Fibrator properties of manifolds

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Abstract

Approximate fibrations form a useful class of maps, in part, because they provide computable relationships involving the domain, image and homotopy fiber. Fibrator properties, which pertain to and depend upon the homotopy fiber, allow instant recognition of approximate fibrations. This is a survey of results about those fibrator properties.

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The aim of this survey is to present a coherent overview of results describing fibrator properties. Fibrators are closed manifolds $N$ which afford instant recognition of approximate fibrations. A surjective map between ANRs has a chance of being an approximate fibration if each point preimage is a copy, up to shape, of the fixed manifold $N$. Work in this area attacks various kinds of converses; more specifically, it seeks to identify those $N$ such that any map $p: M \rightarrow B$ defined on a manifold $M$ and for which each $p^{-1}(b)$ is homeomorphic (or homotopy equivalent, or shape equivalent, or ...) to $N$ must be an approximate fibration. Attention is restricted to maps having finite-dimensional images $B$, and the question is parsed in several ways, most notably, by the codimension of $N$ relative to $M$. Propelling this effort is the belief in the benefits of quick and effortless detection of approximate fibrations.

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Briefly, the paper is organized as follows: Section 1 spells out relevant properties of approximate fibrations; Section 2 defines the fibrator properties studied to date; Section 3 reviews results about codimension-2 fibrator properties; Section 4 reviews the much shorter list of results about codimension-\( k \) fibrator properties, \( k > 2 \); Section 5 does the same in a slightly special PL category, where the results are fairly extensive; Section 6 presents what is known about the \( N \) which are fibrators in all codimensions with respect to PL maps between p.l. triangulated manifolds; Section 7 details fibrator properties of low dimensional manifolds \( N \); Section 8 treats some issues about the structure of manifolds admitting \( N \)-shaped maps and approximate fibrations; and Section 9 offers a collection of unsolved problems.

1. Approximate fibrations

An approximate fibration is a proper map \( p : M \to B \) satisfying the Approximate Homotopy Lifting Property (AHLP): given any space \( X \), any maps \( f : X \times 0 \to M \) and \( F : X \times [0, 1] \to B \) with \( pf = F \), and any open cover \( U \) of \( B \), there exists a map \( \tilde{F} : X \times [0, 1] \to M \) such that \( p\tilde{F} \) is \( U \)-close to \( F \) (i.e., to each \( \langle x, t \rangle \in X \times [0, 1] \) corresponds \( U \in \mathcal{U} \) with \( \{ F(\langle x, t \rangle), p\tilde{F}(\langle x, t \rangle) \} \subset U \)).

This class of maps was introduced and studied by Coram and Duvall [7,8], who proved, among other things, that any two fibers of such a map between connected ANRs are shape equivalent. Approximate fibrations have proved to be valuable because, just as with the more familiar class of fibrations, they give rise not only to an exact sequence relating the shape homotopy groups of the fiber, domain and base space, but also to a Leray spectral sequence useful for analysing similar relations among (co)homology groups. Moreover, unlike fibrations, for fixed \( M \) and \( B \) the collection of all approximate fibrations \( p : M \to B \) is closed in the space of all continuous functions \( f : M \to B \).

Examples. One might expect approximate fibrations to be characterized as those maps which can be approximated by fibrations. However, Husch [39] produced an approximate fibration \( p : M \to S^1 \) which is not even homotopic to a fibration. Chapman–Ferry [2] then produced an approximate fibration \( p : M \to S^2 \) which is homotopic to a fibration but cannot be closely approximated by one. Im [40] constructed a comparable approximate fibration \( p : M \to \mathbb{R} \); he also provided conditions on the homotopy type of the fiber under which such \( p : M \to \mathbb{R} \) can be approximated by locally trivial bundle maps (see Corollary 8.9 here).

