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LINEAR ALGEBRA AND ITS APPLICATIONS

Some comparison theorems for weak nonnegative splittings of bounded operators

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Abstract

The comparison of the asymptotic rates of convergence of two iteration matrices induced by two splittings of the same matrix has arisen in the works of many authors. In this paper we derive new comparison theorems for weak nonnegative splittings and weak splittings of bounded operators in a general Banach space and rather general cones, and in a Hilbert space, which extend some of the results obtained by Woźnicki (Japan J. Indust. Appl. Math. 11(1994) 289-342) and Marek and Szyld (Numer. Math. 44(1984) 23-35). Furthermore, we present new theorems also for bounded operator which extend some results by Csordas and Varga (Numer. Math. 44. (1984) 23-35) for weak nonnegative splittings of matrices. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

Consider the linear system

$$Ax = \boldsymbol{b}. \tag{1}$$

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For the iterative solution of system (1) it is customary to represent the operator A as

$$A = M - N.$$

If M is a nonsingular operator, the iterative method is expressed in the form

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k \geqslant 0. \tag{2}$$

As is well known, the above iterative scheme converges to the unique solution $x = A^{-1}b$ of system (1) for each initial vector $x^{(0)}$, if and only if $\rho(M^{-1}N) < 1$, where $\rho(M^{-1}N)$ is the spectral radius of the *iteration operator* $M^{-1}N$. Also, the asymptotic rate of convergence of the iterative scheme (2) depends on $\rho(M^{-1}N)$, see, for example, [3,17] or [20] for matrices and [4] for bounded operators. Therefore, the spectral radius of the iteration operator plays an important role in the comparison of the speed of convergence of different iterative methods.

The theory of nonnegative splittings, apart from being a useful tool in convergence analysis for the iterative scheme (2), also provides us with some extremely interesting comparison results. They also appear as a generalization of well known comparison theorems introduced by Varga [17] for regular splittings, as well as the lesser known ones introduced by Woźnicki in his unpublished dissertation (1973), although these results are being cited by Csordas and Varga [5].

In recent years, comparison results for splittings of matrices have been studied by different authors such as Csordas and Varga [5], Elsner [6], Mangasarian [9], Miller and Neumann [11], Song [15,16] and Woźnicki [18], among others. On the other hand, Marek and Szyld [10] extend the classical comparison theorems for regular splittings of Varga and Woźnicki to general Banach spaces and rather general cones. Our contribution to the problem extends their approach also for general Banach spaces and rather general cones, using the classification introduced by Woźnicki [18] for nonnegative splittings of matrices as well as the correction and extension for bounded operators of the results in which Woźnicki ([18], Theorems 3.13 and 3.14) introduces new conditions, which have been proved to be false (see Theorems 8 and 10 and Examples 3 and 4).

The outline of this paper is as follows. In Section 2 we present some definitions and technical results that are necessary in the next sections. In Section 3 we give some comparison theorems for weak nonnegative splittings (see Definition 1) of the first or the second type. We also point out the existence of an incorrect condition elsewhere in Woźnicki [18] (Section 3.1), and deduce some comparison results for weak splittings (Section 3.2). Finally, in Section 4 we establish some of the relations existing between the conditions studied in Section 3.

Forthcoming we will present some results that extend the results of Song [16] for weak splittings of the second type, and we will pose some generalizations from the results provided by Miller and Neumann [11].

2. Definitions and theoretical background

Throughout the paper $\mathbb{E}$ denotes a real Banach space, $\mathbb{E}'$ its dual and $\mathscr{B}(\mathbb{E})$ the space of all bounded linear operators mapping $\mathbb{E}$ into itself. We do not specify the norms of these spaces, writting $\|\cdot\|$ in each case. If the norm is defined by $\|\boldsymbol{x}\| = (\langle x, x\rangle)^{1/2}$ where $\langle\cdot,\cdot\rangle$ is an inner product, then $\mathbb{E}$ is called a Hilbert space (see Dunford and Schwartz [21]).

We assume that $\mathbb{E}$ is generated by a normal cone K; that is, using the notation of Marek and Szyld [10], $\mathbb{E} = K - K$ where K has the following properties:
(1) $K + K = K$,
(2) $\alpha K \subset K$ for $\alpha \geqslant 0$,
(3) $K \cap (-K) = \{\mathbf{0}\}$,
(4) $\overline{K} = K$ where $\overline{K}$ denotes the norm-clausure of K,
(5) for $\boldsymbol{x}, \boldsymbol{y} \in K$ there exists $\tau > 0$ such that $\|\boldsymbol{x} + \boldsymbol{y}\| \geqslant \tau\|\boldsymbol{x}\|$.

Let $K' = \{y' \in \mathbb{E}' : y'(x) \geqslant 0 \text{ for all } x \in K\}$, it is easy to see that K' is a normal cone generating $\mathbb{E}'$.

If T is an element of $\mathscr{B}(\mathbb{E})$ the adjoint element of T, denoted by T', is an element of $\mathscr{B}(\mathbb{E}')$ such that $(T'y')(x) = y'(T(x)) = (y'T)(x)$ where x in $\mathbb{E}$ and y' in $\mathbb{E}'$; that is $T'y' = y'T$. Also, an element T of $\mathscr{B}(\mathbb{E})$ is nonsingular if there exists an unique element T^{-1} of $\mathscr{B}(\mathbb{E})$, called the inverse of T, such that $TT^{-1} = T^{-1}T = I$, where I dentoes the identity operator. If in addition $\mathbb{E}$ is a Hilbert space, then $T' \in \mathscr{B}(\mathbb{E})$ for all $T \in \mathscr{B}(\mathbb{E})$ (see Dunford and Schwartz [21]).

Let $\overset{o}{K}$ be the interior of K. An operator T in $\mathscr{B}(\mathbb{E})$ is called K-nonnegative (respectively, K-positive) iif $TK \subseteq K$ (respectively, $T(K\backslash\{0\}) \subseteq \overset{o}{K}$) and we dentoe it by $T \geqslant 0$ (respectively, $T > 0$). Similarly, for T and S in $\mathscr{B}(\mathbb{E})$, we denote $T - S \geqslant 0$ (respectively, $T - S > 0$) by $T \geqslant S$ (respectively, $T > S$). A vector x in $\mathbb{E}$ is called K-nonnegative (respectively, K-positive) if x in K (respectively, x in $\overset{o}{K}$). We denote it by $x \geqslant 0$ (respectively, $x > 0$). Furthermore, if $T \geqslant 0$ then $T' \geqslant 0$ and conversely, if $T' \geqslant 0$ then $T \geqslant 0$ (see Kreĭn and Rutman [7]). Marek and Szyld [10] use the concept of d-interior

$$K^d = \{x \in K : y'(x) > 0 \text{ for all } y' \in K', y' \neq 0\}$$

as a replacement of $\overset{o}{K}$. However, if $\overset{o}{K} \neq \emptyset$, then $\overset{o}{K} = K^d$ (see Kreĭn and Rutman [7]).

In this paper we will say that the operator T is K-irreducible, for simplicity irreducible, if for each nonzero vecotr $x \geqslant 0$, there exists a natural number $m = m(x)$ such that $T^m x > 0$. We say that the operator T in $\mathscr{B}(\mathbb{E})$ has the property of "d" if there exist $x \in K$ and $y' \in K'$ such that $Tx = \rho(T)$ and $T'y' = \rho(T)y'$. This is a a little modification of the same concept introduced by Marek and Syzld [10].

When we consider the particular case $\mathbb{E} = \mathbb{R}^n$, then $\mathscr{B}(\mathbb{E})$ is the space of all $n \times n$ real matrices. Moreover, T' is the transpose matrix of T, K^{d} coincides with the interior of K and all operator have the property "d". If in addition

$K = \mathbb{R}^n_+$, that is, the set of all vectors with nonnegative entries, then $T \geqslant 0$ (respectively, $T > 0$) denotes a matrix with nonnegative (respectively, positive) entries. This particular case will be considered in all examples of this paper. For this case, we can consider inequalities like that $A \leqslant B^T$ where B^T is the transpose matrix B. However, if A and B are operators, we cannnot consider the inequality $A \leqslant B'$ unless $\mathbb{E}$ is a Hilbert adjoint operators, we will assume the $\mathbb{E}$ is a Hilbert space, in the other cases $\mathbb{E}$ will be a Banach space.

Finally in all that follows we will use the following results without any explicit reference to them (see for example [7,4]): if $A, B \in \mathscr{B}(\mathbb{E})$ then $(AB)' = B'A'$, $\rho(AB) = \rho(BA)$, $\rho(A') = \rho(A)$ and if $A > 0$ then A is irreducible. This results for finite dimensional case are well known (see for example [17,20]).

Definition 1. Let A be a bounded operator. The representation

$$A = M - N$$

is called a *splitting* of A if M is a nonsingular operator.

In addition, the splitting is

- *convergent* if the iteration operator $M^{-1}N$ is convergent; that is, if $\rho(M^{-1}N) < 1$,
- *regular* if $M^{-1} \geqslant 0$, and $N \geqslant 0$;
- *nonnegative* if $M^{-1} \geqslant 0$, $M^{-1}N \geqslant 0$, and $NM^{-1} \geqslant 0$;
- *weak nonnegative of the first type* if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$, *weak nonnegative of the second type* if $M^{-1} \geqslant 0$ and $NM^{-1} \geqslant 0$;
- *weak of the first type* if $M^{-1}N \geqslant 0$, *weak of the second type* if $NM^{-1} \geqslant 0$.

The different types of splittings introduced in Definition 1 have been defined by Woźnicki [18] for matrices. However, not all the authors use the same classification. The definition of nonnegative splitting given is the same as the definition of weak regular splitting by Ortega and Rheinboldt [14] although other authors, such as Amedjoe [1], Beauwens [2], Berman and Plemmons [3], Elsner [6], Neumann and Plemmons [12], and O'Leary and White [13] consider weak regular splittings as weak nonnegative splittings of the first type and nonnegative splittings as weak splittings of the first type. Marek and Szyld [10] also use weak regular splittings as weak nonnegative splittings of the first type, and weak splittings as weak splittings of the first type.

