of the \(Q_k\)'s, the cubes \(2Q_j\) are contained in \(\Omega\) and have the bounded overlap property, but the cubes \(4Q_k\) intersect \(F = \mathbb{R}^n \setminus \Omega.\)

Hence
\[ \sum_k |2Q_k| \leq C|\Omega| \leq C a^{-p} \int_{\mathbb{R}^n} |Lf|^p + |B|^{1/2} f|^p. \]

(3.6) and (3.7) are satisfied by the cubes \(2Q_k\).

b) Construction of \(b_k\):
Let \((\chi_k)\) be a partition of unity on \(\Omega\) associated to the covering \((Q_k)\) so that for each \(k\), \(\chi_k\) is a \(C^1\) function supported in \(2Q_k\) with
\[ \|\chi_k\|_\infty + R_k \|\nabla \chi_k\|_\infty \leq c(n), \]
where \(R_k\) is the sidelength of \(Q_k\) and \(\sum \chi_k = 1\) on \(\Omega\). We say that a cube \(Q\) is of type 1 if \(R^2 \int_{Q} |B| > 1\), and is of type 2 if \(R^2 \int_{Q} |B| > 1\).

We apply the gauge transformation on the cubes \(2Q_k\) such that \(Q_k\) is of type 2, hence there exist \(h_k \in L^n(2Q_k, \mathbb{R}^n)\) and a real function \(\phi_k \in H^1(2Q_k)\) such that
\[ h_k = a - \nabla \phi_k \text{ a.e. on } 2Q_k, \]
(3.10)
\[ \left( \int_{2Q_k} |h_k|^n \right)^{1/n} \leq c R_k \left( \int_{2Q_k} |B|^{n/2} \right)^{2/n}. \]
(3.11)

We denote
\[ m_{2Q_k}(e^{i\phi_k} f) = \int_{2Q_k} (e^{i\phi_k} f). \]

Let
\[ b_k = \begin{cases} f \chi_k, & \text{if } Q_k \text{ is of type 1,} \\ (f - e^{-i\phi_k} m_{2Q_k}(e^{i\phi_k} f)) \chi_k, & \text{if } Q_k \text{ is of type 2.} \end{cases} \]
(3.12)

\[ ^2 \text{In fact, the factor } 2 \text{ should be some } c = c(n) > 1 \text{ explicitly given in [23, Chapter 6]. We use this convention to avoid too many irrelevant constants.} \]
c) Proof of estimate (3.5):
Suppose $Q_k$ is of type 1, then

$$R_k^{-p} \leq c \left( \int_{2Q_k} |B| \right)^{p/2} \leq C \int_{2Q_k} |B|^{p/2},$$

where we used $|B|^{p/2} \in RH_{2/p}$ if $p < 2$ (by Proposition 2.1) and the Jensen’s inequality with convex function $t \mapsto t^{p/2}$ if $p \geq 2$.

In order to control $Lb_k$, we have

$$Lb_k = L(f \chi_k) = (Lf) \chi_k + \frac{1}{i} f \nabla \chi_k,$$

then

$$\int_{2Q_k} |Lb_k|^p + R_k^{-p} |b_k|^p \leq C \|\chi_k\|_p^p \int_{2Q_k} |Lf|^p + \|\nabla \chi_k\|_\infty^p \int_{2Q_k} |f|^p + R_k^{-p} \|\chi_k\|_\infty \int_{2Q_k} |f|^p$$

$$\leq C \left\{ \int_{2Q_k} |Lf|^p + R_k^{-p} \int_{2Q_k} |f|^p \right\} \leq C \left\{ \int_{2Q_k} |Lf|^p + \|B\|^{1/2} |f|^p \right\}$$

$$\leq C \alpha^p |Q_k|,$$

where we used the $L^p$ version of the usual Fefferman–Phong inequality (2.5) and the intersection of $4Q_k$ with $F$, hence $\int_{4Q_k} |Lf|^p + \|B\|^{1/2} |f|^p \leq C \alpha^p |4Q_k|$. Then estimation (3.5) holds for the cubes of type 1.

If $Q_k$ is of type 2, $R_k^2 \int_{Q_k} |B| \leq 1$. $|B(x)| dx$ is a doubling measure, then there exists $C > 0$, such that $R_k^2 \int_{Q_{2k}} |B| \leq C$.

$$b_k = (f - e^{-i\phi_k} m_{2Q_k}(e^{i\phi_k} f)) \chi_k.$$  

Let us estimate $Lb_k$. By the gauge invariance, all we require is the estimation of $\tilde{L}(e^{i\phi_k} b_k)$, where

$$\tilde{L} = \frac{1}{i} \nabla - h_k.$$  

We have

$$\tilde{L}(e^{i\phi_k} b_k) = \chi_k(\tilde{L} f_k) + \frac{1}{i} (f_k - m_{2Q_k} f_k) \nabla \chi_k - \left( \int_{2Q_k} f_k \right) \chi_k h_k,$$
where \( f_k = e^{i\phi_k} f \). Then,
\[
\left( \frac{1}{2Q_k} \int |Lb_k|^p \right)^{1/p} \leq C \left\{ \left( \frac{1}{2Q_k} \int |\tilde{L} f_k|^p \right)^{1/p} \|\chi_k\|_\infty + \left( \frac{1}{2Q_k} \int |(f_k - m_{2Q_k} f_k)|^p \right)^{1/p} \|\nabla \chi_k\|_\infty 
+ \left( \frac{1}{2Q_k} \int |h_k|^p \right)^{1/p} \left( \frac{1}{2Q_k} \int |f_k|^p \right)^{1/p} \|\chi_k\|_\infty \right\}.
\]

Using the Poincaré inequality and condition (3.9), we obtain
\[
\left( \frac{1}{2Q_k} \int |Lb_k|^p \right)^{1/p} \leq C \left\{ \left( \frac{1}{2Q_k} \int |\tilde{L} f_k|^p \right)^{1/p} + \left( \frac{1}{2Q_k} \int |\nabla f_k|^p \right)^{1/p} + \left( \frac{1}{2Q_k} \int |h_k|^p \right)^{1/p} \left( \frac{1}{2Q_k} \int |f_k|^p \right)^{1/p} \right\}
\]
\[
\leq C \left\{ \left( \frac{1}{2Q_k} \int |L f_k|^p \right)^{1/p} + \left( \frac{1}{2Q_k} \int \left| \frac{1}{t} \nabla f_k - h_k f_k \right|^p \right)^{1/p} 
+ \left( \frac{1}{2Q_k} \int |h_k|^p \right)^{1/p} \left( \frac{1}{2Q_k} \int |f_k|^p \right)^{1/p} \right\}.
\]

Hence
\[
\left( \frac{1}{2Q_k} \int |Lb_k|^p \right)^{1/p} \leq C \left\{ \left( \frac{1}{2Q_k} \int |\tilde{L} f_k|^p \right)^{1/p} + I + II \right\}.
\]

Next, we apply inequality (3.11) to estimate \( I \). The fact that \(|B|\) is a RH\(_{n/2}\) weight and \(Q_k\) is of type 2 leads:
\[
\left( \frac{1}{2Q_k} \int |h_k|^p \right)^{1/p} \left( \frac{1}{2Q_k} \int |f_k|^p \right)^{1/p} \leq \left( \frac{1}{2Q_k} \int |h_k|^n \right)^{1/n} \left( \frac{1}{2Q_k} \int |f_k|^p \right)^{1/p}
\]
\[
\leq CR_k \left( \frac{1}{2Q_k} \int |B|^{n/2} \right)^{2/n} \left( \frac{1}{2Q_k} \int |f_k|^p \right)^{1/p}
\]
\[
\leq CR_k \left( \frac{1}{2Q_k} \int |B| \right) \left( \frac{1}{2Q_k} \int |f_k|^p \right)^{1/p}
\]
\[
\leq C \left( \frac{1}{2Q_k} \int |B|^{1/2} \right) \left( \frac{1}{2Q_k} \int |f_k|^p \right)^{1/p}.
\]

By Fefferman–Phong inequality (2.5),
\[ I \leq C \left( \frac{\int_{2Q_k} |B|^{p/2} \int_{2Q_k} |f_k|^p}{2Q_k} \right)^{1/p} \leq C \left( \int_{2Q_k} |B|^{p/2} \int_{2Q_k} |f_k|^p \right)^{1/p} \]

\[ \leq C \left( \int_{2Q_k} |\tilde{L} f_k|^p + |B|^{1/2} |f_k|^p \right)^{1/p}. \]

Hence

\[ I \leq C \int_{2Q_k} |\tilde{L} f_k|^p + |B|^{1/2} |f_k|^p. \] (3.13)

To estimate the second term \( II \), first we use the Hölder inequality and the fact that \(|B| \in RH_{n/2}\) and \(Q_k\) is of type 2. Next, we apply Poincaré inequality and the diamagnetic inequality (under our hypothesis, \(f_k \in W^{1,2}(\mathbb{R}^n)\)):

\[ II = \left( \int_{2Q_k} |h_k f_k|^p \right)^{1/p} \leq \left( \int_{2Q_k} |h_k|^{pn/p} \right)^{1/p} \left( \int_{2Q_k} |f_k|^{pn/(n-p)} \right)^{(n-p)/pn} \]

\[ \leq CR_k \left( \int_{2Q_k} |B|^{n/2} \right)^{2/n} \left( \int_{2Q_k} |f_k|^{pn/(n-p)} \right)^{(n-p)/pn} \]

\[ \leq CR_k \left( \int_{2Q_k} |B| \right) \left( \int_{2Q_k} |f_k|^{pn/(n-p)} \right)^{(n-p)/pn} \]

\[ \leq CR_k \left( \int_{2Q_k} |B| \right) \left\{ \left( \int_{2Q_k} |f_k| - m_{2Q_k} (|f_k|) \right)^{pn/(n-p)} + m_{2Q_k} (|f_k|) \right\} \]

\[ \leq C \left\{ R_k^2 \left( \int_{2Q_k} |B| \right) \left( \int_{2Q_k} |\tilde{L} f_k|^p \right)^{1/p} + \left( \int_{2Q_k} |B| \right)^{1/2} \left( \int_{2Q_k} |f_k| \right) \right\} \]

\[ \leq C \left\{ \left( \int_{2Q_k} |\tilde{L} f_k|^p \right)^{1/p} + \left( \int_{2Q_k} |L f_k|^p + |B|^{1/2} |f_k|^p \right)^{1/p} \right\}. \]

Then

\[ II \leq C \left( \int_{2Q_k} |\tilde{L} f_k|^p + |B|^{1/2} |f_k|^p \right)^{1/p}. \] (3.14)

Since \(|L(f)| = |\tilde{L}(f_k)|\), then, by gauge invariance,

\[ \int_{2Q_k} |Lb_k|^p \leq C \left\{ \int_{2Q_k} |L f|^p + |B|^{1/2} |f|^p \right\} \leq c \alpha^p. \]
And by the same argument, we have

$$R_k^{-p} \int_{2Q_k} |b_k|^p = R_k^{-p} \int_{2Q_k} |(f_k - m_{2Q_k} f_k) \chi_k|^p \leq C \alpha^p.$$  

Thus (3.5) is proved.

d) Definition and properties of $|B|^{1/2} g$:

Set $g = f - \sum b_k$. Note that, by (3.7), this sum is locally finite. It is clear that $g = f$ on $F$ and $g = \sum_{k \in J} e^{-i\phi_k} m_{2Q_k} (e^{i\phi_k} f) \chi_k$ on $\Omega$, where $J$ is the set of indices $k$ such that $Q_k$ is of type 2.