Coram and Duvall [8] developed useful recognition criteria for approximate fibrations. One of them can be stated quite simply for PL maps. In that setting each fiber \( p^{-1}(b) \) has a neighborhood \( U_b \) equipped with a retraction

\[
R : U_b \to p^{-1}(b);
\]

\( p \) is an approximate fibration if and only if \( R \) restricts to homotopy equivalences

\[
R| : p^{-1}(c) \to p^{-1}(b)
\]

for all \( c \in B \) sufficiently near \( b \). For purposes here two key issues will be: Is \( R| : p^{-1}(c) \to p^{-1}(b) \) a degree 1 map? If so, does it induce a \( \pi_1 \)-isomorphism?
For the more general $N$-shaped maps $p$, defined in the next section, Goad [32] has shown that $p$ is an approximate fibration if and only if $p$ has a local homotopy product $N$-structure, meaning that each $b \in B$ has a neighborhood $W_b$ and there exists a map $q: p^{-1}(W_b) \to N$ such that $p \times q: p^{-1}(W_b) \to W_b \times N$ is a proper homotopy equivalence.

2. Definitions and notation

Fix a closed, connected $n$-manifold $N$. A proper map $p: M \to B$ defined on an $(n+k)$-manifold $M$ is said to be $N$-shaped if each fiber $p^{-1}(b)$ has the shape of $N$; $N$ itself is called a codimension-$k$ (orientable) fibration if, for every $N$-shaped map $p: M \to B$, where $M$ is an (respectively, orientable) $(n+k)$-manifold, and where $B$ is finite-dimensional, $p$ is an approximate fibration. Without this specific requirement to the contrary, $N$-shaped maps might raise dimension to infinity: Dranishnikov [28] demonstrated the existence of a cell-like map $q: S^7 \to X$ onto an infinite-dimensional space $X$, and, for arbitrary fixed $N$, each fiber of $q \circ \text{proj}: S^7 \times N \to X$ has the shape of $N$.

When $B$ is a polyhedron and $p$ as above is PL, then $p$ is $N$-like if each fiber collapses to an $n$-complex homotopy equivalent to $N$; $N$ is a codimension-$k$ (orientable) PL fibration if, for every $N$-like map $p: M \to B$, where $M$ is an (respectively, orientable) PL $(n+k)$-manifold, $p$ is an approximate fibration. Moreover, $N$ is a PL (orientable) fibration if it is a codimension-$k$ (orientable) PL fibration for all $k > 0$. In the orientable setting we abbreviate by writing simply that $N$ is a codimension-$k$ PL o-fibrator or, in the extreme case, a PL o-fibrator. PL fibrators do exist—a large collection within the class of aspherical manifolds—but no manifolds are known to be codimension-$k$ fibrators (without the PL hypothesis) for all $k$.

Loosely put, everything provable about those PL maps with point preimages being actual copies of $N$ seems provable for $N$-like PL maps. However, the latter concept offers a geometric tameness feature—exposed in Lemma 5.2—not possessed by a more general $N$-shaped map. A remark at the opening of Section 5 best reveals the distinction between $N$-like and $N$-shaped maps. A general theme, not to be taken as mathematically precise, is: ‘Most’ manifolds are PL fibrators. No evidence is at hand to support a similar theme about codimension-$k$ fibrators, $k > 2$, without the PL restriction.

A closed, orientable $n$-manifold $N$ is Hopfian if every degree one map $f: N \to N$ which induces a $\pi_1$-isomorphism is necessarily a homotopy equivalence. It is worth noting that $N$ is Hopfian if either (1) $\pi_1(N)$ is finite, (2) $N$ is aspherical, (3) $n \leq 4$ (Swarup [47], Hausmann [34]) or (4) $\pi_i(N) \cong 0$ for $1 < i < n − 1$ [47].

There is a better-known, similar-sounding group-theoretic term: a group $\Gamma$ is Hopfian if every epimorphism $\Gamma \to \Gamma$ is an automorphism. Finitely presented groups need not be Hopfian, as evidenced by the famous Baumslag–Solitar groups such as $\Gamma = \langle a, b \mid ab^2a^{-1} = b^3 \rangle$; like any finitely presented group, $\Gamma$ is the fundamental group of some closed 4-manifold. As far as we know, fundamental groups of closed 3-manifolds are Hopfian [35].