The necessity to distinguish between splittings of the first type and the second type is motivated by the fact that there exist convergent splittings that are of the second type but not of the first type, and therefore, using the known results for splittings of the first type we cannot ensure the convergence of the splitting. In Example 2 we give two weak nonnegative splittings of the second type that are not weak nonnegative splittings of the first type, ensuring now the convergence of both splittings by Remark 1.

In this paper we will use only weak nonnegative splittings and weak splittings both of the first and second type. As a consequence of Theorem 1, all the results of the paper hold for regular and nonnegative splittings.

Theorem 1. *Let A be a nonsingular operator in a Banach space. Any regular splitting of A is a nonnegative splitting of A. Any nonnegative splitting of A is a weak nonnegative splitting of the first and second type of A. Any weak nonnegative splitting of the first (respectively, second) type of A is a weak splitting of the first (respectively, second) type of A. The converses are not true.*

This theorem is a generalization of Corollaries 3.1 and 6.1 of Woźnicki [18] for a general real Banach space.

In the following lemma we present some results of Marek and Szyld [10], that we will use frequently in all that follows.

Lemma 1. (i) (Corollary 3.2 of [10]) *Let $T \geqslant 0$, and let $\boldsymbol{x} \geqslant 0$ be such that $T\boldsymbol{x} - \alpha\boldsymbol{x} \geqslant 0$. Then $\alpha \leqslant \rho(T)$. Moreover, if T has property* "d" *and $T\boldsymbol{x} - \alpha\boldsymbol{x} > 0$, then $\alpha < \rho(T)$.*

(ii) (Lemma 3.3 of [10]) *Let $T \geqslant 0$ having property* "d", *and let $\boldsymbol{x} > 0$ be such that $\alpha\boldsymbol{x} - T\boldsymbol{x} \geqslant 0$. Then $\rho(T) \leqslant \alpha$. Moreover, if $\alpha\boldsymbol{x} - T\boldsymbol{x} > 0$ then $\rho(T) < \alpha$.*

The convergence theorems that we will see in this section, within the theory of nonnegative splittings, include some results of Berman and Plemmons ([3], Theorem 7.5.2), Song ([15], Lemmas 2.1 and 2.2), and Woźnicki ([18], Theorems 3.2 and 6.1) as particular cases. As in the case of all the results in the theory of nonnegative splittings, these results can be seen as a generalization of the convergence theorems for regular splittings of matrices introduced by Varga [17].

Theorem 2. *Let A be a nonsingular operator in a Banach space and let $A = M - N$ be a weak splitting of the first type with $M^{-1}N$ and $A^{-1}N$ having property* "d". *The following conditions are equivalent:*

(i) $A^{-1}M \geqslant 0$.

(ii) $\rho(M^{-1}N) = (\rho(A^{-1}M) - 1)/\rho(A^{-1}M)$.

(iii) $\rho(M^{-1}N) = \rho(NM^{-1}) < 1$.

(iv) $(I - M^{-1}N)^{-1} \geqslant 0$.

(v) $A^{-1}N \geqslant 0$.

(vi) $A^{-1}N \geqslant M^{-1}N$.

(vii) $\rho(M^{-1}N) = \rho(A^{-1}N)/(1 + \rho(A^{-1}N))$.

Proof. (i) $\rightarrow$ (ii): Since $A^{-1}M \geqslant 0$, for the eigenvalue $\rho(A^{-1}M) \neq 0$ there exists an eigenvector $\boldsymbol{x} \geqslant 0$ (see [7]) such that

$$A^{-1}M\boldsymbol{x} = \rho(A^{-1}M)\boldsymbol{x}.$$

Now from $A = M - N$ we have that

$$M^{-1}N = (A^{-1}M)^{-1}(A^{-1}M - I) \tag{3}$$

and therefore

$$M^{-1}N\boldsymbol{x} = \frac{\rho(A^{-1}M) - 1}{\rho(A^{-1}M)}\boldsymbol{x},$$

and by part (i) of Lemma 1,

$$\frac{\rho(A^{-1}M) - 1}{\rho(A^{-1}M)} \leqslant \rho(M^{-1}N). \tag{4}$$

On the other hand, since $M^{-1}N \geqslant 0$, for the eigenvalue $\rho(M^{-1}N)$ there exists an eigenvector $\boldsymbol{y} \geqslant 0$ such that

$$M^{-1}N\boldsymbol{y} = \rho(M^{-1}N)\boldsymbol{y}.$$

Now, by Eq. (3) we have that

$$\rho(M^{-1}N)\boldsymbol{y} = (A^{-1}M)^{-1}(A^{-1}M - I)\boldsymbol{y}$$

and then by part (i) of Lemma 1,

$$\rho(M^{-1}N) \leqslant \frac{\rho(A^{-1}M) - 1}{\rho(A^{-1}M)}. \tag{5}$$

Hence from inequalities (4) and (5) we obtain (ii).

(ii) $\rightarrow$ (iii): Obvious.

(iii) $\rightarrow$ (iv): Since $M^{-1}N \geqslant 0$, if $\rho(M^{-1}N) < 1$, by Problem 12.M of Brown and Pearcy [4] we have that

$$(I - M^{-1}N)^{-1} = I + M^{-1}N + (M^{-1}N)^2 + \cdots \geqslant 0.$$

(iv) $\rightarrow$ (v): From $A = M - N$ we have that

$$A^{-1}N = (M - N)^{-1}MM^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N \geqslant 0$$

because $(I - M^{-1}N)^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$.

(v) $\leftrightarrow$ (vi): From $A = M - N$ we have that

$$A^{-1}N = M^{-1}N + A^{-1}NM^{-1}N,$$

hence

$$A^{-1}N - M^{-1}N = A^{-1}NM^{-1}N \geqslant 0,$$

because $A^{-1}N \geqslant 0$ and $M^{-1}N \geqslant 0$.

The converse is trivial.

(v) → (vii): Similar to (i) → (ii) using $M^{-1}N = (I + A^{-1}N)^{-1}A^{-1}N$ instead of Eq. (3).

(v) → (i): From $M = A + N$ we have that

$$A^{-1}M = I + A^{-1}N \geqslant 0,$$

because it is a sum of nonnegative operators.

(vii) → (iii): Obvious. □

Remark 1. The above theorem also holds if we replace "first type" by "second type" and operators $A^{-1}M$, $M^{-1}N$ and $A^{-1}N$ by MA^{-1}, NM^{-1} and NA^{-1}, respectively.

Mangasarian [9], and Berman and Plemmons [3] establish a sufficient condition for a splitting of a matrix to be convergent. As a consequence of Theorem 2 and Remark 1, we establish sufficient conditions (see Corollary 1) for a splitting to be convergent, obtaining the result of Mangasarian ([9], Theorem 2), and Berman and Plemmons ([3], Corollary 7.5.4) as a particular case. Previous to this discussion, we will give a technical lemma in which a series of equivalences appear, enabling us to simplify the proof of this result.

Lemma 2. *Let A be a nonsingular operator in a Banach space and let $A = M - N$ be a splitting.*

(i) The following conditions are equivalent

(1) $A'\boldsymbol{y}' \geqslant 0$ implies $N'\boldsymbol{y}' \geqslant 0$.

(2) $A^{-1}N \geqslant 0$.

(ii) The following conditions are equivalent

(1) $A\boldsymbol{y} \geqslant 0$ implies $N\boldsymbol{y} \geqslant 0$.

(2) $NA^{-1} \geqslant 0$.

Proof. (i) Let $\boldsymbol{x}' = A'\boldsymbol{y}'$, then $(A')^{-1}\boldsymbol{x}' = \boldsymbol{y}'$, so we can write condition (i) (1) as

$$\boldsymbol{x}' \geqslant 0 \quad \text{implies} \quad N'(A')^{-1}\boldsymbol{x}' \geqslant 0.$$

So $(A^{-1}N)' = N'(A')^{-1} \geqslant 0$ and then, $A^{-1}N \geqslant 0$.

Conversely, if $A^{-1}N \geqslant 0$ then $N'(A')^{-1} = (A^{-1}N)' \geqslant 0$. Furthermore, if $\boldsymbol{x}' = A'\boldsymbol{y}' \geqslant 0$, we have that

$$N'\boldsymbol{y}' = N'(A')^{-1}\boldsymbol{x}' \geqslant 0.$$

(ii) Let $\boldsymbol{x} = A\boldsymbol{y}$, then $A^{-1}\boldsymbol{x} = \boldsymbol{y}$, so we can write condition (ii) (1) as

$$\boldsymbol{x} \geqslant 0 \quad \text{implies} \quad NA^{-1}\boldsymbol{x} \geqslant 0.$$

Then, $NA^{-1} \geqslant 0$.

Conversely, if $NA^{-1} \geqslant 0$ and $\boldsymbol{x} = A\boldsymbol{y} \geqslant 0$, we have that

$$Ny = NA^{-1}x \geqslant 0. \qquad \square$$

Remark 2. The above lemma holds if we replace N by M in parts (i) and (ii). Also holds if we replace A by M.

Now, as an immediate consequence of Lemma 2, Theorem 2, and Remarks 1 and 2 we obtain the following result.

Corollary 1. *Let A be a nonsingular operator in a Banach space and let $A = M - N$ be a splitting.*

(i) *Assume that $M^{-1}N$ and $A^{-1}N$ have property* "d". *If some of the following conditions hold:*

1. *$A'y' \geqslant 0$ implies $N'y' \geqslant 0$, and $M'y' \geqslant 0$ implies $N'y' \geqslant 0$* (*see [10], Theorem 2 or [3], Corollary 7.5.4 for finite dimensional case*),
2. *$A'y' \geqslant 0$ implies $M'y' \geqslant 0$, and $M'y' \geqslant 0$ implies $N'y' \geqslant 0$,* then $\rho(M^{-1}N) < 1$.