$$\int_{\mathbb{R}^n} ||B|^{1/2} g||^n = \int_{F} ||B|^{1/2} f||^n + \int_{\Omega} ||B|^{1/2} g||^n = I + II.$$  

By construction,

$$I = \int_{F} ||B|^{1/2} g||^n = \int_{F} ||B|^{1/2} f||^n \leq c \alpha^{n-p} (\|Lf\|_p + \|B|^{1/2} f\|_p)^p.$$  

Since $|B|^{1/2} \in RH_n$, and by the $L^1$ Fefferman–Phong inequality (2.5) on $2Q_k$, type 2 cubes, we obtain

$$II = \int_{\Omega} ||B|^{1/2} g||^n \leq c \sum_{k \in J} |Q_k| \left( \int_{2Q_k} ||B|^{1/2} f||^n \right) \leq C \sum_{k \in J} |Q_k| \alpha^n \leq c \alpha^{n-p} \int_{\mathbb{R}^n} |Lf|^p + ||B|^{1/2} f|^p.$$  

Hence

$$\left( \int_{\mathbb{R}^n} ||B|^{1/2} g||^n \right)^{1/n} \leq c \alpha^{n-p} (\|Lf\|_p + \|B|^{1/2} f\|_p)^{p/n}.$$  

(3.15)

e) Estimate of $Lg$:

Let $K$ the set of indices $k$. Let $\xi \in C_0^\infty(\mathbb{R}^n)$, a test function. We know that, for all $k \in K$ such that $x \in 2Q_k$, there exists $C > 0$ such that $d(x, F) > CR_k$. Therefore,

$$\int \sum_{k \in K} |b_k||\xi| \leq C \left( \int \sum_{k \in K} |b_k| \frac{|\xi|}{R_k} \right) \sup_{x \in \mathbb{R}^n} (d(x, F)|\xi(x)|).$$  

The estimate (3.5) gives us

$$\int |b_k|^p \leq CR_k^p \alpha^p |Q_k|.$$
Hence

\[ \int \sum_{k \in K} |b_k||\xi| \leq C\alpha|\Omega| \sup_{x \in \Omega} (d(x, F)|\xi(x)|). \]

We conclude that \( \sum_{k \in K} b_k \) converges in the sense of distributions in \( \mathbb{R}^n \). Then,

\[ \nabla g = \nabla f - \sum_{k \in K} \nabla b_k, \quad \text{in the sense of distributions in } \mathbb{R}^n. \]

Since the sum is locally finite in \( \Omega \) and vanishes on \( F \), then \( ag = af - \sum_{k \in K} ab_k \) holds always everywhere in \( \mathbb{R}^n \). Hence

\[ Lg = Lf - \sum_{k \in K} Lb_k \quad \text{a.e. in } \mathbb{R}^n. \]

**f) Proof of estimate (3.4):**

\[ \sum_{k \in K} \nabla \chi_k(x) = 0 \quad \text{for all } x \in \Omega, \]

then

\[ Lg = (Lf)1_F + \sum_{k \in J} L(e^{-i\phi_k}m_{2Q_k}(e^{i\phi_k}f)\chi_k) \quad \text{a.e. in } \mathbb{R}^n. \]

Since

\[ L(u) = e^{-i\phi_k} \tilde{L}(e^{i\phi_k}u) \quad \text{where } \tilde{L} = \frac{1}{i} \nabla - h_k, \]

then

\[ \sum_{k \in J} L(e^{-i\phi_k}m_{2Q_k}(e^{i\phi_k}f)\chi_k) = \frac{1}{i} \sum_{k \in J} e^{-i\phi_k}m_{2Q_k}(e^{i\phi_k}f)\nabla \chi_k - \sum_{k \in J} e^{-i\phi_k}m_{2Q_k}(e^{i\phi_k}f)\chi_k h_k \]

\[ = G_1 + G_2. \]

Let us estimate \( \|G_2\|_n \). First, we use (3.7):

\[ \|G_2\|_n = \left( \int_{\Omega} \left| \sum_{k \in J} m_{2Q_k}(e^{i\phi_k}f)\chi_k h_k \right|^n \right)^{1/n} \leq C N^{n-1} \left( \int_{k \in J} \int_{2Q_k} \left| m_{2Q_k}(e^{i\phi_k}f)h_k \right|^n \right)^{1/n} \]

\[ \leq C N^{n-1} \left( \sum_{k \in J} |2Q_k| \int_{2Q_k} |h_k|^n \left| m_{2Q_k}(e^{i\phi_k}f) \right|^n \right)^{1/n}. \]

Eq. (2.10) and the fact that \( |B| \) is a \( RH_{n/2} \) weight function and \( Q_k \) is a type 2 cube, yield
\[ \|G_2\|_n \leq C N^{\frac{n-1}{n}} \left( \sum_{k \in J} |2Q_k| R_k^n \left( \frac{1}{2Q_k} \int |B|^{n/2} \right) ^2 |m_2Q_k(e^{i\phi_k} f)|^n \right)^{1/n} \]
\[ \leq C N^{\frac{n-1}{n}} \left( \sum_{k \in J} |2Q_k| \left( R_k \frac{1}{2Q_k} \int |B| |m_2Q_k(e^{i\phi_k} f)| \right) ^n \right)^{1/n} \]
\[ \leq C N^{\frac{n-1}{n}} \left( \sum_{k \in J} |2Q_k| \left( \frac{1}{2Q_k} \int |B|^2 \int |f|^p \right)^{n/p} \right)^{1/n} \]
\[ \leq C N^{\frac{n-1}{n}} \alpha \left( \sum_{k \in J} |2Q_k| \right)^{1/n} \leq C N^{\frac{n-1}{n}} \alpha^{1-\frac{n}{p}} \left( \int |Lf|^p + \|B|^{1/2} f \right)^{1/n} . \]

We obtain
\[ \|G_2\|_n \leq C \alpha^{1-\frac{n}{p}} (\|Lf\|_p + \|B\|^{1/2} f)^{p/n} . \] (3.16)

Recall that \( G_1(x) = \sum_{k \in J} e^{-i\phi_k(x)} m_2Q_k(e^{i\phi_k} f) \nabla \chi_k(x) \). We will estimate \( \|G_1\|_n \). For all \( m \in K \), set \( K_m = \{ l \in K, 2Q_l \cap 2Q_m \neq \emptyset \} \). By construction of Whitney cubes, there exists a constant \( c > 0 \) (we can take \( c = 18 \)) such that for all \( m \in K \), \( 2Q_l \subset cQ_m \), for all \( l \in K_m \). Set \( \hat{Q}_m = cQ_m \).

\[ G_1(x) = \sum_{k \in J} e^{-i\phi_k(x)} m_2Q_k(e^{i\phi_k} f) \nabla \chi_k(x) \]
\[ = \sum_{m \in K} \chi_m(x) \left( \sum_{k \in J \cap K_m} e^{-i\phi_k(x)} m_2Q_k(e^{i\phi_k} f) \nabla \chi_k(x) \right) . \]

It suffices to prove
\[ \int_{2\hat{Q}_m} \left| \sum_{k \in J \cap K_m} e^{-i\phi_k} m_2Q_k(e^{i\phi_k} f) \nabla \chi_k \right|^n \leq C \alpha^n |2\hat{Q}_m| . \] (3.17)

We fix an \( m \), by the gauge transformation of Corollary 2.10, \( \tilde{h}_m = a - \nabla \tilde{\phi}_m \) satisfies (3.11) on \( \hat{Q}_m \).

**First case:** There exists \( k_0 \in J \cap K_m \) such that \( 2Q_{k_0} \) is of type 1.

Since \( |B(x)| \, dx \) is a doubling measure, there exists a constant \( A > 0 \) which depends on \( |B| \), such that for all \( k \in K_m \),
\[ (2R_k)^2 \frac{1}{2Q_k} |B| > A . \]
$|B|^{1/2} \in RH_2$, which means that $R_k^{-1} \leq C \int_{2Q_k} |B|^{1/2}$, for all $k \in K_m$. Then

$$
\int_{2Q_m} \left| \sum_{k \in J \cap K_m} e^{-i\phi_k} m_{2Q_k}(e^{i\phi_k} f) \nabla \chi_k \right|^n \leq C \left( \sum_{k \in J \cap K_m} |Q_k| R_k^{-n} \left( \int_{2Q_k} |f| \right)^n \right)
$$

$$
\leq C \left[ \sum_{k \in J \cap K_m} |Q_k| R_m^{-n} \left( \int_{2Q_k} |f| \right)^n \right]^{1/n} \leq C |Q_m| \alpha,
$$

here we used $|Q_k| \sim |Q_m|$, (3.7), Fefferman–Phong inequality (2.5) and $4Q_m \cap F \neq \emptyset$.

**Second case:** $\forall k \in J \cap K_m$, $2Q_k$ is of type 2.

$$
\sum_{k \in J \cap K_m} e^{-i\phi_k} m_{2Q_k}(e^{i\phi_k} f) \nabla \chi_k = \sum_{k \in J \cap K_m} (e^{-i\phi_k} m_{2Q_k}(e^{i\phi_k} f) - e^{-i\phi_m} m_{2Q_k}(e^{i\phi_m} f)) \nabla \chi_k + \sum_{k \in J \cap K_m} e^{-i\phi_m}(m_{2Q_k}(e^{i\phi_m} f) - m_{\tilde{Q}_m}(e^{i\tilde{\phi}_m} f)) \nabla \chi_k + \sum_{k \in J \cap K_m} e^{-i\phi_m} m_{\tilde{Q}_m}(e^{i\tilde{\phi}_m} f) \nabla \chi_k = I + II + III.
$$

Thus

$$
III = \sum_{k \in K_m} \chi_m e^{-i\phi_m} m_{\tilde{Q}_m}(e^{i\tilde{\phi}_m} f) \nabla \chi_k - \sum_{k \in K_m \setminus J} \chi_m e^{-i\phi_m} m_{\tilde{Q}_m}(e^{i\tilde{\phi}_m} f) \nabla \chi_k.
$$

We know that $\sum_{k \in K_m} \nabla \chi_k(x) = \sum_{k \in K} \nabla \chi_k(x) = 0$, for all $x \in 2Q_m$, and hence the first term in the above expression vanishes.