Generally, analysis of fibration properties here applies most readily to Hopfian manifolds $N$ with Hopfian fundamental groups, simply because a map $f: N \to N$ is a homotopy
equivalence if and only if the (absolute) degree of \( f \) equals 1. In context the surrounding manifold data often assures that the “maps” between fibers arising from shape retractions have degree 1.

3. Codimension-2 fibrators

Analysis of codimension-2 fibrator properties is bolstered by the following two key facts: the manifold nature of the images and the isolated nature of the singularities (with respect to approximate fibrations).

**Theorem 3.1** [25,10]. If \( p : M^{n+2} \to B \) is an \( N^n \)-shaped map defined on an \((n + 2)\)-manifold, then \( B \) is a \( 2 \)-manifold with boundary; if, in addition, both \( M^{n+2} \) and \( N^n \) are orientable, \( B \) is a \( 2 \)-manifold.

**Theorem 3.2** [25]. Suppose \( N^n \) is a Hopfian \( n \)-manifold with Hopfian fundamental group, and \( p : M^{n+2} \to B \) is an \( N^n \)-shaped map, defined on an orientable \((n + 2)\)-manifold. Then \( B \) contains a locally finite set \( F \) such that \( p \) restricts to an approximate fibration \( p^{-1}(B - F) \to B - F \).

In the preferred Hopfian categories, codimension-2 fibrators are fairly well understood. Moreover, among PL manifolds, there is no known distinction between the sets of codimension-2 fibrators and of codimension-2 PL fibrators.

**Theorem 3.3** [14]. Every Hopfian manifold \( N \) such that \( \pi_1(N) \) is Hopfian and \( \chi(N) \neq 0 \) is a codimension-2 \( o \)-fibrator.

A group \( \Gamma \) is hyperhopfian if every endomorphism \( \psi : \Gamma \to \Gamma \) with normal image and cyclic cokernel is necessarily an automorphism. Clearly groups \( \Gamma \) having a cyclic direct factor fail to be hyperhopfian; however, most knot groups [46], fundamental groups of compact hyperbolic manifolds [19], as well as all nontrivial free products of Hopfian groups except \( \mathbb{Z}_2 \ast \mathbb{Z}_2 \) [27,14,33], are hyperhopfian.

Here is why the hyperhopfian property is relevant. If \( p : M^{n+2} \to \mathbb{R}^2 \) is an \( N^n \)-shaped map which restricts to an approximate fibrations over \( \mathbb{R}^2 - \{\text{origin}\} \), it follows that \( p^{-1}(\text{origin}) \) is a shape deformation retract of \( M^{n+2} \) and \( \pi_1(M^{n+2} - p^{-1}(\text{origin})) \) surjects (via \( \text{inclusion}_\# \)) to \( \pi_1(M^{n+2}) \cong \pi_1(N^n) \). The homotopy exact sequence of the approximate fibration

\[
p| : M^{n+2} - p^{-1}(\text{origin}) \to \mathbb{R}^2 - \{\text{origin}\}
\]

assures that \( \pi_1(M^{n+2} - p^{-1}(\text{origin})) \) is a cyclic extension of \( \pi_1(N^n) \) by \( \mathbb{Z} \). If \( \text{inclusion}_\# \) does not carry that \( \pi_1(N^n) \) subgroup of \( \pi_1(M^{n+2} - p^{-1}(\text{origin})) \) onto \( \pi_1(M^{n+2}), \pi_1(N^n) \) will not be hyperhopfian.

**Theorem 3.4** [14]. Every Hopfian manifold \( N \) with hyperhopfian fundamental group is a codimension-2 \( o \)-fibrator.
Theorem 3.5 [23]. If \( \pi_1(N) \) is an Abelian 2-group, then \( N \) is a codimension-2 o-fibrator.