(ii) Assume that NM^{-1} and NA^{-1} have property "d". *If some of the following conditions hold:*

1. *$Ay \geqslant 0$ implies $Ny \geqslant 0$, and $My \geqslant 0$ implies $Ny \geqslant 0$,*
2. *$Ay \geqslant 0$ implies $My \geqslant 0$, and $My \geqslant 0$ implies $Ny \geqslant 0$, then* $\rho(M^{-1}N) < 1$.

The converses of the previous corollary are not satisfied as is shown in the following example; that is, from $\rho(M^{-1}N) < 1$ none of conditions 1 and 2 of parts (i) and (ii) of Corollary 1 can be deduced.

Example 1. Consider the nonsingular matrix

$$A = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & 1 \\ 0 & 0 & \frac{3}{4} \end{bmatrix},$$

and the splitting $A = M - N$, where

$$M = \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & \frac{3}{4} \end{bmatrix}.$$

Then $\rho(M^{-1}N) \approx \frac{2161}{2713} < 1$. However, none of the conditions (1) and (2) of parts (i) and (ii) of Corollary 1 hold (see Lemma 2 and Remark 2 for the corresponding equivalences in conditions (1) and (2) of parts (i) and (ii) of Corollary 1).

Furthermore, we present results that are analogous to Theorem 2 and Remark 1 for the case where the splitting is weak nonnegative of the first type and weak nonnegative of the second type, respectively.

Theorem 3. *Let A be a nonsingular operator in a Banach space and let $A = M - N$ be a weak nonnegative splitting of the first type, with the operators $M^{-1}N$ and $A^{-1}N$ having property* "d". *The following conditions are equivalent:*

(i) $A^{-1} \geqslant 0$.
(ii) $A^{-1} \geqslant M^{-1}$.
(iii) $A^{-1}M \geqslant 0$.
(iv) $\rho(M^{-1}N) = (\rho(A^{-1}M) - 1)/\rho(A^{-1}M)$.
(v) $\rho(M^{-1}N) = \rho(NM^{-1}) < 1$.
(vi) $(I - M^{-1}N)^{-1} \geqslant 0$.
(vii) $A^{-1}N \geqslant 0$.
(viii) $A^{-1}N \geqslant M^{-1}N$.
(ix) $\rho(M^{-1}N) = \rho(A^{-1}N)/(1 + \rho(A^{-1}N))$.

Proof. By Theorems 1 and 2, conditions (iii)–(ix) are equivalent. Therefore, we only need to prove, for example, that (i) $\leftrightarrow$ (ii), (i) $\rightarrow$ (v), and that (vi) $\rightarrow$ (i).

(i) $\leftrightarrow$ (ii): From $M = A + N$ it follows that

$$M^{-1} = A^{-1}(I + NA^{-1})^{-1}.$$

Multiplying both sides of the above equality on the right by $I + NA^{-1}$, we obtain that

$$A^{-1} - M^{-1} = M^{-1}NA^{-1} \geqslant 0$$

because $M^{-1}N \geqslant 0$ and $A^{-1} \geqslant 0$.

The converse is trivial because $M^{-1} \geqslant 0$.

(i) $\rightarrow$ (v): Let $T = M^{-1}N$, for $m = 0, 1, 2, \ldots$ we have that

$$I - T^{m+1} = \sum_{j=0}^{m} T^j(I - T).$$

Since $I - T = M^{-1}A$, it follows that

$$\sum_{j=0}^{m} T^j M^{-1} = (I - T^{m+1})A^{-1} \leqslant A^{-1}.$$

For a vector $\boldsymbol{x} > 0$ it follows that $M^{-1}\boldsymbol{x} > 0$ and $A^{-1}\boldsymbol{x} > 0$.

Let $\boldsymbol{x}_0' \geqslant 0$ such that $\boldsymbol{x}_0'T = T'\boldsymbol{x}_0' = \rho(T)\boldsymbol{x}_0'$. Then

$$\boldsymbol{x}_0'A^{-1}\boldsymbol{x} \geqslant \boldsymbol{x}_0'\sum_{j=0}^{m} T^j M^{-1}\boldsymbol{x} = \sum_{j=0}^{m} (\rho(T))^j \boldsymbol{x}_0' M^{-1}\boldsymbol{x}.$$

Assume that $\rho(T) \geqslant 1$, then

$$\frac{x'A^{-1}x_0}{x'M^{-1}x_0} \geqslant \begin{cases} m+1 & \text{for } \rho(T)=1, \\ \frac{(\rho(T))^{m+1}-1}{\rho(T)-1} & \text{for } \rho(T)>1, \end{cases}$$

which is a contradiction, then $\rho(T) < 1$.

(vi) $\to$ (i): From $A = M - N$ it follows that

$$A^{-1} = (M-N)^{-1}MM^{-1} = (I - M^{-1}N)^{-1}M^{-1} \geqslant 0,$$

since $(I - M^{-1}N)^{-1} \geqslant 0$ and $M^{-1} \geqslant 0$. $\square$

Remark 3. The above theorem also holds if we replace "first type" by "second type" and operators $A^{-1}M$, $M^{-1}N$ and $A^{-1}N$ by MA^{-1}, NM^{-1} and NA^{-1}, respectively.

3. Comparison theorems

Evidently, when solving the system (1) using the iterative scheme (2) it must be ensured that the splitting under analysis is convergent. However, once this convergence is ensured, it is extremely important that it be obtained with the desired precision with the less number of iterations; that is, with the greatest possible speed. Therefore, our aim in this section will be to compare the speed of convergence of two different (and evidently convergent) splittings of the same operator A. In order to do this, it will be necessary to compare the spectral radii of the respective iteration operators.

The next lemma (whose proof is immediate) together with its remark will allow us, by some of the comparison theorems which we will see later, to establish (as an immediate consequence) some interesting results.

Lemma 3. *Let A be a nonsingular operator in a Banach space and let $A = M_1 - N_1 = M_2 - N_2$ be two splittings. Then*

(i) $N_1 \leqslant N_2$ if and only if $M_1 \leqslant M_2$.

(ii) $N_1 < N_2$ if and only if $M_1 < M_2$.

(iii) $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2A^{-1}$ if and only if $A^{-1}M_1A^{-1} \leqslant A^{-1}M_2A^{-1}$.

(iv) $A^{-1}N_1A^{-1} < A^{-1}N_2A^{-1}$ if and only if $A^{-1}M_1A^{-1} < A^{-1}M_2A^{-1}$.

Remark 4. The above lemma also holds if we replace "Banach space" by "Hilbert space", M_2 by M_2', and N_2 by N_2' in parts (i)–(iv), and in addition we assume that $A' = A$.

This section is further divided into two subsections. In the first subsection, the comparison theorems for weak nonnegative splittings, and in the second, the comparison theorems for weak splittings, are presented.

3.1. Weak nonnegative splittings

Firstly, let us go over one of the first comparison results introduced by Varga [17] for regular splittings of matrices.

Theorem 4 (Theorem 3.15 of [17]). *Let A be a nonsingular matrix with $A^{-1} > 0$ and let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A. If*

$$N_1 \leqslant N_2, \quad N_1 \neq N_2,$$

then

$$\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2) < 1.$$

If instead of considering regular splittings we consider weak nonnegative splittings of the same type, the previous result no longer holds as we show in the following example.

Example 2. Let

$$A = \begin{bmatrix} \frac{3}{2} & -1 \\ -1 & \frac{3}{2} \end{bmatrix},$$

it follows then that $A^{-1} > 0$. Consider the splittings $A = M_1 - N_1 = M_2 - N_2$ where

$$N_1 = \begin{bmatrix} \frac{21}{2} & -7 \\ -9 & \frac{27}{2} \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} 11 & -7 \\ -9 & \frac{27}{2} \end{bmatrix}.$$

Both splittings are weak nonnegative of the second type, but not of the first type. Furthermore, $N_1 \leqslant N_2$ and $N_1 \neq N_2$, but

$$\rho(M_1^{-1}N_1) = \frac{9}{10} = \rho(M_2^{-1}N_2).$$

Observe that both splittings are convergent.

However, we can establish the following result for weak nonnegative splittings of the same or different types for bounded operators, similar to Theorem 3.5 of Marek and Szyld [10] for regular and weak nonnegative splittings of the first type, but we require that $N_2 - N_1$ maps the entire cone K into itself; that is $N_1 \leqslant N_2$.

Theorem 5. *Let A be a nonsingular operator in a Banach space with $A^{-1} \geqslant 0$. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of the same or different type. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property* "d" *when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property* "d" *when $A = M_i - N_i$ is of the second type. If*

$$N_1 \leqslant N_2, \tag{6}$$

then

$$\rho(M_1^{-1}N_1) \leqslant \rho(M_2^{-1}N_2) < 1. \tag{7}$$

Moreover, if $A^{-1} > 0$ and

$$N_1 < N_2, \tag{8}$$

then

$$\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2) < 1. \tag{9}$$

Proof. By Theorem 3 and Remark 3, both splittings, either of the same or different types, are convergent.

Suppose that $A = M_1 - N_1$ is of the first type. Since $A^{-1} \geqslant 0$, from inequality (6) and part (vii) of Theorem 3, we have that

$$0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2. \tag{10}$$

For the eigenvalue $\rho(A^{-1}N_1)$ there exists an eigenvector $x \geqslant 0$ such that $A^{-1}N_1\boldsymbol{x} = \rho(A^{-1}N_1)\boldsymbol{x}$, then from inequality (10) we have that

$$A^{-1}N_2\boldsymbol{x} - \rho(A^{-1}N_1)\boldsymbol{x} \geqslant 0,$$

so by part (i) of Lemma 1

$$\rho(A^{-1}N_1) \leqslant \rho(A^{-1}N_2). \tag{11}$$

If $A = M_2 - N_2$ is also of the first type, from (11), part (ix) of Theorem 3 and the monotonicity of the function $\alpha/(\alpha + 1)$, inequality (7) follows. On the other hand, if $A = M_2 - N_2$ is of the second type, then inequality (11) is equivalent to

$$\rho(A^{-1}N_1) \leqslant \rho(N_2A^{-1}). \tag{12}$$

So, from part (ix) of Theorem 3 and from Remark 3 (really, from parts (ix) and (v) of Theorem 3 together with the changes proposed in Remark 3) inequality (7) follows.