Since $2Q_k$, with $k \in K_m \setminus J$, are type 1 cubes, then we obtain using the same procedure as in the first case

$$
\int_{2Q_m} |III|^n \leq C |Q_m| \alpha.
$$

Now we will control the $L^\infty$ norm of $II$,

$$
\left| \sum_{k \in J \cap K_m} e^{-i\phi_m}(m_{2Q_k} e^{i\tilde{\phi}_m} f - m_{\tilde{Q}_m} e^{i\tilde{\phi}_m} f) \nabla \chi_k(x) \right|
$$

$$
\leq \sum_{k \in J \cap K_m} \left| m_{2Q_k} e^{i\tilde{\phi}_m} f - m_{\tilde{Q}_m} e^{i\tilde{\phi}_m} f \right| \| \nabla \chi_k \|_\infty
$$

$$
\leq C \sum_{k \in J \cap K_m} \left| m_{2Q_k} e^{i\tilde{\phi}_m} f - m_{\tilde{Q}_m} e^{i\tilde{\phi}_m} f \right| R_k^{-1},
$$
since
\[ |m_2Q_k(e^{i\tilde{\phi}_m} f) - m_{\tilde{Q}_m}(e^{i\tilde{\phi}_m} f)| \leq CR_m \alpha, \] (3.18)
then
\[ \left| \sum_k e^{-i\tilde{\phi}_m(x)} (m_2Q_k(e^{i\tilde{\phi}_m} f) - m_{\tilde{Q}_m}(e^{i\tilde{\phi}_m} f)) \nabla \chi_k(x) \right| \leq C N \alpha. \]

It suffices to prove (3.18):
\[ |m_2Q_k(e^{i\tilde{\phi}_m} f) - m_{\tilde{Q}_m}(e^{i\tilde{\phi}_m} f)| \leq C |m_{\tilde{Q}_m}(e^{i\tilde{\phi}_m} f - m_{2Q_k}(e^{i\tilde{\phi}_m} f))| \]
\[ \leq C R_k \left( |\nabla(e^{i\tilde{\phi}_m} f)|^p \right)^{1/p} \]
\[ \leq C \tilde{R}_m \left\{ (m_{\tilde{Q}_m}(|L(e^{i\tilde{\phi}_m} f)|)^p \right)^{1/p} + (m_{\tilde{Q}_m}(|m_{1/2}e^{i\tilde{\phi}_m} f)|)^p \right)^{1/p} \}
\[ \leq C \tilde{R}_m \left\{ (m_{\tilde{Q}_m}(|Lf|)^p \right)^{1/p} + (m_{\tilde{Q}_m}(|B^{1/2} f|)^p \right)^{1/p} \}
\]

where \( \tilde{L} = \nabla - \tilde{h}_m \) and \( L(f) = e^{-i\tilde{\phi}_m} \tilde{L}(e^{i\tilde{\phi}_m} f) \).

Lastly we estimate \( I \):
\[ e^{-i\tilde{\phi}_m(x)} m_2Q_k(e^{i\tilde{\phi}_m} f) - e^{-i\tilde{\phi}_m(x)} m_{2Q_k}(e^{i\tilde{\phi}_m} f) \]
\[ = e^{-i\tilde{\phi}_m(x)} \int_{2Q_k} e^{i\tilde{\phi}(y)} f(y) dy - e^{-i\tilde{\phi}_m(x)} \int_{2Q_k} e^{i\tilde{\phi}_m(y)} f(y) dy \]
\[ = \int_{2Q_k} (e^{i(\tilde{\phi}_m(y) - \phi_k(x))} - e^{i(\tilde{\phi}_m(y) - \tilde{\phi}_m(x))}) f(y) dy. \]

Next, we use the following inequality
\[ |e^{i(\tilde{\phi}_m(y) - \phi_k(x))} - e^{i(\tilde{\phi}_m(y) - \tilde{\phi}_m(x))}| \leq \left| (\phi_k(y) - \phi_k(x)) - (\tilde{\phi}_m(y) - \tilde{\phi}_m(x)) \right|, \]
and we obtain
\[ |e^{i(\tilde{\phi}_m(y) - \phi_k(x))} - e^{i(\tilde{\phi}_m(y) - \tilde{\phi}_m(x))}| \]
\[ \leq |(\phi_k - \tilde{\phi}_m)(y) - m_{2Q_k}(\phi_k - \tilde{\phi}_m) + m_{2Q_k}(\phi_k - \tilde{\phi}_m) - (\phi_k - \tilde{\phi}_m)(x)|. \]

Therefore
\[ \int_{2Q_k} \left( \int_{2Q_k} |e^{i(\tilde{\phi}_m(y) - \phi_k(x))} - e^{i(\tilde{\phi}_m(y) - \tilde{\phi}_m(x))} f(y) dy \right)^n dx \]
\[ \leq |2Q_k| \left( \int_{2Q_k} |f(y)| \left| (\phi_k - \tilde{\phi}_m)(y) - m_{2Q_k}(\phi_k - \tilde{\phi}_m) \right| dy \right)^n \]
\[
+ \left\{ \int_{2Q_k} |f(y)| \, dy \right\}^{\frac{n}{\pi-1}} \int_{2Q_k} |(\phi_k - \tilde{\phi}_m)(x) - m_{2Q_k}(\phi_k - \tilde{\phi}_m)|^n \, dx = |2Q_k|X^n + Y.
\]

We apply the Hölder and Poincaré inequalities. Then, we use (3.11), and the fact that $|B|$ is in $RH_{n/2}$ and $2Q_k$ is of type 2.

\[
X \leq \left( \int_{2Q_k} |f(y)|^{\frac{n}{\pi-1}} \, dy \right)^\frac{n-1}{n} \left( \int_{2Q_k} |(\phi_k - \tilde{\phi}_m)(y) - m_{2Q_k}(\phi_k - \tilde{\phi}_m)|^n \, dy \right)^{\frac{1}{n}}
\]

\[
\leq CR_k \left( \int_{2Q_k} |f(y)|^{\frac{n}{\pi-1}} \, dy \right)^\frac{n-1}{n} \left( \int_{2Q_k} |(\nabla(\phi_k - \tilde{\phi}_m)(y)|^n \, dy \right)^{\frac{1}{n}}.
\]

Moreover, by construction

\[
\nabla(\phi_k - \tilde{\phi}_m) = \tilde{h}_m - h_k,
\]

then

\[
X \leq CR_k \left( \int_{2Q_k} |(\tilde{h}_m - h_k)(y)|^n \, dy \right)^{\frac{1}{n}} \left( \int_{2Q_k} |f(y)|^{\frac{n}{\pi-1}} \, dy \right)^{\frac{n-1}{n}}
\]

\[
\leq CR_k \left( \int_{2Q_k} |(\tilde{h}_m - h_k)(y)|^n \, dy \right)^{\frac{1}{n}} \left( \int_{2Q_k} |f(y)|^{\frac{n}{\pi-1}} \, dy \right)^{\frac{n-1}{n}}
\]

\[
\leq CR_k^2 \int_{2Q_k} |B| \left[ \left( \int_{2Q_k} |f(y)| - m_{2Q_k}(|f|)|^{\frac{n}{\pi-1}} \, dy \right)^{\frac{n-1}{n}} + Cm_{2Q_k}(|f|) \right]
\]

\[
\leq CR_k^2 \int_{2Q_k} |B| \left[ \int_{2Q_k} |Lf(y)| \, dy + m_{2Q_k}(|f|) \right] \leq C \left[ \alpha |Q_k|^{1/n} + R_k^2 \int_{2Q_k} |B| \int_{2Q_k} |f| \right]
\]

\[
\leq CR_k \left[ \alpha + \left( \int_{2Q_k} |Lf(y)| + |B|^{1/2} |f(y)| \, dy \right) \right] \leq CR_k \alpha.
\]

We use the same arguments to estimate $Y$:

\[
Y = \left\{ \int_{2Q_k} |f(y)| \, dy \right\}^{\frac{n}{\pi-1}} \int_{2Q_k} |(\phi_k - \tilde{\phi}_m)(x) - m_{2Q_k}(\phi_k - \tilde{\phi}_m)|^n \, dx
\]

\[
\leq CR_k^n \left\{ \int_{2Q_k} |f(y)| \, dy \right\}^{\frac{n}{\pi-1}} \int_{2Q_k} |\nabla(\phi_k - \tilde{\phi}_m)|^n
\]
\[
\leq CR_k^n |Q_k| \int_{Q_k} |\mathbf{h}_m - h_k|^n \left\{ \int_{Q_k} |f(y)| \, dy \right\}^n
\]
\[
\leq |Q_k| R_k^n \left\{ R_k \int_{Q_k} |B| \int_{Q_k} |f(y)| \, dy \right\}^n
\]
\[
\leq |Q_k| R_k^n \left\{ \int_{Q_k} |Lf(y)| + |B|^{1/2} f(y) \, dy \right\}^n \leq |Q_k| R_k^n \alpha^n.
\]

We obtain
\[
\int_{Q_m} |I|^n \leq C \sum_{k \in J \cap K_m} \int_{Q_k} \left| (e^{-i\phi_k(x)} m_{Q_k} e^{i\phi_k f} - e^{-i\tilde{\phi}_m(x)} m_{Q_k} e^{i\tilde{\phi}_m f}) \nabla \chi_k(x) \right|^n \, dx
\]
\[
\leq C \sum_{k \in J \cap K_m} R_k^{-n} |Q_k| R_k^n \alpha^n = C \alpha \sum_{k \in J \cap K_m} |Q_k| \leq C |Q_m| \alpha.
\]

By integration on \( \Omega \) and using (3.6), we get
\[
\|G_1\|_n \leq C \alpha^{1 - \frac{n}{p}} (\|Lf\|_p + \|B|^{1/2} f\|_p)^{p/n}.
\] (3.19)

\( Lg = (Lf)1_F + G_1 + G_2 \), a.e.

Since \(|Lf| \leq C \alpha \) on \( F \), then estimates (3.19) and (3.16) imply
\[
\|Lg\|_n \leq C \alpha^{1 - \frac{n}{p}} (\|Lf\|_p + \|B|^{1/2} f\|_p)^{p/n}.
\] (3.20)

Then
\[
\|Lg\|_n + \|B|^{1/2} g\|_n \leq C \alpha^{1 - \frac{n}{p}} (\|Lf\|_p + \|B|^{1/2} f\|_p)^{p/n}.
\]

Thus (3.4) is proved. \( \square \)

3.2. Estimates for weak solution

Throughout this section we will assume that \( u \) is a weak solution of \( H(a, 0)u = 0 \) in \( 4Q \), where \( Q \) is a cube centered at \( x_0 \in \mathbb{R}^n \) with sidelength \( R \). The constants are independent of \( u \) and \( Q \).

**Lemma 3.4.** (See [20, Lemma 1.11].) Let \( B \) satisfy (1.9). Then, for all \( k > 0 \), there exists a constant \( C_k > 0 \) such that
\[
|u(x_0)| \leq \frac{C_k}{(1 + Rm(x_0, |B|))^k} \left( \int_{Q(x_0, R)} |u|^2 \right)^{1/2}.
\] (3.21)

This lemma leads to the following proposition:
Proposition 3.5. Under the hypothesis (1.9), for all $q > 2$, there exists a constant $C > 0$ such that

$$\left( \int_\mathbb{Q} \abs{m(\cdot, |B|)u}^q \right)^{1/q} \leq C \left( \int_{3\mathbb{Q}} \abs{m(\cdot, |B|)u}^2 \right)^{1/2}. \tag{3.22}$$

Proof. Fix $q > 2$

$$\left( \int_\mathbb{Q} \abs{m(x, |B|)u(x)}^q \, dx \right)^{1/q} \leq \left\{ 1 + Rm(x_0, |B|) \right\}^{k_0} \int_\mathbb{Q} \abs{u}^q \right)^{1/q} \leq C \left( 1 + Rm(x_0, |B|) \right)^{k_0} \left( \int_{3\mathbb{Q}} \abs{u}^2 \right)^{1/2} \leq \left\{ 1 + Rm(x_0, |B|) \right\}^{k_0 - k + (k_0/k_0 + 1)} \frac{C_k m(x_0, |B|)}{\left( 1 + Rm(x_0, |B|) \right)^{k_0}} \left( \int_{3\mathbb{Q}} \abs{u}^2 \right)^{1/2} \leq C \left( \int_{3\mathbb{Q}} \abs{m(\cdot, |B|)u}^2 \right)^{1/2}.$$

Here we used Lemma 2.2 and the fact that $u$ satisfies Proposition 2.1 with arbitrary $k$. \hfill \square

Lemma 3.6. (See [20, Lemma 2.7].) Suppose $B$ satisfies (1.9). For any integer $k > 0$, there exists $C_k > 0$, such that

$$\abs{Lu(x_0)} \leq \frac{C_k}{\left( 1 + Rm(x_0, |B|) \right)^k} \frac{1}{R} \left( \int_{Q(x_0, 2R)} \abs{u}^2 \right)^{1/2}. \tag{3.23}$$

Remark 3.7. The proof of this lemma is based on the following inequality interesting in its own right:

If $2 \leq p < q \leq \infty$ and $1/q - 1/p > -2/n$, then

$$\left( \int_{\frac{1}{2}Q} \abs{Lu}^q \right)^{1/q} \leq C \left( \int_{\frac{1}{2}Q} \abs{Lu}^2 \right)^{1/2} + CR^2 \left( \int_{\frac{1}{2}Q} \abs{\nabla B \abs{u}}^p \right)^{1/p} + CR^2 \left( \int_{\frac{1}{2}Q} \abs{B \abs{Lu}}^p \right)^{1/p}. \tag{3.24}$$
Remark 3.8. (See [20].) Let $\Gamma_0(x, y)$ be the kernel of $H(a, 0)^{-1}$. Under assumptions (1.9), for all $k > 0$, there exists a constant $C_k > 0$ such that

$$\left| L_j^x \Gamma_0(x, y) \right| \leq \frac{C_k}{1 + |x - y|m(x, |B|)} \frac{1}{|x - y|^{n-1}}, \quad (3.25)$$

for all $x, y \in \mathbb{R}^n, x \neq y$, where $L_j^x = \frac{\partial}{\partial x_j} - a_j(x)$.