By deriving the result in case \( \pi_1(N) \) is a direct product of copies of the cyclic group of order \( 2^r \), Chinen was the first to suggest 3.5 might be true. He also obtained:

Theorem 3.6 [5]. Suppose \( \pi_1(N) \) is the direct product of an odd order group with a finite number of cyclic groups, all of the same order \( 2^r, r > 0 \), and suppose \( N_{\text{odd}} \) is the cover of \( N \) corresponding to the odd order subgroup. Then \( N \) is a codimension-2 fibrator provided \( N_{\text{odd}} \) is a codimension-2 o-fibrator.

Theorem 3.7 [23]. Suppose \( N \) is a Hopfian manifold such that either

- \( \pi_1(N) \) is residually finite, torsion-free and \( H_1(N) \) is a finite cyclic group, or
- \( \pi_1(N) \) is finite and \( H_1(N) \) is a direct sum of copies of \( \mathbb{Z}_2 \).

Then \( N \) is a codimension-2 o-fibrator.

Example 3.8 [11]. No closed manifold \( N \) which admits a regular, cyclic covering map \( \Theta : N \to N \) can be a codimension-2 fibrator. Let \( \theta : N \to N \) be a fixed point free action of the cyclic group \( \mathbb{Z}_m \) on \( N \) having \( N \) as its orbit space, and let \( \phi : \mathbb{R}^2 \to \mathbb{R}^2 \) be a rotation about the origin through angle \( 2\pi/m \). Consider the orbit space \( M^{n+2} \) of \( N^n \times \mathbb{R}^2 \) under the (free) \( \mathbb{Z}_m \)-action generated by \( \theta \times \phi \). The map \( p : M^{n+2} \to \mathbb{R}^2 \) induced by the composition of the projection \( N^n \times \mathbb{R}^2 \to \mathbb{R}^2 \) with the orbit map \( \mathbb{R}^2 \to \mathbb{R}^2/\phi \cong \mathbb{R}^2 \) is an \( N^n \)-like map which fails to be an approximate fibration. One can see that the retraction of \( M^{n+2} \) to the image of \( N \times 0 \) restricts to a degree \( m \) map on the other fibers of \( p \).

Let \( h : N^{n-1} \to N^{n-1} \) be a periodic homeomorphism; then the mapping torus \( N^n \) of \( h \) admits a regular, cyclic self-covering map \( \Theta : N^n \to N^n \) and, hence, fails to be a codimension-2 fibrator. Similarly, if \( W \) is a twisted I-bundle over a compact \( (n-1) \)-manifold, then \( N^n = W \cup \phi W \) (where \( \phi \) denotes the identity \( \partial W \to \partial W \)) admits a 2–1 covering \( \theta : N^n \to N^n \) and, again, fails to be a codimension-2 fibrator. In particular, \( RP^n \# RP^n \), which is the double of the twisted I-bundle over \( RP^n \), is not a codimension-2 fibrator.

Example 3.9. For certain Lens spaces \( L^n, L^n \times S^0 \) fails to be a codimension-2 o-fibrator [17]. Local knottedness plays a key role: if \( p : M^{n+2} \to B \) is an \( L^n \times S^0 \)-shaped map and all its fibers are closed \( n \)-manifolds which are locally flat in \( M^{n+2} \), then \( p \) must be an approximate fibration.

After Chinen [3,4] made substantial progress about codimension-2 fibrators in the non-orientable setting, Kim introduced the following useful property: an \( n \)-manifold \( N \) is said to be strongly Hopfian if, in the orientable case, \( N \) is Hopfian or, in the nonorientable case, \( N_H \) is Hopfian, where \( N_H \) denotes the cover of \( N \) determined by the intersection in \( \pi_1(N) \) of all its index-2 subgroups. Kim [42,41] derived analogs of Theorems 3.3 and 3.4 for strongly Hopfian manifolds.
Theorem 3.10 [42]. Every strongly Hopfian manifold $N$ such that $\pi_1(N)$ is residually finite and $\chi(N) \neq 0$ is a codimension-2 fiberator.

Theorem 3.11 [41]. Every strongly Hopfian manifold $N$ with hyperhopfian fundamental group is a codimension-2 fiberator.