Next, if $A = M_1 - N_1$ is of the second type, by a similar argument using

$$0 \leqslant N_1A^{-1} \leqslant N_2A^{-1}$$

instead of inequality (10), inequality (7) follows.

If $A^{-1} > 0$, and $A = M_1 - N_1$ is of the first type, from inequality (8) and part (vii) of Theorem 3 we have that

$$0 \leqslant A^{-1}N_1 < A^{-1}N_2, \tag{13}$$

so $A^{-1}N_2$ is irreducible. Then, see [7], there exists a vector $y > 0$ for the eigenvalue $\rho(A^{-1}N_2) > 0$ such that $A^{-1}N_2 y = \rho(A^{-1}N_2)y$. Therefore, from inequality (13) we have that

$$\rho(A^{-1}N_2)y - A^{-1}N_1 y > 0,$$

and then by part (ii) of Lemma 1 we obtain that

$$\rho(A^{-1}N_1) < \rho(A^{-1}N_2). \tag{14}$$

Now, as in the previous part, using (14) instead of (12), inequality (9) follows. On the other hand, if $A = M_1 - N_1$ is of the second type, by a similar argument using

$$0 \leqslant N_1 A^{-1} < N_2 A^{-1}$$

instead of inequality (13), inequality (9) follows. □

The first part of the previous theorem was established, for weak nonnegative splittings of matrices of the same type by Woźnicki ([18], Theorem 3.4). As an immediate consequence of parts (i) and (ii) of Lemma 3, Theorem 5 also holds if we replace inequalities (6) and (8) by $M_1 \leqslant M_2$ and $M_1 < M_2$, respectively.

The following theorem was introduced by Csordas and Varga [5] for regular splittings of matrices and later Woźnicki [18] proved that they still hold for weak nonnegative splittings of matrices of different types. Now we extend it for bounded operators.

Theorem 6. *Let A be a nonsingular operator in a Banach space with $A^{-1} \geqslant 0$ and let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of different types. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property "d" when $A = M_i - N_i$ is of the first type; however, assume that $N_i M_i^{-1}$ and $N_i A^{-1}$ have the property "d" when $A = M_i - N_i$ is of the second type. If*

$$A^{-1}N_1A^{-1} \leqslant A^{-1}N_2A^{-1}, \tag{15}$$

then inequality (7) holds.

Moreover, if $A^{-1} > 0$ and

$$A^{-1}N_1A^{-1} < A^{-1}N_2A^{-1}, \tag{16}$$

then inequality (9) holds.

Proof. Since $A^{-1} \geqslant 0$, by Theorem 3 and Remark 3 both splittings are convergent. Suppose that $A = M_1 - N_1$ is of the first type, then by Theorem 3 and Remark 3 we have that $A^{-1}N_1 \geqslant 0$ and $N_2A^{-1} \geqslant 0$. Now for the eigenvalue $\rho(A^{-1}N_1)$ there exists an eigenvector $\boldsymbol{x}_1' \geqslant 0$ such that

$$\boldsymbol{x}_1'A^{-1}N_1 = (A^{-1}N_1)'\boldsymbol{x}_1' = \rho(A^{-1}N_1)\boldsymbol{x}_1'. \tag{17}$$

Hence, from inequality (15) we have that

$$\rho(A^{-1}N_1)\boldsymbol{x}_1'A^{-1} \leqslant \boldsymbol{x}_1'A^{-1}N_2A^{-1}.$$

Now considering $\boldsymbol{y}_1' = \boldsymbol{x}_1'A^{-1} \geqslant 0$ we can write the above inequality as

$$\rho(A^{-1}N_1)\boldsymbol{y}_1' \leqslant \boldsymbol{y}_1'N_2A^{-1},$$

so by part (i) of Lemma 1 we have that

$$\rho(A^{-1}N_1) \leqslant \rho(N_2A^{-1}),$$

then using this inequality instead of inequality (12), inequality (7) follows in a similar way.

If $A = M_1 - N_1$ is of the second type, then $N_1A^{-1} \geqslant 0$ and $A^{-1}N_2 \geqslant 0$. Now for the eigenvalue $\rho(A^{-1}N_1)$ there exists an eigenvector $\boldsymbol{x}_1 \geqslant 0$ such that

$$N_1A^{-1}\boldsymbol{x}_1 = \rho(N_1A^{-1})\boldsymbol{x}_1. \tag{18}$$

Hence, from inequality (15) we have that

$$\rho(N_1A^{-1})A^{-1}\boldsymbol{x}_1 \leqslant A^{-1}N_2A^{-1}\boldsymbol{x}_1.$$

Now for $\boldsymbol{y}_1 = A^{-1}\boldsymbol{x}_1 \geqslant 0$ we can write the above inequality as

$$\rho(N_1A^{-1})\boldsymbol{y}_1 \leqslant A^{-1}N_2\boldsymbol{y}_1.$$

Then by part (i) of Lemma 1 we have that

$$\rho(A^{-1}N_1) = \rho(N_1A^{-1}) \leqslant \rho(A^{-1}N_2).$$

Now using this inequality instead of inequality (11), inequality (7) follows in a similar way. If $A^{-1} > 0$, by a similar argument, we obtain inequality (9). □

Remark 5. As an immediate consequence of parts (iii) and (iv) of Lemma 3, if we replace inequalities (15) and (16) by $A^{-1}M_1A^{-1} \leqslant A^{-1}M_2A^{-1}$, and $A^{-1}M_1A^{-1} < A^{-1}M_2A^{-1}$, respectively, then Theorem 9 also holds.

Theorem 4 together with the following result for matrices and regular splittings (which was introduced by Woźnicki in 1973 in his dissertation, published in [18]), are the basic comparison results in the theory of regular splittings from which the comparison results in the theory of nonnegative splittings have been developed. Later, Elsner [6] proved the above theorem, for matrices, for the case where one of the splittings is regular and the other is weak nonnegative of the first type; Marek and Szyld [10] extend the result for bounded operators and Woźnicki [18] proved, for matrices, the same theorem for weak nonnegative splittings of different types. In the case where the splittings are weak nonnegative of the same type he shows that the result does not hold unless the matrix A and at least one of the matrices M_1 and M_2 are symmetric.

Theorem 7. *Let A be a nonsingular operator in a Banach space with $A^{-1} \geqslant 0$ and let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of different type. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property* "d" *when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property* "d" *when $A = M_i - N_i$ is of the second type. If*

$$M_1^{-1} \geqslant M_2^{-1}, \tag{19}$$

then inequality (7) holds.

Moreover, if $A^{-1} > 0$ and

$$M_1^{-1} > M_2^{-1} \tag{20}$$

then inequality (9) holds.

Proof. Suppose that $A = M_1 - N_1$ is of the first type, and that we state inequality (19) as

$$M_1^{-1}AA^{-1} \geqslant A^{-1}AM_2^{-1}. \tag{21}$$

Then, by part (iii) of Theorem 3 and Remark 3,

$$0 \leqslant A^{-1}M_1 \quad \text{and} \quad 0 \leqslant M_2A^{-1}.$$

Multiplying both sides of inequality (19) on the left by $A^{-1}M_1$, and on the right by M_2A^{-1}, we obtain that

$$A^{-1}M_2A^{-1} \geqslant A^{-1}M_1A^{-1}. \tag{22}$$

Now, by Remark 5 inequality (7) follows.

If $A = M_2 - N_2$ is of the second type, using inequality (21) as

$$A^{-1}AM_1^{-1} \geqslant M_2^{-1}AA^{-1}, \tag{23}$$

using part (iii) of Remark 3 (really using part (iii) of Theorem 3 together with the changes proposed in Remark 3) instead of Theorem 3 to obtain

$$0 \leqslant M_1A^{-1} \quad \text{and} \quad 0 \leqslant A^{-1}M_2,$$

and multiplying both sides of inequality (23) on the left by $A^{-1}M_2$ and on the right by M_1A^{-1}, we obtain inequality (22), and again, by Remark 5 inequality (7) follows.

If $A^{-1} > 0$, following the same argument as above, we obtain

$$A^{-1}M_2A^{-1} > A^{-1}M_1A^{-1}.$$

Then, by Remark 5 inequality (9) follows. □

One of the aims of generalizing the comparison results for regular splittings, is to extend the class of matrices and iterative methods based on the iterative scheme (2) to which these results can be applied. Csordas and Varga [5] generalized Theorems 4 and 7 in this way, contributing with new results for regular splittings under weaker conditions. Miller and Neumann [11] prove a theorem which generalizes some of the results of Csordas and Varga [5] and those of Beauwens [2] for weak splittings of the first type, and Song [15,16] introduces a series of results also for weak splittings of the first type, which generalize all the results of the above-mentioned authors. From a theoretical point of view, it is clear that by weakening the conditions, a greater class of iterative methods to which these results are applied is obtained. However, from a practical point of view, they are less useful because they are very difficult to check. Woźnicki [18] attempts to introduce a new type of conditions (using the transpose matrix into one of the matrices of ones of the splittings), which, without being weaker than those already known, are useful from a practical point of view, and at the same time, allow us to obtain a larger class of iterative methods.

In this sense, we establish the following results for bounded operators in a Hilbert space using in this case the adjoint operator into one of the operators of ones of the splittings.

Theorem 8. *Let A be a nonsingular operator in a Hilbert space with $A^{-1} \geqslant 0$ and $A' = A$. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of the same or different type. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property* "d" *when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property* "d" *when $A = M_i - N_i$ is of the second type. If*

$$N_1 \leqslant N_2' \tag{24}$$

then inequality (7) holds.