Using inequalities (3.23) and (3.24), we obtain the following technical lemma, necessary for the proof of Theorem 1.3:

Lemma 3.9. Under assumptions (1.9), for any $q > 2$, there exists a constant $C = C_q > 0$ such that

$$\left( \frac{1}{Q} \int |Lu|^q \right)^{1/q} \leq C \left( \frac{1}{3Q} \int |Lu|^2 + |m(\cdot, |B|)u|^2 \right)^{1/2}, \quad (3.26)$$

and

$$|Lu(x_0)| \leq C \left( \frac{1}{3Q} \int |Lu|^2 + |m(\cdot, |B|)u|^2 \right)^{1/2}. \quad (3.27)$$

Proof. According to the type of the cube $Q$, we would use (3.23) or (3.24) to prove our lemma.

First case: $R^2 \frac{1}{Q} |B| \leq 1$. By the definition of $m(\cdot, |B|)$, it follows that $R \leq \frac{1}{m(x_0, |B|)}$. Using (1.9) and (3.24) we have for all $2 \leq p < q \leq \infty$ and $1/q - 1/p > -2/n$

$$\left( \frac{1}{\frac{1}{3}Q} \int |Lu|^q \right)^{1/q} \leq C \left( \frac{1}{\frac{1}{3}Q} \int |Lu|^2 \right)^{1/2} + CR^2 \left( \frac{1}{\frac{1}{3}Q} \int |m(x, |B|)^3 u(x)||^p dx \right)^{1/p}$$

$$+ CR^2 \left( \frac{1}{\frac{1}{3}Q} \int |m(x, |B|)^2 Lu(x)||^p dx \right)^{1/p}.$$}

Since $R < \frac{1}{m(x_0, |B|)}$, then by Lemma 2.2,

$$\forall x \in Q, \quad m(x, |B|) \approx m(x_0, |B|).$$

Hence:

$$\left( \frac{1}{\frac{1}{3}Q} \int |Lu|^q \right)^{1/q} \leq C \left( \frac{1}{\frac{1}{3}Q} \int |Lu|^2 \right)^{1/2} + CR^2 m(x_0, |B|)^2 \left( \frac{1}{\frac{1}{3}Q} \int |m(x, |B|)u(x)||^p dx \right)^{1/p}$$

$$+ CR^2 m(x_0, |B|)^2 \left( \frac{1}{\frac{1}{3}Q} \int |Lu|^p \right)^{1/p}.$$
We control $R$ by $\frac{1}{m(x_0, |B|)}$ and we obtain
\[
\left( \int_{\frac{1}{2}Q} |Lu|^q \right)^{1/q} \leq C \left\{ \left( \int_{\frac{1}{4}Q} |Lu|^2 \right)^{1/2} + \left( \int_{\frac{1}{4}Q} (|m(\cdot, |B|)u|)^p \right)^{1/p} + \left( \int_{\frac{1}{4}Q} |Lu|^p \right)^{1/p} \right\}.
\]
By iterating the inequality 3.5, it follows that for any $2 < q \leq +\infty$,
\[
\left( \int_{\frac{1}{2}Q} |Lu|^q \right)^{1/q} \leq C \left\{ \left( \int_{\frac{1}{4}Q} |Lu|^2 \right)^{1/2} + \left( \int_{\frac{1}{4}Q} (|m(\cdot, |B|)u|)^2 \right)^{1/2} \right\}.
\]

**Second case:** $R^2 f_Q |B| > 1$. We use Lemma 3.6 to get the following inequality:
\[
|Lu(x_0)| \leq \frac{C}{R} \left( \int_{\frac{1}{2}Q} |u|^2 \right)^{1/2}.
\]
Now we apply Fefferman–Phong inequality (2.5). As
\[
\min \left( \int_{\frac{1}{2}Q} |B|, \frac{1}{R^2} \right) \sim \min \left( \int_{\frac{1}{4}Q} |B|, \frac{1}{R^2} \right) = \frac{1}{R^2},
\]
the inequality takes the following form
\[
|Lu(x_0)| \leq C \left( \int_{Q(x_0, 2R)} |Lu|^2 + |B||u|^2 \right)^{1/2} \leq C \left( \int_{\frac{1}{2}Q} |Lu|^2 + |m(\cdot, |B|)u|^2 \right)^{1/2}.
\]
The last step uses (1.9). \( \square \)

3.2.1. Some important tools
Reverse Hölder inequalities previously established will be used to prove Theorem 1.3. The primary tool is the following criterion for $L^p$ boundedness [2]. A slightly weaker version appears in Shen [18].

**Theorem 3.10.** Let $1 \leq p_0 < q_0 \leq \infty$. Suppose that $T$ is a bounded sublinear operator on $L^{p_0}(\mathbb{R}^n)$. Assume that there exist constants $\alpha_2 > \alpha_1 > 1$, $C > 0$ such that
\[
\left( \int_{\frac{1}{2}Q} |Tf|^{q_0} \right)^{\frac{1}{q_0}} \leq C \left\{ \left( \int_{\alpha_1 Q} |Tf|^{p_0} \right)^{\frac{1}{p_0}} + (S|f|)(x) \right\},
\]
for all cube $Q$, $x \in Q$ and all $f \in L_{\text{comp}}^{\infty}(\mathbb{R}^n)$ with support in $\mathbb{R}^n \setminus \alpha_2 Q$, where $S$ is a positive operator. Let $p_0 < p < q_0$. If $S$ is bounded on $L^p(\mathbb{R}^n)$, then, there is a constant $C$ such that
\[
\|Tf\|_p \leq C \|f\|_p
\]
for all $f \in L_{\text{comp}}^{\infty}(\mathbb{R}^n)$. 

An important step to prove the $L^p$ boundedness of Riesz transforms via the application of the previous theorem, is the control of the term $m(\cdot, |B|)u$ on the reverse Hölder type estimates established earlier. The following result enables such a control:

**Theorem 3.11.** Under assumptions (1.9), for all $1 < p < \infty$, there exists a constant $C > 0$, depending on $B$, such that

$$\|m(\cdot, |B|)H(a, 0)^{-1/2}(f)\|_p \leq C\|f\|_p.$$  

(3.29)

for all $f \in C_0^\infty(\mathbb{R}^n)$.

This result is a consequence of the $L^p$ boundedness of $m(\cdot, |B|)^2H(a, 0)^{-1}$ for all $1 < p < \infty$ (see [20, Theorem 3.1]). We shall use complex interpolation relying on the fact that for all $y \in \mathbb{R}$, the imaginary power of Schrödinger operator $H^y$ has a bounded extension on $\mathbb{R}^n$, $1 < p < \infty$. This result due to Hebisch [12] follows from the Gaussian estimates on the heat kernel $e^{-tH}$ proved by [8]. Here, $H^y$ is defined as a bounded operator on $L^2(\mathbb{R}^n)$ by functional calculus (see [1] for more details).

**Remark 3.12.** Under assumptions (1.12), it is clear that $VH(a, V)^{-1}$ and $H(a, 0)H(a, V)^{-1}$ are $L^p$ bounded for all $1 \leq p < \infty$.

3.3. Proof of Theorem 1.3

It is known that $LH(a, 0)^{-1/2}$ is $L^p$ bounded for all $p \leq 2$. Thus, we consider $p > 2$. We need the following lemma before we start the proof of our theorem:

**Lemma 3.13.** Under assumption (1.9), the $L^p$ boundedness of $LH(a, 0)^{-1/2}$ is equivalent to that of $LH(a, 0)^{-1}L^*$ and $L(H(a, 0)^{-1}m(\cdot, |B|))$. 

**Proof.** If $LH(a, 0)^{-1/2}$ is $L^p$ bounded. By [22] and [7], $LH(a, 0)^{-1/2}$ is $L^p$ bounded for all $1 < p \leq 2$. By duality, $H(a, 0)^{-1/2}L^*$ is then $L^q$ bounded for all $q \geq 2$. Hence, $LH(a, 0)^{-1}L^*$ is $L^p$ bounded. Due to Theorem 3.11, $H(a, 0)^{-1/2}m(\cdot, |B|)$ is $L^p$ bounded, then $LH(a, 0)^{-1}m(\cdot, |B|)$ is also $L^p$ bounded.

Reciprocally, if $LH(a, 0)^{-1}L^*$ and $LH(a, 0)^{-1}m(\cdot, |B|)$ are $L^p$ bounded, then their adjoints $LH(a, 0)^{-1}L^*$ and $m(\cdot, |B|)H(a, 0)^{-1}L^*$ are bounded on $L^p$. Thus, if $F \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$, $\|H(a, 0)^{-1/2}L^*F\|_{p'} = \|H(a, 0)^{-1/2}H(a, 0)^{-1}L^*F\|_{p'}$, where we used assumption (1.9) and inequality (3.1), and thus we obtain

$$\|H(a, 0)^{-1/2}L^*F\|_{p'} \leq C\|LH(a, 0)^{-1}L^*F\|_{p'} + \|m(\cdot, |B|)H(a, 0)^{-1}L^*F\|_{p'} \leq C\|F\|_{p'}.$$  

Hence, $LH(a, 0)^{-1/2}$ is $L^p$ bounded.  