4. Codimension-$k$ fibrators

Theorem 4.1 [26]. Every $(k - 1)$-connected closed manifold $N$ is a codimension-$k$ fiberator.

Corollary 4.2. $S^k$ is a codimension-$k$ fiberator.

Theorem 4.3 [20]. If $p : M \to B$ is an $N^n$-shaped map onto a finite-dimensional space $B$, then $B$ contains a dense open subset $V$ such that $p : p^{-1}(V) \to V$ is an approximate fibration.

Theorem 4.4 [29]. $B$ is locally simply connected.

A generalized $k$-manifold is a locally compact, finite-dimensional ANR $X$ such that $H_*(X, X - \{x\}; \mathbb{Z}) \cong H_*(\mathbb{R}^k, \mathbb{R}^k - \{\text{origin}\}; \mathbb{Z})$ for all $x \in X$.

Theorem 4.5 [20]. Let $p : M \to B$ be an $N^n$-shaped map from an $(n + k)$-manifold $M$ onto a finite-dimensional space $B$. Then $\dim B = k$. Furthermore, a necessary condition for $p$ to be an approximate fibration is that $B$ be a generalized $k$-manifold.

All generalized manifolds do arise in precisely this fashion. A generalized manifold $X^k$ can be a Cartesian factor of some manifold, in which case, for each closed $n$-manifold $N, n > 1, X^k \times N$ is also a manifold [30] and clearly $\text{proj} : X^k \times N \to X^k$ is a particularly nice $N$-shaped map; in any event, when $k \geq 3$ there is an $S^n$-like map $M^{n+k} \to X^k$ for some $n \leq k$.

$N$-shaped images need not all be generalized manifolds, however. For example, there is an $S^1$-like PL map from $S^1 \times \mathbb{R}^4$ onto the open cone over $RP^3$.

Theorem 4.6 [45]. Suppose $N$ is a Hopfian $n$-manifold, $\pi_1(N)$ is a Hopfian group, and $p : M \to B$ is an $N$-shaped map from an $(n + k)$-manifold onto a finite-dimensional space $B$. Then $B$ contains a dense open subset $V$ such that $B - V$ separates no connected open subset of $B$ and $p : p^{-1}(V) \to V$ is an approximate fibration.

Theorem 4.7 [24]. Suppose $N$ is a Hopfian $n$-manifold, $\pi_1(N)$ is a Hopfian group, and $p : M \to B$ is an $N$-shaped map from an $(n + k)$-manifold onto a finite-dimensional space $B$ such that the $i$th cohomology sheaf $\mathcal{H}^i[p; \mathbb{Z}]$ is locally constant for $i \leq k$. Then $p$ is an approximate fibration.

Corollary 4.8. A Hopfian manifold $N$ with Hopfian fundamental group is a codimension-$k$ fiberator if $H_i(N) \cong 0$ for $0 < i \leq k$. 
In fact, with a bit of extra work, one can also establish the codimension-$k$ fibration conclusion in 4.7 when $N$ merely satisfies $H_i(N) \cong 0$ for $0 < i < k$. Consequently, homology $n$-spheres (i.e., $n$-manifolds $\Sigma^n$ for which $H_\ast(\Sigma^n) \cong H_\ast(S^n)$) are essentially as effective as $S^n$ at inducing approximate fibrations.

**Corollary 4.9.** If the homology $n$-sphere $\Sigma^n$ is a Hopfian manifold and has Hopfian fundamental group, then $\Sigma^n$ is a codimension-$n$ fibration.

### 5. Codimension-$k$ PL fibrators

Here is a list of the known codimension-2 fibrators which fail to be PL fibrators: $S^m$; $RP^m$ [18]; the orientable $S^m$-bundle over $RP^m$; those 3-manifolds $N^3$ covered by $S^3$ that arise as coset spaces with respect to the quaternionic group structure of $S^3$; and Cartesian products involving any of the above as a factor.