Moreover, if $A^{-1} > 0$ and

$$N_1 < N_2', \tag{25}$$

then inequality (9) holds.

Proof. As in Theorem 5, both splittings are convergent by Theorem 3 and Remark 3.

Suppose that $A = M_1 - N_1$ is of the first type. Since $A^{-1} \geqslant 0$, from inequality (24) and part (vii) of Theorem 3, we have that

$$0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2' = (N_2A^{-1})', \tag{26}$$

because $A' = A$. Moreover, for the eigenvalue $\rho(A^{-1}N_1)$ there exist an eigenvector $x_1' \geqslant 0$ such that $(A^{-1}N_1)'x_1' = \rho(A^{-1}N_1)x_1'$. Then from inequality (26) we have that

$$(A^{-1}N_2)'x_1' - \rho(A^{-1}N_1)x_1' \geqslant 0.$$

Then by part (i) of Lemma 1 we have that

$$\rho(A^{-1}N_1) \leqslant \rho(A^{-1}N_2).$$

If $A = M_2 - N_2$ is of the second type, taking into account that $\rho(A^{-1}N_2) = \rho(N_2A^{-1})$ and using Remark 3 by similar argument inequality (7) holds.

If $A = M_1 - N_1$ is of the second type, then from inequality (24) and Remark 3 we obtain that

$$0 \leqslant N_1A^{-1} \leqslant N_2'A^{-1} = (A^{-1}N_2)'.$$

Now by similar argument inequality (7) holds.

If $A^{-1} > 0$, by a similar argument, we obtain inequality (9). $\square$

As an immediate consequence of Remark 4 (really, parts (i) and (ii) of Lemma 3 together with the changes proposed in Remark 4) Theorem 8 also holds if we replace inequalities (24) and (25) by $M_1 \leqslant M_2'$ and $M_1 < M_2'$, respectively.

Theorem 8 was established by Woźnicki ([18], Theorem 3.13) for weak nonnegative splittings of matrices of different types, without the hypothesis $A' = A$; that is, A symmetric, but this hypothesis is necessary as we can see in the following example.

Example 3. Let

$$A = \begin{bmatrix} 2 & -\frac{1}{2} & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & -\frac{3}{2} & 2 \end{bmatrix}.$$

It follows then that $A^{-1} \geqslant 0$. Consider the splittings $A = M_1 - N_1 = M_2 - N_2$, where

$$N_1 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

Clearly $N_1 \leqslant N_2^{\mathrm{T}}$. Both splittings are regular, and, therefore, weak nonnegative of the same type. Nevertheless

$$\rho(M_1^{-1}N_1) = \frac{3}{8} > \frac{9}{49} = \rho(M_2^{-1}N_2).$$

The comparison condition (27) with N_2 instead of N_2' was established by Csordas and Varga [5] for regular splittings for matrices and later Woźnicky [18] proved that it still holds for weak nonnegative splittings of different types also for matrices, but without the assumption $A = A'$. Here we present the following result for weak nonnegative splittings of the same type for operators in a Hilbert space.

Theorem 9. *Let A be a nonsingular operator in a Hilbert space with $A^{-1} \geqslant 0$ and $A' = A$. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of the same type. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property* "d" *when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property* "d" *when $A = M_i - N_i$ is of the second type. If*

$$A^{-1}N_1A^{-1} \leqslant A^{-1}N_2'A^{-1}, \tag{27}$$

then inequality (7) holds.

Moreover, if $A^{-1} > 0$ and

$$A^{-1}N_1A^{-1} < A^{-1}N_2'A^{-1}, \tag{28}$$

then inequality (9) holds.

Proof. The proof follows in a similar way to the proof of Theorem 6 taking into account that $N_2'A^{-1} = (A^{-1}N_2)'$ because $A = A'$. □

Remark 6. As an immediate consequence of Remark 4 (really, parts (iii) and (iv) of Lemma 3 together with the changes suggested in Remark 4), if we replace inequalities (27) and (28) by $A^{-1}M_1A^{-1} \leqslant A^{-1}M_2'A^{-1}$, and $A^{-1}M_1A^{-1} < A^{-1}M_2'A^{-1}$, respectively, then Theorem 9 also holds.

Theorem 10. *Let A be a nonsingular operator in a Hilbert space with $A^{-1} \geqslant 0$ and $A' = A$. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of the same type. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property* "d" *when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property* "d" *when $A = M_i - N_i$ is of the second type. If*

$$M_1^{-1} \geqslant (M_2^{-1})', \tag{29}$$

then inequality (7) holds.

Moreover, if $A^{-1} > 0$ and

$$M_1^{-1} > (M_2^{-1})' \tag{30}$$

then inequality (9) holds.

Proof. Suppose that both splittings are of the first type, and that we state inequality (29) as

$$M_1^{-1}AA^{-1} \geqslant A^{-1}A(M_2^{-1})'. \tag{31}$$

Then, by part (iii) of Theorem 3,

$$0 \leqslant A^{-1}M_1 \quad \text{and} \quad 0 \leqslant (A^{-1}M_2)' = M_2'A^{-1}$$

because $A' = A$. Multiplying both sides of inequality (31) on the left by $A^{-1}M_1$, and on the right by $M_2'A^{-1}$, we obtain that

$$A^{-1}M_2'A^{-1} \geqslant A^{-1}M_1A^{-1}. \tag{32}$$

Now, by Remark 6 inequality (7) follows.

If both splittings are of the second type, using inequality (29) as

$$A^{-1}AM_1^{-1} \geqslant (M_2^{-1})'AA^{-1}, \tag{33}$$

using Remark 3 (really, part (iii) of Theorem 3 together with the changes proposed in Remark 3) instead of Theorem 3 to obtain

$$0 \leqslant M_1A^{-1} \quad \text{and} \quad 0 \leqslant (M_2A^{-1})' = A^{-1}M_2',$$

and multiplying both sides of inequality (33) on the left by $A^{-1}M_2'$ and on the right by M_1A^{-1}, we obtain inequality (32), and again, by Remark 6 inequality (7) follows.

If $A^{-1} > 0$, following the same argument as above, we obtain

$$A^{-1}M_2'A^{-1} > A^{-1}M_1A^{-1}.$$

Then, by Remark 6 inequality (9) follows. □

The above theorem was also established by Woźnicki ([18], Theorem 3.14) for matrices without the hypothesis $A' = A$, that is, A symmetric, but this hypothesis is necessary as we can see in the following example.

Example 4. Let

$$A = \begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & 2 \end{bmatrix}.$$

It follows then that $A^{-1} \geqslant 0$. Consider the splittings $A = M_1 - N_1 = M_2 - N_2$, where

$$M_1 = \begin{bmatrix} 2 & -\frac{3}{4} \\ -\frac{1}{10} & 2 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} \frac{201}{100} & 0 \\ -\dfrac{1}{2} & \frac{201}{100} \end{bmatrix}.$$

Both splittings are regular, and, therefore, weak nonnegative of the same type. Furthermore,

$$M_1^{-1} = \begin{bmatrix} \frac{80}{157} & \frac{30}{157} \\ \frac{4}{157} & \frac{80}{157} \end{bmatrix} \geqslant \begin{bmatrix} \frac{100}{201} & \frac{5000}{40401} \\ 0 & \frac{100}{201} \end{bmatrix} = (M_2^{-1})^{\mathrm{T}}.$$

Nevertheless

$$\rho(M_1^{-1}N_1) \approx \frac{740}{3587} > \frac{280}{2097} \approx \rho(M_2^{-1}N_2).$$

Examples 3 and 4 not only show that Theorems 3.13 and 3.14 of Woźnicki [18] are false for weak nonnegative splittings of the same type, but also for regular splittings. Furthermore, as a consequence of these examples, Corollaries 3.3 and 3.4 and Theorem 3.15 of Woźnicki [18] are also false in general.

If, instead of considering weak nonnegative splittings of the same type, we consider weak nonnegative splittings of different types, Theorems 9 and 10 no longer hold as we show in the following example.

Example 5. Let

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

It follows then that $A^{-1} \geqslant 0$. Consider the splittings $A = M_1 - N_1 = M_2 - N_2$, where

$$M_1 = \begin{bmatrix} \frac{5}{4} & -\frac{9}{5} \\ -\frac{5}{4} & \frac{18}{5} \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} \frac{173}{150} & -\frac{73}{75} \\ -\frac{41}{25} & \frac{148}{45} \end{bmatrix}.$$

It follows then that $A = M_1 - N_1$ is a weak nonnegative splitting of the first (but not of the second) type, and $A = M_2 - N_2$ is a weak nonnegative splitting of the second (but not of the first) type. Now, it is easy to check that inequalities (27) and (28), (29) and (30) hold, and inequalities (7) and (9) do not hold.

The results which we have seen up to now would form part of what Woźnicki [18] calls natural conditions; that is, conditions which are easy or immediate to check, provided that we compare matrices which we either know or could easily know. It is, therefore, evident that introducing the transposed matrices into the conditions of comparison, even though the results seen up to now are not valid for nonsymmetric matrices (as proposed erroneously by Woźnicki), it allows us to apply these results to a greater number of iterative methods based on the iterative scheme (2). However, the application that Woźnicki [18,19] proposes for the above theorems related to choosing a more efficient splitting of A in the Gauss–Seidel method when A is a nonsymmetric matrix, is no longer of interest since those theorems are not true for nonsymmetric matrices as we have showed in Examples 3 and 4.

3.2. Weak splittings

If we replace "weak nonnegative splitting" by "weak splitting" in Section 3.1 we can ensure that $\rho(M_1^{-1}N_1) \leqslant \rho(M_2^{-1}N_2)$, but we cannot ensure that both splittings are convergent. This is due to the fact that if the splittings are weak nonnegative, the condition "$A^{-1} \geqslant 0$" guarantees convergence (see Theorem 3 and Remark 3). However, in the case where the splittings are weak, this condition does not guarantee convergence as we show in the following example (see also Theorem 2 and Remark 1).