We will need the following result:

**Proposition 3.14.** Under assumption (1.9) for all $2 < p < \infty$ there exists $C_p$ such that for any $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ and any $F \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$,

$$\|m(\cdot, |B|)H(a, 0)^{-1}m(\cdot, |B|)f\|_p \leq C_p\|f\|_p, \quad \text{and} \quad \|m(\cdot, |B|)H(a, 0)^{-1}L^*F\|_p \leq C_p\|F\|_p.$$
Proof. This is a direct consequence of Theorem 3.11 and the $L^p$ boundedness of $LH(a, 0)^{-1/2}$ for all $1 < p \leq 2$. \hfill \square

It suffices therefore to prove the following result:

Proposition 3.15. Under assumption (1.9), for all $2 < p < \infty$, there exists $C_p$ such that for any $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ and any $F \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$, we have

$$\|LH(a, 0)^{-1}m(\cdot, |B|)f\|_p \leq C_p\|f\|_p$$

and

$$\|LH(a, 0)^{-1}L^*F\|_p \leq C_p\|F\|_p.$$  

Proof. Fix a cube $Q$ and let $F \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$ supported away from $4Q$. Set $H = H(a, 0)$. We use the same argument for $L_sL_kH_0^{-1}$. Then $H$ is well defined on $\mathbb{R}^n$. In particular, the support condition on $F$ implies that $u$ is a weak solution of $Hu = 0$ in $4Q$. Hence $|u|^2$ is subharmonic on $4Q$, and by Lemma 3.9, we obtain that for all $q > 2$, there exists a constant $C > 0$ such that

$$\left(\frac{1}{Q}\int |LH^{-1}L^*F|^q\right)^{1/q} \leq C\left(\frac{1}{Q}\int |LH^{-1}L^*F|^2 + |m(\cdot, |B|)H^{-1}L^*F|^2\right)^{1/2}.$$  

Thus (3.28) holds with $T = LH^{-1}L^*$, $q_0 = q$, $p_0 = 2$ and

$$SF = (M(|m(\cdot, |B|)H^{-1}L^*F|)^2)^{1/2},$$

where $M$ is the maximal Hardy–Littlewood operator. Since $S$ is $L^p$ bounded for all $2 < p < \infty$, then by Proposition 3.14, $T$ is $L^p$ bounded by Theorem 3.10.

We use the same argument for $LH^{-1}m(\cdot, |B|)$. \hfill \square

Proof of Theorem 1.10 with $V = 0$. Set $H_0 = H(a, 0)$ and $m = m(\cdot, |B|)$.

$$L_sL_kH_0^{-1} = L_sH_0^{-1}L_k + L_s[L_k, H_0^{-1}].$$

Let $j \geq 1$, $L_jH_0^{-1/2}$ is $L^p$ bounded for all $1 < p < \infty$, then $L_sH_0^{-1}L_k$ is $L^p$ bounded for $1 < p < \infty$. We know that

$$L_sH_0^{-1}b_{kj}L_jH_0^{-1} = L_sH_0^{-1}m\frac{b_{kj}}{m^2}mL_jH_0^{-1},$$

$$L_sH_0^{-1}\partial_jb_{kj}H_0^{-1} = L_sH_0^{-1}m\frac{\partial_jb_{kj}}{m^2}m^2H_0^{-1}.$$ 

Here, $b_{kj}$ and $\partial_jb_{kj}$ are the operators of multiplication by $b_{kj}$ and $\partial_jb_{kj}$.

Next, we use the assumptions $|b_{kj}| \leq Cm^2$ and $|\partial_jb_{kj}| \leq Cm^3$ and the fact that $L_sH_0^{-1}m$, $mL_jH_0^{-1}$, and $m^2H_0^{-1}$ are $L^p$ bounded for all $p > 1$. Thus, $L_sH_0^{-1}b_{kj}L_jH_0^{-1}$ and $L_sH_0^{-1}\partial_jb_{kj}L_jH_0^{-1}$ are $L^p$ bounded. Hence, $L_s[L_k, H_0^{-1}]$ is $L^p$ bounded. The $L^p$ boundedness of $L_sL_kH_0^{-1}$, for all $1 < p < \infty$, follows easily. \hfill \square
4. Schrödinger operator with electric potential on $A_\infty$

In this section, we will add the electric potential $V$ to the pure magnetic Schrödinger operator previously studied. If we take some sharp hypothesis on $V$, as condition (1.12), the approach to study the Riesz transforms will be identical, all we have to do is to replace the weight function $|B|$ by $V + |B|$ and then Theorem 1.10 easily follows. Now a natural step is to improve the conditions on $V$ and extend this result to the Schrödinger operators with an electric potential contained in $A_\infty$.

To prove such a result, we will start by giving some reverse Hölder type estimates of weak solutions. We will also use the reverse inequalities of Theorem 1.6, which are always established through Calderón–Zygmund decomposition similar to Section 3.1. We will use an equivalent approach to that of [1]. We study $H(a, V)$ considering it as a “perturbation” of $H(a, 0)$. By the Kato–Simon inequality, we will establish some maximal estimates using the $L^p$ boundedness of operators $V(-\Delta + V)^{-1}$ and $\Delta(-\Delta + V)^{-1}$ proved in [1].

4.1. Estimates for weak solution

Fix an open set $\Omega$. A subharmonic function on $\Omega$ is a function $v \in L^1_{\text{loc}}(\Omega)$ such that $\Delta v \geq 0$ in $D'(\Omega)$.

**Lemma 4.1.** Suppose $a \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. If $u$ is a weak solution of $H(a, V)u = 0$ in $\Omega$, then $|u|^2$ is a subharmonic function and

$$\Delta |u|^2 = 2|Lu|^2 + 2V|u|^2. \tag{4.1}$$

**Proof.** Since

$$\Delta |u|^2 = \Delta(u\bar{u}) = 2\text{Re}(\Delta u\bar{u}) + 2|\nabla u|^2,$$

and $H(a, V)u = 0$, then

$$\Delta u = \sum_{k=1}^n \left( iak \frac{\partial u}{\partial x_k} + i \frac{\partial}{\partial x_k} (a_k u) \right) + |a|^2 u + Vu.$$

It follows that

$$\Delta |u|^2 = 2\text{Re} \left( \sum_{k=1}^n \left( iak \frac{\partial u}{\partial x_k} + i \frac{\partial}{\partial x_k} (a_k u) \right) \bar{u} + |a|^2 u \bar{u} + Vu \bar{u} \right) + 2|\nabla u|^2$$

$$= 2\text{Re} \left( \sum_{k=1}^n iak \frac{\partial u}{\partial x_k} \bar{u} + i \frac{\partial}{\partial x_k} (a_k u) \bar{u} \right) + 2|a|^2 |u|^2 + 2V|u|^2 + 2|\nabla u|^2$$

$$= 2\text{Re} \left( \sum_{k=1}^n iak \frac{\partial u}{\partial x_k} \bar{u} + i \frac{\partial}{\partial x_k} (a_k |u|^2) - iak \frac{\partial u}{\partial x_k} \bar{u} \right) + 2|a|^2 |u|^2 + 2V|u|^2 + 2|\nabla u|^2$$

$$= 4\text{Im}(a\nabla u \bar{u}) + 2|a|^2 |u|^2 + 2|\nabla u|^2 + 2V|u|^2 = 2|Lu|^2 + 2V|u|^2. \quad \square$$
The main technical lemma is interesting in its own right. For a detailed proof see [4] and [1]. It states that a form of the mean value inequality for subharmonic functions still holds if the Lebesgue measure is replaced by a weighted measure of Muckenhoupt type. More precisely,

**Lemma 4.2.** Let \( \omega \in RH_q \) for some \( 1 < q \leq \infty \) and let \( 0 < s < \infty \) and \( r > q \) (if \( q = \infty \), \( r = \infty \)) such that \( \omega \in RH_r \). Then there exists a constant \( C \geq 0 \) depending only on \( \omega \), \( r \), \( p \), \( s \) and \( n \), such that for any cube \( Q \) and any non-negative subharmonic function \( f \) in a neighborhood of \( 2Q \) we have for all \( 1 < \mu \leq 2 \),

\[
\left( \frac{1}{Q} \int \omega f^s r \right)^{1/r} \leq C \int_{\mu Q} \omega f^s, \quad \text{for } r < +\infty
\]

and

\[
\sup_{\overline{Q}} f \leq \frac{C}{\int_{\mu Q} \omega} \int_Q \omega f^s, \quad \text{for } r = +\infty.
\]

Throughout this section we will assume \( V \in RH_q \) with \( 1 < q \leq +\infty \) and \( B \) satisfies the assumption (1.9) and \( u \) is a weak solution of \( H(a, V)u = 0 \) in \( 4Q \). All the constants are independent of \( Q \) and \( u \) but they may depend on \( V \) and \( q \).

First we give three important results that are the main tools for the proof of Theorem 1.3:

**Proposition 4.3.** There exists a constant \( C > 0 \) such that

\[
\left( \frac{1}{Q} \int |V^{1/2}u|^{2q} \right)^{1/2q} \leq C \left( \frac{1}{3Q} \int |V^{1/2}u|^2 \right)^{1/2}.
\]

**Proof.** It follows directly from Lemmas 4.2 and 4.1. \( \Box \)

**Proposition 4.4.** Set \( \bar{q} = \inf(q^*, 2q) \). For all \( 1 < \mu \leq 2 \) and \( k > 0 \), there is a constant \( C \) such that

\[
\left( \frac{1}{Q} \int |Lu|^\bar{q} \right)^{1/\bar{q}} \leq \frac{C}{(1 + R^2 \int_{\mu Q} V)^k} \left( \int_{\mu Q} |Lu|^2 + |m(\cdot, |B|)u|^2 + V|u|^2 \right)^{1/2}.
\]

**Proposition 4.5.** Let \( n/2 \leq q < n \), for all \( 1 < \mu \leq 2 \), there is a constant \( C \) such that

\[
\left( \frac{1}{Q} \int |Lu|^{q^*} \right)^{1/q^*} \leq C \left( \int_{\mu Q} |Lu|^{2q} + |m(\cdot, |B|)u|^{2q} \right)^{1/2q}.
\]

If \( q \geq n \) then there is a constant \( C \) such that

\[
\sup_Q |Lu| \leq C \left( \int_{\mu Q} |Lu|^{2q} + |m(\cdot, |B|)u|^{2q} \right)^{1/2q}.
\]
The next lemma will be useful to prove Propositions 4.4 and 4.5.

**Lemma 4.6.** For all $1 \leq \mu < \mu' \leq 2$ and $k > 0$, there is a constant $C$ such that
\[
\int_{\mu Q} |u|^2 \leq \frac{C}{(1 + R^2 f_Q V)^k} \left( \int_{\mu' Q} |u|^2 \right)
\]
and
\[
\int_{\mu Q} (|Lu|^2 + V|u|^2) \leq \frac{C}{(1 + R^2 f_Q V)^k} \left( \int_{\mu' Q} (|Lu|^2 + V|u|^2) \right).
\]

**Proof.** There is nothing to prove if $R^2 f_Q V \leq 1$. We assume $R^2 f_Q V > 1$. The well-known Caccioppoli type argument yields for $1 \leq \mu < \mu' \leq 2$
\[
\int_{\mu Q} |Lu|^2 + V|u|^2 \leq \frac{C}{R^2} \int_{\mu' Q} |u|^2.
\]  
(4.3)

The improved Fefferman–Phong inequality (2.8) and the fact that the averages of $V$ on $\mu Q$ with $1 \leq \mu \leq 2$ are all uniformly comparable imply for some $\beta > 0$,
\[
\frac{1}{R^2} \int_{\mu Q} |u|^2 \leq \frac{C}{(R^2 f_Q V)^\beta} \int_{\mu Q} |Lu|^2 + V|u|^2.
\]

The desired estimates follow readily by iterating these two inequalities. \( \square \)

**Lemma 4.7.** For all $1 < \mu \leq 2$ and $k > 0$, there is a constant $C$ such that
\[
\left( R \int_{Q} V \right)^2 \int_{Q} |u|^2 \leq \frac{C}{(1 + R^2 f_Q V)^k} \left( \int_{\mu Q} V|u|^2 \right).
\]

**Proof.** Using Lemma 4.6 with $k > 1$ and $1 < \mu' < \mu$ and subsequently Lemma 4.2, we have
\[
\left( R \int_{Q} V \right)^2 \int_{Q} |u|^2 \leq \frac{C f_Q V \mu' Q |u|^2}{(1 + R^2 f_Q V)^{k-1}} \leq \frac{C f_Q V \sup_{\mu' Q} |u|^2}{(1 + R^2 f_Q V)^{k-1}} \leq \frac{C f_Q V|u|^2}{(1 + R^2 f_Q V)^{k-1}}. \quad \square
\]

**Lemma 4.8.** For all $1 < \mu \leq 2$, $k > 0$ and $n < p < \infty$, there is a constant $C$ such that
\[
\left( R \int_{Q} V \right)^2 \int_{Q} |u|^2 \leq \frac{C}{(1 + R^2 f_Q V)^k} \left( \int_{\mu Q} |Lu|^p \right)^{2/p}.
\]
Proof. If \( f_{\mu Q} |Lu|^p = \infty \), there is nothing to prove. Assume, therefore, that \( f_{\mu Q} |Lu|^p < \infty \).
Let \( 1 < v < \mu \) and \( \eta \) be a smooth non-negative function, bounded by 1, equal to 1 on \( vQ \) with support on \( \mu Q \) and whose gradient is bounded by \( C/R \) and Laplacian by \( C/R^2 \).
Integrating the equation \( H(a, 0)u + Vu = 0 \) against \( \bar{u}\eta^2 \).