**Remark.** The collection of PL manifolds which are nonfibrators definitely exceeds the collection of PL manifolds which are not PL fibrators. No homology $n$-sphere $\Sigma^n$ is a codimension-$(n+1)$ fibration. The point is that, for the open cone $W$ on $\Sigma^n$, there is a proper $\Sigma^n$-shaped map $p: W \times \Sigma^n \to W$ whose point preimages are either $c \times \Sigma^n$, $c$ being the cone point of $W$, or $r \Sigma^n \times s$, $r \Sigma^n \subset W$ being one of the cone levels and $s \in \Sigma^n$. Clearly $p$ is not an approximate fibration, since every retraction $W \times \Sigma^n \to c \times \Sigma^n$ restricts to a degree 0 map on all other fibers $p^{-1}(w) = r \Sigma^n \times s$.

**Lemma 5.1** (Codimension reduction [13]). Let $p: M \to B$ be a PL $N$-shaped map defined on the PL $(n+k)$-manifold $M$ and $b \in B$. Then each $b \in B$ has a PL neighborhood $S = b \ast L \subset B$ such that $p^{-1}(S)$ is a regular neighborhood of $p^{-1}(b)$ in $M$ and $p^{-1}(L) = \partial(p^{-1}(S))$ is a PL $(n+k-1)$-manifold.

As a consequence, when seeking to determine whether a codimension-$(k-1)$ PL fibration has the same property in codimension-$k$, one can split the restricted retraction $R|: p^{-1}(c) \to p^{-1}(b)$ into two parts, one involving the approximate fibration $p|: p^{-1}(L) \to L$ and the other, the composition of the homotopy equivalence (deformation retraction) $R: p^{-1}(S) \to p^{-1}(b)$ with the inclusion $p^{-1}(L) \to p^{-1}(S)$. In this PL category, in order for $p$ to be an approximate fibration, $B$ must be not merely a generalized $k$-manifold, but an actual manifold—a topological manifold endowed with a triangulation as a simplicial complex, not necessarily a PL manifold. A significant component of any effort to establish codimension-$k$ PL fibration properties often is devoted to proving that all links $L$ of vertices $v \in B$ are homotopy $(k-1)$-spheres. Being certain that its image is a manifold is a powerful boost toward establishing that a map $p$ is an approximate fibration, as the subsequent section, like Section 3, makes evident.

**Lemma 5.2.** Let $p: M \to B$ be a PL $N$-like map defined on the PL $(n+k)$-manifold $M$ and $b \in B$. Then, for the PL neighborhood $S = b \ast L \subset B$ of Lemma 5.1,
inclusion_#: \pi_i(p^{-1}(L)) \to \pi_i(p^{-1}(S)) is an isomorphism for \( i \leq k - 2 \) and an epimorphism for \( i = k - 1 \).

A manifold \( N \) is \( t \)-aspherical if \( \pi_i(N) = 0 \) for \( 1 < i \leq t \) (i.e., if the universal cover of \( N \) is \( t \)-connected). If \( N \) is \( t \)-aspherical and \( f : N \to N \) induces a \( \pi_1 \)-isomorphism, then \( f_* : H_i(N) \to H_i(N) \) is an isomorphism whenever \( i \leq t \), by the Whitehead theorem.

**Theorem 5.3** [15]. If \( N \) is a codimension-2 PL o-fibrator, \( N \) is \( t \)-aspherical, and \( \pi_1(N) \) is finite, then \( N \) is a codimension-\((t + 1)\)-PL o-fibrator.

Chinen improved 5.3 to:

**Theorem 5.4** [4]. If \( N \) is a codimension-2 PL o-fibrator, \( N \) is \( (t - 1) \)-aspherical, and both \( \pi_t(N) \) and \( \pi_1(N) \) are finite, then \( N \) is a codimension-\((t + 1)\)-PL o-fibrator.