Example 6. Consider the monotone matrix A of Example 3, and let

$$M = \begin{bmatrix} -1 & \frac{1}{4} & 0 \\ \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & \frac{3}{2} & -2 \end{bmatrix}.$$

For the splitting $A = M - N$ we have that $NM^{-1} = \operatorname{diag}(3, 2, 3) \geqslant 0$. So, the splitting is weak of the second type and it is clearly not convergent.

If we consider weak splittings substituting "$A^{-1} \geqslant 0$" for "convergent splittings", then we cannot establish any relation between the spectral radii of the iteration operators as we can see in the following example.

Example 7. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}.$$

Let $A = M_1 - N_1 = M_2 - N_2 = M_3 - N_3$ be three splittings, where

$$N_1 = \begin{bmatrix} -1 & -1 \\ 5 & -1 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} 2 & -1 \\ 6 & -1 \end{bmatrix}, \qquad \text{and} \quad N_3 = \begin{bmatrix} -1 & -1 \\ 6 & 0 \end{bmatrix}.$$

All the splittings are weak of the first type, $N_1 \leqslant N_2$, $N_1 \leqslant N_3$, and

$$\rho(M_1^{-1}N_1) \approx \tfrac{829}{938} > \tfrac{2}{3} = \rho(M_2^{-1}N_2),$$
$$\rho(M_1^{-1}N_1) \approx \tfrac{829}{938} < \tfrac{1220}{1353} \approx \rho(M_3^{-1}N_3).$$

Observe that all the splittings are convergent.

Therefore, for the results of the previous subsection to be true for weak splittings, it is necessary to add the condition that the splittings are convergent. In Woźnicki [18] the theorems for weak nonnegative splittings for matrices also appear for weak splittings (Theorems 6.8–6.10 as well as Corollary 6.2 of [18]). However, Examples 3 and 4 show once again that these results are not true in general.

Let us now propose the following comparison conditions for weak (of the same or different types) and convergent splittings for the operator A in a Banach space. We can consider these comparison conditions as being within the group of the so-called natural conditions.

Theorem 11. *Let A be a nonsingular operator in a Banach space and let $A = M_1 - N_1 = M_2 - N_2$ be two weak splittings of the same or different types. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property* "d" *when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property* "d" *when $A = M_i - N_i$ is of the second type. Also assume that both splittings are convergent. If in addition some of the following conditions hold*

(i) $0 \leqslant A^{-1}N_1 < A^{-1}N_2$,
(ii) $0 \leqslant N_1A^{-1} < A^{-1}N_2$,
(iii) $0 \leqslant N_1A^{-1} < N_2A^{-1}$,
(iv) $0 \leqslant A^{-1}N_1 < N_2A^{-1}$,

then inequality (9) holds.

Proof. Using that $A^{-1}N_2$ (respectively, N_2A^{-1}) is irreducible, there exists $\boldsymbol{x} > 0$ such that $A^{-1}N_2\boldsymbol{x} = \rho(A^{-1}N_2)\boldsymbol{x}$ (respectively, $N_2A^{-1}\boldsymbol{x} = \rho(N_2A^{-1})\boldsymbol{x}$), then by part (ii) of Lemma 1 we have that

$$\rho(N_1A^{-1}) = \rho(A^{-1}N_1) < \rho(A^{-1}N_2) = \rho(N_2A^{-1}).$$

Now, by part (vii) of Theorem 2 and/or Remark 1 (really, parts (vii) of Theorem 2 together with the changes suggested in Remark 1) inequality (9) follows. □

If in the above theorem we replace "Banach space" by "Hilbert space", $A^{-1}N_2$ by $(A^{-1}N_2)'$ and N_2A^{-1} by $(N_2A^{-1})'$ then, we obtain, in a similar way, the following result.

Theorem 12. *Let A be a nonsingular operator in a Hilbert space and let $A = M_1 - N_1 = M_2 - N_2$ be two weak splittings of the same or different type. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property* "d" *when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property* "d" *when $A = M_i - N_i$ is of the second type. Also assume that both splittings are convergent. If in addition some of the following conditions hold*

(i) $0 \leqslant A^{-1}N_1 < (A^{-1}N_2)'$,
(ii) $0 \leqslant N_1A^{-1} < (A^{-1}N_2)'$,
(iii) $0 \leqslant N_1A^{-1} < (N_2A^{-1})'$,
(iv) $0 \leqslant A^{-1}N_1 < (N_2A^{-1})'$,

then inequality (9) holds.

It is easy to prove that each one of the conditions (i)–(iv) of Theorems 11 and 12 with N_i, $i = 1, 2$, is equivalent to the one analogous with M_i.

Remark 7. If in parts (i)–(iv) of Theorems 11 and 12 we replace "$<$" by "$\leqslant$", then inequality (7) holds. These results were established by Woźnicki ([18], Theorems 6.2 and 6.11) for matrices with the additional hypothesis $A^{-1} \geqslant 0$, but this hypothesis is not necessary.

4. Relations between the comparison conditions

In Section 3, we have seen conditions which, given two splittings of the same operator, would allow us to know which one converges faster. In this section we present some theorems in which some relations (ordered from the most restrictive to the weakest) between the different hypotheses in the comparison theorems of Section 3 are established. In this way, when we are interested in comparing the spectral radii of two splittings of the same operator, we will start from the most restrictive (usually the simplest) to the weakest (usually the most difficult), until one of them allows us to give an appropriate answer.

Csordas and Varga ([5], Proposition 1) propose some relations between comparison conditions for regular splittings of matrices. Woźnicki ([18], Lemma 5.1) also presents some relations for nonnegative splittings of matrices, but does not always take into account whether the weakest condition allows us to establish the comparison result under the hypotheses for which they are established or not. This is, after the convergence of the splitting, our objective.

Theorem 13. *Let A be a nonsingular operator in a Banach space with $A^{-1} \geqslant 0$ and let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of different type. For $i = 1,2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property* "d" *when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property* "d" *when $A = M_i - N_i$ is of the second type.*

(i) If $N_1 \leqslant N_2$, then $M_1^{-1} \geqslant M_2^{-1}$.

(ii) If $M_1^{-1} \geqslant M_2^{-1}$, then $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2A^{-1}$.

(iii) If $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2A^{-1}$, then inequality (7) holds.

Proof. (i) From part (i) of Lemma 3, $M_1 \leqslant M_2$. Now, multiplying both sides on the left by $M_1^{-1} \geqslant 0$, and on the right by $M_2^{-1} \geqslant 0$, it follows that $M_1^{-1} \geqslant M_2^{-1}$.

(ii) If $A = M_1 - N_1$ is of the first type and hence $A = M_2 - N_2$ is of the second type, by Theorem 3 and Remark 3, we have $A^{-1}N_1 \geqslant 0$ and $N_2A^{-1} \geqslant 0$ respectively.

Now, from $M_i = A + N_i$, $i = 1,2$, we have that

$$\begin{aligned}(I + A^{-1}N_1)^{-1}A^{-1} &= (A + N_1)^{-1}AA^{-1} \\ &\geqslant A^{-1}A(A + N_2)^{-1} = A^{-1}(I + N_2A^{-1})^{-1},\end{aligned}$$

and multiplying both sides on the left by $I + A^{-1}N_1 \geqslant 0$, and on the right by $I + N_2A^{-1} \geqslant 0$, it follows that $A^{-1}N_2A^{-1} \geqslant A^{-1}N_1A^{-1}$.

(iii) See Theorem 6. □

Observe that part (i) of Theorem 13 also holds for weak nonnegative splittings of the same type. This is not true for parts (ii) and (iii) as we can see in Example 8. Example 8 also shows that the converses of Theorem 13 are not true.

Theorem 14. *Let A be a nonsingular operator in a Banach space with $A^{-1} \geqslant 0$ and let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of different type. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property "d" when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property "d" when $A = M_i - N_i$ is of the second type.*

(i) If $N_1 \leqslant N_2$ and $A = M_1 - N_1$ is of the first (respectively, second) type, then $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2$ (respectively, $0 \leqslant N_1A^{-1} \leqslant N_2A^{-1}$).

(ii) If $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2$ or $0 \leqslant N_1A^{-1} \leqslant N_2A^{-1}$, then $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2A^{-1}$.

(iii) If $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2A^{-1}$, then inequality (7) holds.

Proof. Part (i) follows by Theorem 3 (Remark 3), part (ii) is trivial, and part (iii) is part (iii) of Theorem 13. □

Observe that part (i) of Theorem 14 is valid regardless of the type of the splitting $A = M_2 - N_2$. Furthermore part (ii) is also valid regardless of the type of the splittings.

Example 8 shows that the converses of Theorem 14 are not true. It also shows that under the hypotheses of Theorems 13 and 14, no relation exists between the inequalities $M_1^{-1} \geqslant M_2^{-1}$ and $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2$ (respectively, $0 \leqslant N_1A^{-1} \leqslant N_2A^{-1}$) when $A = M_1 - N_1$ is a weak nonnegative splitting of the first (respectively, second) type.

Theorem 15. *Let A be a nonsingular operator in a Banach space with $A^{-1} \geqslant 0$ and let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of the same or different type. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property "d" when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property "d" when $A = M_i - N_i$ is of the second type.*

(a) Assume that $A = M_1 - N_1$ is of the first type.

(i) If $N_1 \leqslant N_2$, then $A^{-1}N_1 \leqslant A^{-1}N_2$.

(ii) If $A^{-1}N_1 \leqslant A^{-1}N_2$, then inequality (7) holds.

(b) Assume that $A = M_1 - N_1$ is of the second type.

(i) If $N_1 \leqslant N_2$, then $N_1A^{-1} \leqslant N_2A^{-1}$.

(ii) If $N_1A^{-1} \leqslant N_2A^{-1}$, then inequality (7) holds.