Since

\[
H(a, V)u = \sum_{j=1}^n L_j^* L_j u + Vu,
\]

\[
\int H(a, V)u\bar{u}\eta^2 = \sum_{j=1}^n \int L_j u L_j (u\eta^2) + \int V|u|^2 \eta^2,
\]

then

\[
\int |Lu|^2 \eta^2 + V|u|^2 \eta^2 = 2 \int Lu \cdot \nabla \bar{u}\eta,
\]

hence

\[
\int V|u|^2 \eta^2 \leq \frac{C}{R} \left( \int |Lu|^2 \right)^{1/2} \left( \int |u|^2 \eta^2 \right)^{1/2},
\]

\[
X \leq C \left( R^2 \int_Q V \right)^{1/2} |\mu Q|^{1/2} Y^{1/2} Z^{1/2}
\]

(4.4)

where we set \( X = (R^2 \int_Q V) \int V|u|^2 \eta^2 \), \( Y = (f_{\mu Q} |Lu|^p)^{2/p} \) and \( Z = \int_Q V \int |u|^2 \eta^2 \). By Morrey’s embedding theorem and diamagnetic inequality (2.3), \( u \) is Hölder continuous with exponent \( \alpha = 1 - n/p \). Hence for all \( x, y \in \mu Q \), we have

\[
||u(x)| - |u(y)|| \leq C \left( \frac{|x - y|}{R} \right)^\alpha \left( \int_{\mu Q} |\nabla |u||^p \right)^{1/p} \leq C \left( \frac{|x - y|}{R} \right)^\alpha RY^{1/2}.
\]

We pick \( y \in \overline{Q} \) such that \( |u(y)| = \inf_Q |u| \). Then

\[
Z = \int_Q V \int |u|^2 \eta^2 \leq 2 \left( \int_Q V \right) \inf_Q |u| \int \eta^2 + 2 \left( \int_Q V \right) \int |u(x)| - |u(y)||^2 \eta^2(x) dx
\]

\[
\leq 2 \left( \int_Q (V|u|^2) \right) \int \eta^2 + C \left( \int_Q V \right) R^2 Y \int \left( \frac{|x - y|}{R} \right)^{2\alpha} \eta^2(x) dx
\]

\[
\leq C \left( \int_Q (V|u|^2) \right) |Q| + C \left( \int_Q V \right) R^2 Y |\mu Q|
\]

\[
\leq C \int_Q V|u|^2 \eta^2 + C \left( \int_Q V \right) R^2 Y |\mu Q|.
\]
where, in the penultimate inequality, we used the support condition on $\eta$ and $0 \leq \eta \leq 1$, and in the last, $\eta = 1$ on $Q$. Using the previous inequalities, we obtain

$$X \leq C|\mu Q|^{1/2}Y^{1/2}\left(CX + C\left(R^2 \int_Q V\right)^2 |\mu Q|Y\right)^{1/2},$$

which, as $2ab \leq \epsilon^{-1}a^2 + \epsilon b^2$ for all $a, b \geq 0$ and $\epsilon > 0$, implies

$$X \leq C\left(1 + R^2 \int_Q V\right)^2 |\mu Q|Y.$$

Next, let $1 < \nu' < \nu$. Using $\eta = 1$ on $\nu Q$ Lemmas 4.2 and 4.6

$$\int_Q V|u|^2\eta^2 \geq \int_Q V|u|^2 \geq C \int_Q V \int_{\nu' Q} |u|^2 \geq C\left(\int_Q V\right)\left(1 + R^2 \int_Q V\right)^k \int_Q |u|^2,$$

hence

$$X \geq C\left(R \int_Q V\right)^2 \left(1 + R^2 \int_Q V\right)^k \int_Q |u|^2.$$

The upper and lower bounds for $X$ yield the lemma. 

**Lemma 4.9.** Let $q < n$, there exists a constant $C > 0$ such that

$$\left(\int_Q |Lu|^{q^*}\right)^{1/q^*} \leq C\left(\frac{1}{R} + R \int_Q V\right)\left(\int_{3Q} |u|^2\right)^{1/2}. \quad (4.5)$$

Consider $q \geq n$, there is a constant $C > 0$ such that

$$\sup_Q |Lu| \leq C\left(\frac{1}{R} + R \int_Q V\right)\left(\int_{3Q} |u|^2\right)^{1/2}. \quad (4.6)$$

**Proof.** Set $\phi \in C_0^\infty(2Q)$, with $\phi \equiv 1$ in $Q$, $|\nabla \phi| \leq C/R$ and $|\nabla^2 \phi| \leq C/R^2$.

Since

$$H(a, 0)(u \phi) = \frac{2}{i}Lu \nabla \phi - u \Delta \phi - V u \phi,$$

then

$$u(x)\phi(x) = \int_{\mathbb{R}^n} \Gamma_0(x, y) \left[\frac{2}{i}Lu(y)\nabla \phi(y) - u(y)\Delta \phi(y) - V(y)u(y)\phi(y)\right] dy.$$
By (3.25), we obtain for all $x_0 \in Q$

$$|Lu(x_0)| \leq \frac{C}{R^n} \int_{2Q} |Lu(y)| \, dy + \frac{C}{R^{n+1}} \int_{2Q} |u(y)| \, dy + C \int_{2Q} \frac{V(y)|u(y)|}{|x_0 - y|^{n-1}} \, dy.$$ 

Using Caccioppoli type inequality, it follows that

$$|Lu(x_0)| \leq \frac{C}{R} \left( \int_{2Q} |u(y)|^2 \, dy \right)^{1/2} + C \int_{2Q} \frac{V(y)|u(y)|}{|x_0 - y|^{n-1}} \, dy.$$ 

If $q < n$,

$$\left( \int_Q |Lu|^{q^*} \right)^{1/q^*} \leq \frac{C}{R} \sup_{2Q} |u| + C \left( \int_{2Q} \left\{ \int_{2Q} \frac{V(y)|u(y)|}{|x_0 - y|^{n-1}} \, dy \right\}^{q^*} \right)^{1/q^*}.$$ 

By Hardy–Littlewood–Sobolev inequality, we obtain

$$\left( \int_Q |Lu|^{q^*} \right)^{1/q^*} \leq \frac{C}{R} \sup_{\frac{Q}{2}} |u| + CR \left( \int_Q |V|^q \right)^{1/q} \sup_{2Q} |u|$$

$$\leq \frac{C}{R} \sup_{\frac{Q}{2}} |u| + CR \int_Q |V| \sup_{\frac{Q}{2}} |u|.$$ 

(4.7)

Subharmonicity of $|u|^2$ yields

$$\left( \int_Q |Lu|^{q^*} \right)^{1/q^*} \leq C \left( \frac{1}{R} + R \int_Q V \right) \left( \int_{\frac{Q}{2}} |u|^2 \right)^{1/2}.$$ 

If $q \geq n$

$$\sup_Q |Lu| \leq \frac{C}{R} \sup_{2Q} |u| + C \sup_{2Q} |u(y)| \sup_{x \in Q} \left( \int_{2Q} \frac{V(y)}{|x - y|^{n-1}} \, dy \right)$$

$$\leq \frac{C}{R} \sup_{2Q} |u| + C \frac{R^{n-1}}{2Q} \sup_{2Q} |u| \int_{2Q} V(y) \, dy.$$ 

Here we used Hölder inequality with $V \in L^q(2Q)$ and the fact that $V \in RH_q$. Hence, inequality (4.6) holds. □
Lemma 4.10. Let $1 < \mu \leq 2$ and $k > 0$, if $n/2 \leq q < n$, then there is a constant $C$ such that
\[
\left( \frac{1}{Q} \int |Lu|^q \right)^{1/q} \leq \frac{C}{R(1 + R^2 \int_Q V)^k} \left( \sup_{\mu Q} |u| \right).
\]
If $q \geq n$, then there is a constant $C$ such that
\[
\sup_Q |Lu| \leq \frac{C}{R(1 + R^2 \int_Q V)^k} \left( \sup_{\mu Q} |u| \right).
\]

Proof. It suffices to combine Lemma 4.9 with Lemma 4.6. \qed

4.1.1. Proof of Proposition 4.4

Proof. We assume $q > \frac{2n}{n+2}$.

Let $v$ be a weak solution of $H(a,0)v = 0$ in $2Q$ with $v = u$ on $\partial(2Q)$ and set $w = u - v$ on $2Q$. Since $w = 0$ on $\partial(2Q)$, we have
\[
\left( \frac{1}{2Q} \int |Lw|^2 \right)^{1/2} \leq \left( \frac{1}{2Q} \int |Lu|^2 \right)^{1/2}.
\]
By estimates of Lemma 3.9, we have for all $2 \leq p \leq \infty$ and in particular for $p = \bar{q}$,
\[
\left( \frac{1}{Q} \int |Lv|^p \right)^{1/p} \leq C \left( \frac{1}{\frac{Q}{2}} \int |Lv|^2 + \frac{1}{\frac{Q}{2}} \int |m(\cdot, |B|)v|^2 \right)^{1/2}.
\]
The subharmonicity of $|v|^2$ and $|u|^2$ implies
\[
\frac{1}{\frac{Q}{2}} \int |v|^2 \leq \sup_{\frac{Q}{2}} |v|^2 = \sup_{\partial(\frac{Q}{2})} |v|^2 = \sup_{\partial(\frac{Q}{2})} |u|^2 \leq C \frac{1}{\frac{Q}{3}} \int |u|^2.
\]
Hence
\[
\left( \frac{1}{\frac{Q}{2}} \int |m(x, |B|)v(x)|^2 \, dx \right)^{1/2} \leq \{1 + Rm(x_0, |B|)\}^{k_0} \left( \frac{1}{\frac{Q}{2}} \int |v|^2 \right)^{1/2}
\]
\[
\leq \frac{C_k \{1 + Rm(x_0, |B|)\}^{k_0} m(x_0, |B|)}{\{1 + Rm(x_0, |B|)\}^k} \left( \frac{1}{\frac{Q}{3}} \int |u|^2 \right)^{1/2}.
\]
\[ \leq \left\{ 1 + Rm(x_0, |B|) \right\}^{k_0-k+(k_0/k_0+1)} \frac{C_{k}m(x_0, |B|)}{1 + Rm(x_0, |B|)} \left( \int_{3Q} |u|^2 \right)^{1/2} \]
\[ \leq C \left( \int_{3Q} |m(\cdot, |B|)u|^2 \right)^{1/2}. \]