A group \( \Gamma \) is said to be cohopfian if every monomorphism \( \psi : \Gamma \to \Gamma \) is an automorphism; it is said to be normally cohopfian if every monomorphism \( \psi : \Gamma \to \Gamma \) with normal image is an automorphism; and it is said to be sparsely Abelian if every homomorphism \( \psi : \Gamma \to \Gamma \) with normal image and Abelian kernel is an automorphism. For brevity we say that \( \Gamma \) has Property NCSA if it is both normally cohopfian and sparsely Abelian. Except for \( \mathbb{Z}_2 \ast \mathbb{Z}_2 \), free products of Hopfian groups have Property NCSA. Property NCSA has a notable benefit, which can be seen by examining the homotopy exact sequence of \( p^{-1}(L) \to L \):

**Lemma 5.5.** Suppose \( N^n \) is a codimension-\((k - 1)\) PL fibrator; \( k > 2 \), \( \pi_1(N^n) \) has Property NCSA, and \( p : M \to B \) is a PL \( N \)-like map defined on a PL \((n + k)\)-manifold \( M \). Then each link \( L \) of a vertex \( b \) in \( B \) is simply-connected and, for all \( x \in L \), the inclusion-induced homomorphism \( \pi_1(p^{-1}(x)) \to \pi_1(p^{-1}(L)) \) is an isomorphism. Consequently, the usual retractions \( R : p^{-1}(S) \to p^{-1}(b) \), upon restriction, induce isomorphisms \( (R|_b) : \pi_1(p^{-1}(c)) \to \pi_1(p^{-1}(b)) \) for all \( c \in S \).

**Theorem 5.6** [22]. If \( \pi_1(N) \) has Property NCSA, where \( N \) is a strongly Hopfian, \( t \)-aspherical, codimension-2 fibrator (respectively, o-fibrator), then \( N \) is a codimension-\((t + 1)\) PL fibrator (respectively, o-fibrator).

**Remark.** \( S^{t+1} \times A \), where \( A \) denotes any aspherical (orientable) manifold, shows that 5.6 is sharp.

**Corollary 5.7.** If the closed, orientable, aspherical manifold \( N \) is a codimension-2 fibrator (respectively, o-fibrator) and \( \pi_1(N) \) has Property NCSA, then \( N \) is a PL fibrator (respectively, o-fibrator).

**Corollary 5.8.** Every closed, orientable, hyperbolic manifold \( N \) is a PL fibrator.

**Corollary 5.9.** If \( N_e \) is an aspherical \( n \)-manifold with Hopfian fundamental group \( (e = 1, 2) \), then \( N_1 \# N_2 \) is a codimension-\((n - 1)\) PL o-fibrator.
Theorem 5.10 [22]. If $\pi_1(N^n)$ has Property NCSA, where $N^n$ is a Hopfian, $t$-aspherical, codimension-$2$ fibrator satisfying any one of the following conditions:

- $\beta_s(N^n) > 0$ for some $s$, $n/2 \leq s \leq t$, or
- $\beta_{t+1}(N^n) > 1$ and $t \geq n/2$, or
- $\beta_{t+1}(N^n) > 2$ and $t + 1 = n/2$,

then $N^n$ is a codimension-$(2t + 2)$ PL o-fibrator. If, in addition, $\chi(N^n) \not= 0$, then $N^n$ is a codimension-$(2t + 3)$ PL o-fibrator.

Corollary 5.11. If $N_e$ is an orientable, aspherical $n$-manifold with Hopfian fundamental group $(e = 1, 2)$ and if either $\beta_s(N_1) > 0$ for some $s$, $1 < s < n - 1$, or $\beta_1(N_1) > 1$, then $N_1 \# N_2$ is a codimension-$(2n - 2)$ PL o-fibrator.

Theorem 5.12 [16]. Suppose (1) $N^n$ is a Hopfian $n$-manifold which is a codimension-$(k - 1)$ PL o-fibrator, (2) $\pi_1(N^n)$ is Hopfian and has Property NCSA, and (3) either $H^k(N^n)$ or $\bigcup_{i=2}^{k} H^i(N^n)$ is in the subring of $H^*(N^n)$ generated by $H^1(N^n)$. Then $N^n$ is a PL o-fibrator.