Proof. Parts (a)(i) and (b)(i) are trivial. Parts (a)(ii) and (b)(ii) follow from Remark 7 (really from parts (i) and (iii), respectively, of Theorem 11 together with the changes proposed in Remark 7). □

Observe that parts (a)(i) and (b)(i) of Theorem 15 are valid regardless of the type of the splittings. This is not true for parts (a)(ii) and (b)(ii) as we can see in

Example 8. Example 8 also shows that the converses of Theorem 15 are not true.

Example 8. For the monotone matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ -1 & 0 & 2 \end{bmatrix}$$

consider the splittings $A = P_i - Q_i$, $i = 1, 2, \ldots, 15$, where

$$P_1 = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & -2 \\ -2 & 0 & 2 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 2 & -2 \\ -\frac{3}{2} & 0 & \frac{3}{2} \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 2 & -4 \\ -1 & 0 & 4 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 1 & -\frac{9}{5} & \frac{9}{5} \\ 0 & \frac{8}{5} & -\frac{18}{5} \\ -1 & \frac{2}{5} & \frac{18}{5} \end{bmatrix}, \quad P_5 = \begin{bmatrix} 1 & -2 & 1 \\ -\frac{1}{3} & 4 & -2 \\ -\frac{1}{2} & \frac{1}{4} & 2 \end{bmatrix}, \quad P_6 = \begin{bmatrix} \frac{8}{9} & -\frac{20}{9} & \frac{10}{9} \\ \frac{2}{9} & \frac{40}{9} & -\frac{20}{9} \\ -\frac{4}{9} & \frac{10}{9} & \frac{40}{9} \end{bmatrix},$$

$$P_7 = \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}, \quad P_8 = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & -2 \\ -1 & 0 & 2 \end{bmatrix}, \quad P_9 = \begin{bmatrix} 1 & -\frac{20}{9} & \frac{10}{9} \\ \frac{2}{9} & \frac{40}{9} & -\frac{20}{9} \\ -\frac{5}{9} & \frac{10}{9} & \frac{40}{9} \end{bmatrix},$$

$$P_{10} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & -3 \\ -\frac{1}{2} & 0 & 2 \end{bmatrix}, \quad P_{11} = \begin{bmatrix} 1 & -2 & \frac{3}{4} \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{3}{4} \\ 0 & 2 & -2 \\ 0 & -\frac{1}{2} & 3 \end{bmatrix},$$

$$P_{13} = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & -2 \\ -3 & 2 & 2 \end{bmatrix}, \quad P_{14} = \begin{bmatrix} \frac{119}{100} & -2 & \frac{11}{10} \\ -\frac{9}{50} & 4 & -3 \\ -\frac{21}{50} & 0 & \frac{19}{10} \end{bmatrix}, \quad P_{15} = \begin{bmatrix} 1 & -4 & 4 \\ 0 & 2 & -2 \\ -1 & 0 & 2 \end{bmatrix}.$$

For $i = 1, 2, 3, 4, 6, 9, 11, 13$, the splittings are weak nonnegative of the first type. For $i = 5, 10, 12, 14, 15$, the splittings are weak nonnegative of the second type. For $i = 7$, the splittings is nonnegative, and for $i = 8$ the splittings is regular.

Let $M_1 = P_1$ and $M_2 = P_2$ it follows that $M_1^{-1} \geqslant M_2^{-1}$, but $A^{-1}N_1A^{-1} \not\leqslant A^{-1}N_2A^{-1}$. Now for $M_1 = P_3$ and $M_2 = P_4$ we have that $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2A^{-1}$ but $\rho(M_1^{-1}N_1) \not\leqslant \rho(M_2^{-1}N_2)$. Hence, parts (ii) and (iii) of Theorem 13 are not true for weak nonnegative splittings of the same type.

Let $M_1 = P_5$ and $M_2 = P_6$, it follows then that $M_1^{-1} \geqslant M_2^{-1}$, but $N_1 \not\leqslant N_2$. Now, for $M_1 = P_5$ and $M_2 = P_7$, we have that $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2A^{-1}$, but $M_1^{-1} \not\geqslant M_2^{-1}$. Finally, for $M_1 = P_8$ and $M_2 = P_5$, we have that $\rho(M_1^{-1}N_1) \leqslant \rho(M_2^{-1}N_2)$, but $A^{-1}N_1A^{-1} \not\leqslant A^{-1}N_2A^{-1}$. Hence the converses of Theorem 13 are not true.

Let $M_1 = P_{11}$ and $M_2 = P_{12}$ then we have that $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2$ but $N_1 \nleqslant N_2$. Next, if we take $M_1 = P_7$ and $M_2 = P_6$, then $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2A^{-1}$, but $A^{-1}N_1 \nleqslant A^{-1}N_2$ and $N_1A^{-1} \nleqslant N_2A^{-1}$. Hence, the converses of parts (i) and (ii) of Theorem 14 are not true. Part (iii) of Theorem 14 is part (iii) of Theorem 13.

Let $M_1 = P_3$ and $M_2 = P_5$, it follows then that $M_1^{-1} \geqslant M_2^{-1}$ but $A^{-1}N_1 \nleqslant A^{-1}N_2$. For $M_1 = P_5$ and $M_2 = P_6$, we have that $M_1^{-1} \geqslant M_2^{-1}$ but $N_1A^{-1} \nleqslant N_2A^{-1}$. Now, for $M_1 = P_{11}$ and $M_2 = P_{12}$, it follows that $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2$, but $M_1^{-1} \ngeqslant M_2^{-1}$. Finally for $M_1 = P_{15}$ and $M_2 = P_8$, we have that $0 \leqslant N_1A^{-1} \leqslant N_2A^{-1}$, but $M_1^{-1} \ngeqslant M_2^{-1}$. Hence, there is no relation between the inequalities $M_1^{-1} \geqslant M_2^{-1}$, and $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2$ (respectively, $0 \leqslant N_1A^{-1} \leqslant N_2A^{-1}$) when $A = M_1 - N_1$ is a weak nonnegative splitting of the first (respectively, second) type under the hypotheses of Theorems 13 and 14.

Let $M_1 = P_6$ and $M_2 = P_9$ then we have that $A^{-1}N_1 \leqslant A^{-1}N_2$ but $N_1 \nleqslant N_2$. Next, for $M_1 = P_8$ and $M_2 = P_{10}$ we have that $\rho(M_1^{-1}N_1) \leqslant \rho(M_2^{-1}N_2)$ but $A^{-1}N_1 \nleqslant A^{-1}N_2$. Hence, the converses of part (a) of Theorem 15 are not true.

Now, for $M_1 = P_{10}$ and $M_2 = P_5$ we have that $N_1A^{-1} \leqslant N_2A^{-1}$ but $N_1 \nleqslant N_2$. Finally, for $M_1 = P_{10}$ and $M_2 = P_7$ we have that $\rho(M_1^{-1}N_1) \leqslant \rho(M_2^{-1}N_2)$ but $N_1A^{-1} \nleqslant N_2A^{-1}$. Hence, the converses of part (b) of Theorem 15 are not true.

Let $M_1 = P_{10}$ and $M_2 = P_{14}$, then we have that $A^{-1}N_1 \leqslant A^{-1}N_2$ but $\rho(M_1^{-1}N_1) \nleqslant \rho(M_2^{-1}N_2)$. So part (a)(ii) of Theorem 15 is not true if the splitting $A = M_1 - N_1$ is weak nonnegative of the second type.

Finally, let $M_1 = P_1$ and $M_2 = P_{13}$, it follows then that $N_1A^{-1} \leqslant N_2A^{-1}$ but $\rho(M_1^{-1}N_1) \nleqslant \rho(M_2^{-1}N_2)$. So, part (b)(ii) of Theorem 15 is not true if the splitting $A = M_1 - N_1$ is weak nonnegative of the first type.

Remark 8. Parts (ii) and (iii) of Theorem 13, and Theorems 14 and 15 also hold if we replace "weak nonnegative" by "weak and convergent". This is not true for part (i) of Theorem 13 (see Marek and Szyld [10, Example 4.1]).

If, in the inequalities of Theorems 13–15 we replace "Banach space" by "Hilbert space", M_2 by M_2', N_2 by N_2', "different types" by "name types", and assume, in addition, that $A = A'$, then we obtain, in a similar way, Theorems 16–18, respectively.

Theorem 16. *Let A be a nonsingular operator in a Hilbert space with $A^{-1} \geqslant 0$ and $A = A'$. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of the same type. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property* "d" *when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property* "d" *when $A = M_i - N_i$ is of the second type.*

(i) If $N_1 \leqslant N_2'$, then $M_1^{-1} \geqslant (M_2^{-1})'$.

(ii) If $M_1^{-1} \geqslant (M_2^{-1})'$, then $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2'A^{-1}$.

(iii) If $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2'A^{-1}$, then inequality (7) holds.

Observe that part (i) of Theorem 16 also holds for weak nonnegative splittings of different types. This is not true for parts (ii) and (iii), as we can see in Example 9. Furthermore, the converses of Theorem 16 are not true as we can see in Example 10.

Theorem 17. *Let A be a nonsingular operator in a Hilbert space with $A^{-1} \geqslant 0$ and $A' = A$. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of the same type. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property "d" when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property "d" when $A = M_i - N_i$ is of the second type.*

(i) *If $N_1 \leqslant N_2$ and $A = M_1 - N_1$ is of the first (respectively, second) type, then $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2'$ (respectively, $0 \leqslant N_1A^{-1} \leqslant N_2'A^{-1}$).*

(ii) *If $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2'$ or $0 \leqslant N_1A^{-1} \leqslant N_2'A^{-1}$, then $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2'A^{-1}$.*

(iii) *If $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2'A^{-1}$, then inequality (7) holds.*

Observe that part (i) of Theorem 17 is valid regardless of the type of the splitting $A = M_2 - N_2$. Furthermore, part (ii) is valid regardless of the type of the splittings.