Where we used Lemmas 2.2 and 4.6 for an arbitrary \( k \). It follows
\[ \left( \int_{Q} |Lu|^p \right)^{1/p} \leq C \left( \int_{3Q} |Lu|^2 + \int_{3Q} |m(\cdot, |B|)u|^2 \right)^{1/2}. \]

Let \( 1 < \mu < 2 \) and \( \eta \) be a smooth non-negative function, bounded by 1, equal to 1 on \( \Phi \) with support contained in \( \mu \Phi \) and whose gradient is bounded by \( C/R \) and Laplacian by \( C/R^2 \). As \( H(a, 0)w = H(a, 0)u = -Vu \) on \( 2\Phi \), we have
\[ H(a, 0)(w\eta) = \frac{2}{i}Lw \nabla \eta - w \Delta \eta - Vu \eta. \]

Hence
\[ L(w\eta)(x) = \int_{\mathbb{R}^n} L^\Phi \Gamma_0(x, y) \left[ \frac{2}{i}L(w)(y)\nabla \eta(y) - w(y)\Delta \eta(y) - (Vu \eta)(y) \right] dy \]
\[ = I + II + III, \]
with \( \Gamma_0 \) the kernel of \( H(a, 0)^{-1} \). We know by (3.25), \(|L^\Phi \Gamma_0(x, y)| \leq C|x - y|^{-n}\). Since \( \tilde{q} \leq q^* \), then
\[ \left( \int_{Q} |Lw|^\tilde{q} \right)^{1/\tilde{q}} \leq \left( \int_{Q} |Lw|^q \right)^{1/q}. \]

Using support conditions on \( \eta \), we obtain the following estimates for all \( x \in \Phi \),
\[ |I| \leq C \left( \int_{2\Phi} |Lw|^2 \right)^{1/2} \leq C \left( \int_{2\Phi} |Lu|^2 \right)^{1/2} \]
and
\[ |II| \leq \frac{C}{R} \int_{2\Phi} |w| \leq C \left( \int_{2\Phi} |\nabla |w|^2 \right)^{1/2} \leq C \left( \int_{2\Phi} |Lw|^2 \right)^{1/2} \leq C \left( \int_{2\Phi} |Lu|^2 \right)^{1/2}. \]

Above we used the Poincaré and the diamagnetic inequality (2.3).\(^3\)

\(^3\) We consider the function \( \tilde{w} \) defined as \( \tilde{w} = w, \) on \( \Phi \), \( \tilde{w} = 0, \) on \( \mathbb{R}^n \setminus \Phi \). Then \( L(\tilde{w}) = 12\Phi L(w) \) as \( w \) vanishes on \( \partial 2\Phi \).
It follows by Hardy–Littlewood–Sobolev inequality,
\[
\left( \int_{\mathbb{R}^n} |\nabla^{i} f|^q \right)^{1/q^*} \leq C \left( \int_{\mathbb{R}^n} |Vu\eta|^q \right)^{1/q} \leq C \left( \int_{\mu Q} |V|^q \right)^{1/q} \sup_{\mu Q} |u|.
\]

Since \( V \in RH_q \), then
\[
\left( \int_{Q} |\nabla^{i} f|^q \right)^{1/q^*} \leq CR \int_{\mu Q} V \sup_{\mu Q} |u|.
\]
\( (4.8) \)

Now, if \( \mu < \mu' < 2 \), subharmonicity of \( |u|^2 \) and Lemma 4.2 yield
\[
R \int_{\mu Q} V \sup_{\mu Q} |u| \leq CR \int_{\mu Q} V \left( \int_{\mu' Q} |u|^2 \right)^{1/2},
\]
which by Lemma 4.7 is bounded by \( C(f_{2Q}(V|u|^2))^{1/2} \). Gathering the estimates obtained for \( Lv \) and \( Lw \), the lemma is proved. \( \square \)

4.1.2. Proof of Proposition 4.5

**Proof.** Assume \( q > n/2 \) (it includes \( q = \frac{n}{2} \) via the self-improvement of reverse Hölder classes). The previous lemma shows that \( \int_{\mu Q} |Lu|^q \) is finite for all \( 1 < \mu' \leq \mu \). As \( \tilde{q} = 2q > n \), Lemma 4.8 applies and using it for \( k = 0 \) instead of Lemma 4.7 in the argument of Proposition 4.4, we obtain,
\[
\left( \int_{Q} |Lw|^q \right)^{1/q^*} \leq C \left( \int_{\mu Q} |Lw|^{2q} \right)^{1/2q}.
\]
Next, we know that
\[
\left( \int_{Q} |Lu|^q \right)^{1/q^*} \leq C \left( \int_{\mu Q} |Lu|^{2q} + |m(\cdot, |B|)u|^{2q} \right)^{1/2q}.
\]
Hence
\[
\left( \int_{Q} |Lu|^q \right)^{1/q^*} \leq C \left( \int_{\mu Q} |Lu|^{2q} + |m(\cdot, |B|)u|^{2q} \right)^{1/2q}.
\]
\( \square \)

4.1.2. Proof of Proposition 4.5

**Proof.** Assume \( q > n/2 \) (it includes \( q = \frac{n}{2} \) via the self-improvement of reverse Hölder classes). The previous lemma shows that \( \int_{\mu Q} |Lu|^q \) is finite for all \( 1 < \mu' \leq \mu \). As \( \tilde{q} = 2q > n \), Lemma 4.8 applies and using it for \( k = 0 \) instead of Lemma 4.7 in the argument of Proposition 4.4, we obtain,
\[
\left( \int_{Q} |Lw|^q \right)^{1/q^*} \leq C \left( \int_{\mu Q} |Lw|^{2q} \right)^{1/2q}.
\]
Next, we know that
\[
\left( \int_{Q} |Lu|^q \right)^{1/q^*} \leq C \left( \int_{\mu Q} |Lu|^{2q} + |m(\cdot, |B|)u|^{2q} \right)^{1/2q}.
\]
Hence
\[
\left( \int_{Q} |Lu|^q \right)^{1/q^*} \leq C \left( \int_{\mu Q} |Lu|^{2q} + |m(\cdot, |B|)u|^{2q} \right)^{1/2q}.
\]
\( \square \)

4.2. Maximal inequalities

**Proof of Theorem 1.8.** The proof of this theorem is identical to that of Theorem 1.1 in [1]. First we prove an \( L^1 \) inequality, then we establish some reverse Hölder type estimates, then finally we apply Theorem 3.10.
Lemma 4.11. Let \( f \in L^\infty_{\text{comp}}(\mathbb{R}^n) \) and \( u = H(a, V)^{-1}f \). Then,

\[
\int_{\mathbb{R}^n} V|u| \leq \int_{\mathbb{R}^n} |f|, \tag{4.9}
\]

and

\[
\int_{\mathbb{R}^n} |H(a, 0)u| \leq 2 \int_{\mathbb{R}^n} |f|. \tag{4.10}
\]

Proof. \( V \geq 0 \), by Kato–Simon inequality (2.4), we have

\[
|H(a, V)^{-1}f| \leq H(0, V)^{-1}|f|.
\]

We know, by [1] that

\[
\int_{\mathbb{R}^n} VH(0, V)^{-1}|f| \leq \int_{\mathbb{R}^n} |f|.
\]

Thus, inequality (4.9) holds, and inequality (4.10) follows by difference. \( \square \)

Proof of the \( L^p \) maximal inequality: Assume \( V \in RH_q \) with \( q > 1 \). \( VH(a, V)^{-1} \). We know that this operator is bounded on \( L^1(\mathbb{R}^n) \), so we apply Theorem 3.10 through the reverse Hölder inequality verified by any weak solution. Set \( Q \) a fixed cube and \( f \in L^\infty(\mathbb{R}^n) \) a function with compact support in \( \mathbb{R}^n \setminus 4Q \). Then \( u = H(a, V)^{-1}f \) is well defined in \( \mathcal{V} \) and it is a weak solution of \( H(a, 0)u + Vu = 0 \) in \( 4Q \).

Since \( |u|^2 \) is subharmonic, by Lemma 4.2 with \( w = V, f = |u|^2 \) and \( s = 1/2 \), we obtain

\[
\left( \frac{1}{Q} \int |Vu|^q \right)^{1/q} \leq C \frac{1}{2Q} \int |Vu|.
\]

Thus (3.28) holds with \( T = VH(a, V)^{-1}, p_0 = 1, q_0 = q, S = 0, \alpha_1 = 2 \) and \( \alpha_2 = 4 \). Hence \( VH(a, V)^{-1} \) is bounded on \( L^p(\mathbb{R}^n) \) for all \( 1 < p < q \) by Theorem 3.10. Due to the properties of \( RH_q \) weights, we can replace \( q \) by \( q + \epsilon \). Taking the difference, we obtain the same result for \( H(a, 0)H(a, V)^{-1} \). This completes the proof of Theorem 1.8. \( \square \)

Remark 4.12. Theorem 1.11 is a consequence of Theorems 1.10 and 1.8:

\[
L_5 L_k H(a, V)^{-1} = L_5 L_k H(a, 0)^{-1} H(a, 0) H(a, V)^{-1}.
\]

4.3. Proof of Theorem 1.4

Using Theorem 1.3 and the Corollary 1.9, we can establish a first result:

Theorem 4.13. Under the assumptions of Theorem 1.4, there exists an \( \epsilon > 0 \) such that \( LH(a, V)^{-1/2} \) is \( L^p \) bounded for all \( 1 < p < 2q + \epsilon \), where \( \epsilon \) depends only on \( V \).
Proof.

\[ LH(a, V)^{-1/2} = LH(a, 0)^{-1/2} H(a, 0)^{1/2} H(a, V)^{-1/2}. \]

Remark 4.14. Using the same argument, we obtain that \( m(\cdot, |B|) H(a, V)^{-1/2} \) is \( L^p \) bounded for \( 1 \leq p < 2q + \epsilon \).

Now, we have to control the term \( m(\cdot, |B|)u \) appearing in the previous estimates. It suffices to study the \( L^p \) boundedness of the operator \( m(\cdot, |B|) H(a, V)^{-1/2} \). The result of Remark 4.14 is not enough, we will improve it through the following theorem:

Theorem 4.15. Let \( a \in L^2_{\text{loc}}(\mathbb{R}^n) \), \( V \in RH_q \), \( 1 < q \leq +\infty \) and we assume (1.9). Then, for all \( 1 \leq p \leq \infty \), there is a constant \( C_p \), such that

\[ \| m(\cdot, |B|)^{1/2} H(a, V)^{-1/2} f \|_p \leq C \| f \|_p \]  

(4.11)

for all \( f \in C_0^\infty(\mathbb{R}^n) \).

By complex interpolation, we obtain

Corollary 4.16. Suppose \( a \in L^2_{\text{loc}}(\mathbb{R}^n) \) and \( V \in RH_q \), \( 1 < q \leq +\infty \). We also assume (1.9). Then, for all \( 1 \leq p < \infty \), there is a constant \( C_p \), such that

\[ \| m(\cdot, |B|) H(a, V)^{-1/2} f \|_p \leq C \| f \|_p. \]  

(4.12)

for all \( f \in C_0^\infty(\mathbb{R}^n) \).

We will apply Theorem 3.10 to prove Theorem 4.15 for \( p > 2 \) and we will need the following lemma:

Lemma 4.17. Under assumptions of Theorem 4.15, let \( u \) be a weak solution of \( H(a, V)u = 0 \) in \( 4Q \) centered at \( x_0 \in \mathbb{R}^n \) and of sidelength \( 4R \). Then, for any integer \( k > 0 \), there exists a constant \( C_k \) such that

\[ |u(x_0)| \leq \frac{C_k}{(1 + Rm(x_0, |B|))^{1+k}} \left( \int_{3Q} |u|^2 \right)^{1/2}. \]  

(4.13)

Proof. We will use the results obtained in the absence of electric potential \( V \). For \( f \in C_0^\infty(\mathbb{R}^n) \),

\[ \| m(\cdot, |B|) f \|_2 \leq C \| H(a, 0)^{1/2} f \|_2 \leq C \| Lf \|_2. \]  

(4.14)

Consider \( \phi \) a smooth non-negative function, bounded by 1, equal to 1 on \( Q \) with support in \( \frac{3}{7} Q \) and whose gradient is bounded by \( C/R \).
We apply inequality (4.14) to $u\phi$ and we obtain
\[
\int_{\mathbb{R}^n} |m(\cdot, |B|)u\phi|^2 \leq C \int_{\mathbb{R}^n} |L(u\phi)|^2.
\]

This gives
\[
\int_{Q} |m(\cdot, |B|)u|^2 \leq C \int_{\frac{3}{2}Q} |\phi Lu|^2 + \int_{\frac{3}{2}Q} |u \nabla \phi|^2,
\]
\[
\int_{Q} |m(\cdot, |B|)u|^2 \leq C \int_{\frac{3}{2}Q} |Lu|^2 + C \int_{\frac{3}{2}Q} |u|^2 + C \int_{\frac{2}{2Q}} |u|^2,
\]
where we used Caccioppoli type inequality. Now, Lemma 2.2 yields
\[
\int_{Q} |m(\cdot, |B|)u|^2 \leq C \int_{\frac{3}{2}Q} |Lu|^2 + C \int_{\frac{3}{2}Q} |u|^2 \leq C \int_{\frac{2}{2Q}} |u|^2,
\]
then
\[
|u(x_0)| \leq C \left( \int_{Q} |u|^2 \right)^{1/2} \leq \frac{C_k}{\{1 + Rm(x_0, |B|)\}^{k/(k_0 + 1)}} \left( \int_{\frac{3}{2}Q} |u|^2 \right)^{1/2}. \quad \square
\]

**Proposition 4.18.** Under assumptions of Theorem 4.15, let $u$ be a weak solution of $H(a, V)u = 0$ in $4Q$, for all $s > 2$, there exists a constant $C > 0$ such that
\[
\left( \int_{Q} m(\cdot, |B|)^{2}|u|^{s} \right)^{1/s} \leq C \left( \int_{\frac{3}{2}Q} m(\cdot, |B|)^{2}|u|^{2} \right)^{1/2}. \tag{4.15}
\]

The proof is similar to that of Proposition 3.5.

**Proof of Theorem 4.15.** We have
\[
m(\cdot, |B|)^2 H(a, V)^{-1} = m(\cdot, |B|)^2 H(a, 0)^{-1} H(a, 0) H(a, V)^{-1}.
\]

It follows by Theorem 1.8 that $H(a, 0) H(a, V)^{-1}$ is $L^{p}$ bounded for $1 \leq p \leq q + \epsilon$. We know also that $m(\cdot, |B|)^2 H(a, 0)^{-1}$ is $L^{p}$ bounded for $1 < p < \infty$. Hence $m(\cdot, |B|)^2 H(a, V)^{-1}$ is bounded on $L^{p} (\mathbb{R}^n)$ for all $1 < p < q + \epsilon$. In particular it is $L^{2}$ bounded. Then we apply Theorem 3.10 to study the behaviour of this operator on $L^{p} (\mathbb{R}^n)$. Fix a cube $Q$ and let $f \in C_0^\infty (\mathbb{R}^n, C)$ compact support contained in $\mathbb{R}^n \setminus 4Q$. Then $u = H(a, V)^{-1} f$ is well defined on $\mathbb{R}^n$. Due to the
support conditions on $f$, $u$ is a weak solution of $H(a, V)u = 0$ on $4Q$. It follows by Proposition 4.18 that, for all $s > 2$, there is a constant $C$, independent of $Q$ and $F$, such that
\[
\left( \int_Q |m(\cdot, |B|)^2 H(a, V)^{-1} f|^s \right)^{1/s} \leq C \left( \int_{3Q} |m(\cdot, |B|)^2 H(a, V)^{-1} f|^2 \right)^{1/2}.
\] (4.16)

Then (3.28) holds with $T = m(\cdot, |B|)^2 H(a, V)^{-1}$, $q_0 = s$, $p_0 = 2$ and $T$ is $L^p$ bounded by Theorem 3.10. □

Remark 4.19. Note that we can prove Corollary 4.16 by a proof analogous to that of Theorem 4.15. In fact, under hypotheses of Corollary 4.16, if $u$ is a weak solution of $H(a, V)u = 0$ in the cube $4Q$ centered at $x_0 \in \mathbb{R}^n$ of sidelength $4R$. Then, for all $s > 2$, there exists a constant $C > 0$ such that
\[
\left( \int_Q |m(\cdot, |B|) u|^s \right)^{1/s} \leq C \left( \int_{3Q} |m(\cdot, |B|)^2 u|^2 \right)^{1/2}.
\] (4.17)

Proof of Theorem 1.4. We know that for $p \leq 2$ and without conditions on $V$ operators $LH(a, V)^{-1/2}$ and $V^{1/2} H(a, V)^{-1/2}$ are $L^p$ bounded. We would therefore limit ourselves to cases where $p > 2$.

The following lemma allows the reduction of the problem.

Lemma 4.20. Under the assumptions of Theorem 1.4, $LH(a, V)^{-1/2}$ is $L^p$ bounded if and only if $LH(a, V)^{-1} L^*$ and $LH(a, V)^{-1} V^{1/2}$ are $L^p$ bounded.

The proof of this lemma is similar to that of Lemma 3.13.

We also use the following results:

Proposition 4.21. Assume $V \in RH_q$ with $1 < q \leq \infty$, then there is an $\epsilon > 0$ such that for all $p$ with $2 < p < 2(q + \epsilon)$, there exists a constant $C_p$ depending on $V$, such that $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ and $F \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$,
\[
\| V^{1/2} H(a, V)^{-1} V^{1/2} f \|_p \leq C_p \| f \|_p, \quad \| V^{1/2} H(a, V)^{-1} L^* F \|_p \leq C_p \| F \|_p.
\]

Proof. Fix a cube $Q$ in $\mathbb{R}^n$ and let $f \in C_0^\infty(\mathbb{R}^n)$ supported away from $4Q$. Then $u = H(a, V)^{-1} V^{1/2} f$ is well defined on $\mathbb{R}^n$ with $\| V^{1/2} u \|_2 + \| Lu \|_2 \leq \| f \|_2$, by construction of $H(a, V)$ and
\[
\int_{\mathbb{R}^n} Vu \varphi + \nabla u \cdot \nabla \varphi = \int_{\mathbb{R}^n} V^{1/2} f \varphi
\]
for all $\varphi \in L^2$ with $\| V^{1/2} \varphi \|_2 + \| \nabla \varphi \|_2 < \infty$. In particular, the support condition on $f$ implies that $u$ is a weak solution of $H(a, V)u = 0$ in $4Q$, hence $|u|^2$ is subharmonic on $4Q$. Consider $r$
such that $V \in RH_r$ and note that $V^{1/2} \in RH_{2r}$. By Lemma 4.2 with $\omega = V^{1/2}$, $f = |u|^2$ and $s = 1/2$, we have

\[
\left( \frac{\int_Q (V^{1/2}|u|)^{2r}}{\mu_Q} \right)^{1/2r} \leq C \int_{\mu_Q} (V^{1/2}|u|).
\]

Hence (3.28) holds with $T = V^{1/2}H(a, V)^{-1}V^{1/2}$, $q_0 = 2r$, $p_0 = 2$ and $S = 0$. By Theorem 3.10, $V^{1/2}H^{-1}V^{1/2}$ is then $L^p$ bounded for $2 < p < 2r$.

We use the same argument to obtain that $V^{1/2}H(a, V)^{-1}L^*$ is $L^p$ bounded for $2 < p < 2r$. □

To prove Theorem 1.4, it suffices to prove the following result:

**Proposition 4.22.** Assume $V \in RH_q$ with $q > 1$. If $2 < p < q^* + \epsilon$ for an $\epsilon > 0$ which depends on the $RH_q$ constant of $V$, then for all $f \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ and $F \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$,

\[
\| LH(a, V)^{-1}V^{1/2}f \|_p \leq C_p \| f \|_p, \quad \| LH(a, V)^{-1}L^*F \|_p \leq C_p \| F \|_p.
\]

**Proof.** Assume $q < n/2$. Fix a cube $Q$ and let $F \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)$ supported away from $4Q$. Set $H = H(a, V)$, $u = H^{-1}L^*F$ is well defined on $\mathbb{R}^n$. As before, the support condition on $F$, implies that $u$ is a weak solution of $Hu = 0$ on $4Q$. Proposition 4.4 implies for all $p \leq q^*$

\[
\left( \int_Q |LH^{-1}L^*F|^p \right)^{1/p} \leq C \left( \int_{3Q} |LH^{-1}L^*F|^2 + |m(\cdot, |B|)H^{-1}L^*F|^2 + |V^{1/2}H^{-1}L^*F|^2 \right)^{1/2}.
\]  

Then (3.28) holds with

\[
T = LH^{-1}L^*, \quad q_0 = q^*, \quad p_0 = 2 \quad \text{and} \quad SF = (M(m(\cdot, |B|)H^{-1}L^*F + V^{1/2}H^{-1}L^*F)^2)^{1/2},
\]

where $M$ is the maximal Hardy–Littlewood operator. Since $S$ is $L^p$ bounded for all $1 < p < 2q$ and $q^* \leq 2q$, then $T$ is bounded on $L^p(\mathbb{R}^n, \mathbb{C}^n)$, $p < q^*$. By the self-improvement of reverse Hölder estimates we can replace $q$ by a slightly larger value and, therefore, $L^p$ boundedness for $p < q^* + \epsilon$ holds.4

Assume next that $n/2 \leq q < n$, then $q^* \geq 2q$. We follow the same argument used for $p < n/2$, and we obtain first that $LH^{-1}L^*$ is $L^p$ bounded for $q \leq 2q$.

We can improve this result by Proposition 4.5: in fact, inequality (3.28) holds with $T = LH^{-1}L^*$, $q_0 = q^*$, $p_0 = 2q$ and $S = M(|m(\cdot, |B|)H^{-1}L^*|^2)^{1/2}$. Since $S$ is $L^p$ bounded for all

4 Thanks to Theorem 4.13, we can improve the range of $p$: $1 < p < 2q + \epsilon$.  

\(1 < p < \infty\) then \(T\) is bounded on \(L^p(\mathbb{R}^n, C^n)\), \(p < q^*\). Again, by self-improvement of the RH\(_q\) condition, it holds for \(p < q^* + \varepsilon\).

Finally, if \(q \geq n\), then \(LH^{-1}V^{1/2}\) is \(L^p\) bounded for \(2 < p < \infty\). And this ends the proof. \(\Box\)

References