Example 10 shows that the converses of Theorem 17 are not true. Example 10 also shows that under the hypotheses of Theorems 16 and 17, no relation exists between the inequalities $M_1^{-1} \geqslant (M_2^{-1})'$ and $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2'$ (respectively, $0 \leqslant N_1A^{-1} \leqslant N_2'A^{-1}$) when $A = M_1 - N_1$ is a weak nonnegative splitting of the first (respectively, second) type.

Theorem 18. *Let A be a nonsingular operator in a Hilbert space with $A^{-1} \geqslant 0$ and $A = A'$. Let $A = M_1 - N_1 = M_2 - N_2$ be two weak nonnegative splittings of the same or different type. For $i = 1, 2$, assume that $M_i^{-1}N_i$ and $A^{-1}N_i$ have the property "d" when $A = M_i - N_i$ is of the first type; however, assume that $N_iM_i^{-1}$ and N_iA^{-1} have the property "d" when $A = M_i - N_i$ is of the second type.*

(a) Assume that $A = M_1 - N_1$ is of the first type.

(i) If $N_1 \leqslant N_2'$, then $A^{-1}N_1 \leqslant A^{-1}N_2'$.

(ii) If $A^{-1}N_1 \leqslant A^{-1}N_2'$, then inequality (7) holds.

(b) Assume that $A = M_1 - N_1$ is of the second type.

(i) If $N_1 \leqslant N_2'$, then $N_1A^{-1} \leqslant N_2'A^{-1}$.

(ii) If $N_1A^{-1} \leqslant N_2'A^{-1}$, then inequality (7) holds.

Observe that parts (a)(i) and (b)(i) are valid regardless of the type of the splittings. This is not true for parts (a)(ii) and (b)(ii), as we can see in Example 9 below. Furthermore, the converses of Theorem 18 are not true as we can see in Example 10.

Example 9. Consider the symmetric and monotone matrix A of Example 5, and consider the splittings $A = P_i - Q_i$, $i = 1, 2, \dots, 7$, where

$$P_1 = \begin{bmatrix} \frac{5}{4} & -\frac{9}{5} \\ -\frac{5}{4} & \frac{18}{5} \end{bmatrix}, \quad P_2 = \begin{bmatrix} \frac{173}{150} & -\frac{73}{75} \\ -\frac{41}{25} & \frac{148}{45} \end{bmatrix}, \quad P_3 = \begin{bmatrix} \frac{7}{4} & -\frac{7}{4} \\ -1 & 2 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad P_5 = \begin{bmatrix} \frac{17}{10} & -\frac{9}{10} \\ -\frac{17}{10} & 2 \end{bmatrix}, \quad P_6 = \begin{bmatrix} \frac{7}{4} & -1 \\ -\frac{7}{4} & 2 \end{bmatrix},$$

$$P_7 = \begin{bmatrix} \frac{17}{10} & -\frac{17}{10} \\ -\frac{9}{10} & 2 \end{bmatrix},$$

For $i = 1, 4, 5, 6$ the splittings are weak nonnegative of the first type. For $i = 2, 3, 7$ the splittings are weak nonnegative of the second type.

Let $M_1 = P_3$ and $M_2 = P_4$, it follows then that $M_1^{-1} \geqslant (M_2^{-1})^{\mathrm{T}}$, but $A^{-1}N_1A^{-1} \nleqslant A^{-1}N_2^{\mathrm{T}}A^{-1}$. Now, for $M_1 = P_1$ and $M_2 = P_2$, we have that $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2^{\mathrm{T}}A^{-1}$, but $\rho(M_1^{-1}N_1) \nleqslant \rho(M_2^{-1}N_2)$. Hence, parts (i) and (ii) of Theorem 16 are not true for weak nonnegative splittings of different types.

Let $M_1 = P_3$ and $M_2 = P_5$, then we have that $A^{-1}N_1 \leqslant A^{-1}N_2^{\mathrm{T}}$, but $\rho(M_1^{-1}N_1) \nleqslant \rho(M_2^{-1}N_2)$. So, part (a)(ii) of Theorem 18 is not true if $A = M_1 - N_1$ is weak nonnegative of the second type.

Let $M_1 = P_6$ and $M_2 = P_7$, then we have that $N_1A^{-1} \leqslant N_2^{T}A^{-1}$, but $\rho(M_1^{-1}N_1) \nleqslant \rho(M_2^{-1}N_2)$. So, part (b)(ii) of Theorem 18 is not true if $A = M_1 - N_1$ is weak nonnegative of the first type.

Example 10. For the symmetric and monotone matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

consider the splittings $A = P_i - Q_i$, $i = 1, 2, \ldots, 12$, where

$$P_1 = \begin{bmatrix} \frac{200}{31} & -\frac{100}{31} & 0 \\ -\frac{90}{31} & \frac{200}{31} & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad P_5 = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad P_6 = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

$$P_7 = \begin{bmatrix} 3 & -\frac{3}{2} & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad P_8 = \begin{bmatrix} 4 & -1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad P_9 = \begin{bmatrix} \frac{15}{4} & -\frac{3}{2} & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$P_{10} = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad P_{11} = \begin{bmatrix} \frac{10}{3} & -1 & 0 \\ -\frac{2}{3} & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & \frac{17}{5} & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

For $i = 1, 7, 9, 11$ the splittings are nonnegative. For $i = 2, 8, 10$ the splitting are weak nonnegative of the first type. For $i = 3, 5, 6, 12$ the splittings are regular. For $i = 4$ the splitting is weak nonnegative of the second type.

For $M_1 = P_2$ and $M_2 = P_1$ we have that $M_1^{-1} \geqslant (M_2^{-1})^{\mathrm{T}}$ but $N_1 \nleqslant N_2^{\mathrm{T}}$. Next, for $M_1 = P_3$ and $M_2 = P_1$ we have that $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2^{\mathrm{T}}A^{-1}$ but $M_1^{-1} \ngeqslant (M_2^{-1})^{\mathrm{T}}$. Finally, for $M_1 = P_3$ and $M_2 = P_4$ we have that $\rho(M_1^{-1}N_1) \leqslant \rho(M_2^{-1}N_2)$ but $A^{-1}N_1A^{-1} \nleqslant A^{-1}N_2^{\mathrm{T}}A^{-1}$. Hence, the converses of Theorem 16 are not true.

Let $M_1 = P_8$ and $M_2 = P_9$, then we have that $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2^{\mathrm{T}}$, but $N_1 \nleqslant N_2^{\mathrm{T}}$. Let $M_1 = P_7$ and $M_2 = P_1$, then $A^{-1}N_1A^{-1} \leqslant A^{-1}N_2^{\mathrm{T}}A^{-1}$, but $A^{-1}N_1 \nleqslant A^{-1}N_2^{\mathrm{T}}$, and $N_1A^{-1} \nleqslant N_2^{\mathrm{T}}A^{-1}$. Hence, the converses of parts (i) and (ii) of Theorem 17 do not hold. Part (iii) of Theorem 17 is part (iii) of Theorem 16.

Let $M_1 = P_6$ and $M_2 = P_{10}$, it follows then that $M_1^{-1} \geqslant (M_2^{-1})^{\mathrm{T}}$ but $A^{-1}N_1 \nleqslant A^{-1}N_2^{\mathrm{T}}$. For $M_1 = P_4$ and $M_2 = P_{12}$, we have that $M_1^{-1} \geqslant (M_2^{-1})^{\mathrm{T}}$ but $N_1A^{-1} \nleqslant N_2^{\mathrm{T}}A^{-1}$. Now, for $M_1 = P_3$ and $M_2 = P_5$, it follows that $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2^{\mathrm{T}}$, but $M_1^{-1} \ngeqslant (M_2^{-1})^{\mathrm{T}}$. Finally for $M_1 = P_3$ and $M_2 = P_{11}$, we have that $0 \leqslant N_1A^{-1} \leqslant N_2^{\mathrm{T}}A^{-1}$, but $M_1^{-1} \ngeqslant (M_2^{-1})^{\mathrm{T}}$. Hence, there is no relation between the inequalities $M_1^{-1} \geqslant (M_2^{-1})^{\mathrm{T}}$, and $0 \leqslant A^{-1}N_1 \leqslant A^{-1}N_2^{\mathrm{T}}$ (respectively, $0 \leqslant N_1A^{-1} \leqslant N_2^{\mathrm{T}}A^{-1}$) when $A = M_1 - N_1$ is a weak nonnegative splitting of the first (respectively, second) type under the hypotheses of Theorems 16 and 17.

For $M_1 = P_3$ and $M_2 = P_5$ we have that $A^{-1}N_1 \leqslant A^{-1}N_2^{\mathrm{T}}$ but $N_1 \nleqslant N_2^{\mathrm{T}}$. Now, for $M_1 = P_6$ and $M_2 = P_2$ we have that $\rho(M_1^{-1}N_1) \leqslant \rho(M_2^{-1}N_2)$ but $A^{-1}N_1 \nleqslant A^{-1}N_2^{\mathrm{T}}$. Hence the converses of part (a) of Theorem 18 are not true.

Next, for $M_1 = P_4$ and $M_2 = P_1$ we have that $N_1A^{-1} \leqslant N_2^{\mathrm{T}}A^{-1}$ but $N_1 \nleqslant N_2^{\mathrm{T}}$. Finally, for $M_1 = P_6$ and $M_2 = P_5$ we have that $\rho(M_1^{-1}N_1) \leqslant \rho(M_2^{-1}N_2)$ but $N_1A^{-1} \nleqslant N_2^{\mathrm{T}}A^{-1}$. Hence the converses of part (b) of Theorem 18 are not true.

Remark 9. Parts (ii) and (iii) of Theorem 16, and Theorems 17 and 18 also hold if we replace "weak nonnegative" by "weak and convergent". This is not true for part (i) of Theorem 16 (see [10], Example 4.1, with M_2 as M_2^{T} and N_2 as N_2^{T}).

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