Corollary 5.13. If $N_1 \# N_2$ is a Hopfian $n$-manifold, where $N_1, N_2$ are PL manifolds with nontrivial, Hopfian fundamental groups and where $H^n(N_1)$ is in the subring of $H^*(N_1)$ generated by $H^1(N_1)$, then $N_1 \# N_2$ is a PL o-fibrator.

6. PL o-fibrators

The closed manifold $N$ is called a codimension-$k$ PL $m$-fibrator (respectively, mo-fibrator) if for every (respectively, orientable) PL $(n + k)$-manifold $M^{n+k}$ and every $N$-like PL map $p : M^{n+k} \to B$ to another p.l. triangulated manifold $B$, $p$ is an approximate fibration. The “$m$” of this abbreviation is meant to connote restricting to the setting of manifold target.

Lemma 6.1 [22]. If $N$ is a codimension-$(k - 1)$ PL mo-fibrator and either $H_{k-1}(N) \cong 0$ or $\beta_k(N) > 1$, then $N$ is a codimension-$k$ PL mo-fibrator.

Corollary 6.2 [44]. If $N^n$ is a codimension-$(n + 1)$ PL mo-fibrator, it is a PL mo-fibrator.

Theorem 6.3. If $N$ is a $t$-aspherical, codimension-2 PL fibrator and $\pi_1(N)$ is finite, then $N$ is a codimension-$(t + 1)$ PL $m$-fibrator.

Theorem 6.4 [22]. If the $n$-manifold $N$ is a codimension-$(k - 1)$ PL $m$-fibrator satisfying any one of the following conditions:

- $\beta_s(N) > 0$ for some $s$, $n/2 \leq s < k - 1$, 
Theorem 7.1 [16]. Every codimension-2 PL o-fibrator $N^3$ with $\pi_1(N^3) \neq \mathbb{Z}_2 * \mathbb{Z}_2$ is a codimension-3 PL o-fibrator.

Theorem 7.2 [16]. If $N^3$ is a codimension-2 PL o-fibrator such that $\beta_1(N^3) > 0$ and $\pi_1(N^3)$ is normally cohopfian, then $N^3$ is a codimension-4 PL o-fibrator.

Theorem 7.3 [21]. Every connected sum $N^3$ of nonsimply connected 3-manifolds with residually finite fundamental groups such that $\pi_1(N^3) \neq \mathbb{Z}_2 * \mathbb{Z}_2$ is a codimension-4 PL o-fibrator.

Theorem 7.4 [15]. Let $N^3$ be a closed aspherical 3-manifold possessing some geometric structure. Then $N^3$ is a PL fibrator if and only if it is a codimension-2 PL fibrator.
Theorems 7.3 and 7.4 combine to assure that, for the most part, if $N^3$ is a codimension-2 but not a codimension-4 PL o-fibrator, then $\pi_1(N^3)$ is finite.

Among orientable 4-manifolds, the orientable $S^2$-bundle over $P^2$ and products of $S^2$ with orientable surfaces are the only known codimension-2 but not codimension-3 fibrators. When $\pi_1(N^3)$ is finite and $\chi(N^4) \neq 0$, $N^4$ almost always has strong fibrator properties, because finite-sheeted covers of $N^4$ typically have high rank first or second homology groups.

**Theorem 7.5** [21]. Suppose the closed orientable $4$-manifold $N^4$ satisfies: $\pi_1(N^4) \neq 1$ is finite and $\beta_2(N^4) > 0$ in case $\pi_1(N^4) \cong \mathbb{Z}_2$. Then $N^4$ is a codimension-5 PL o-fibrator.

**Theorem 7.6** [21]. Suppose the closed orientable $4$-manifold $N^4$ satisfies: $\beta_1(N^4) = 0$; $\beta_2(N^4) > 2$; and $\pi_1(N^4)$ is Hopfian and has Property NCSA. Then $N^4$ is a codimension-5 PL o-fibrator.

Nontrivial connected sums of orientable 4-manifolds also tend to carry rich fibrator properties.

**Theorem 7.7** [21]. Suppose $N^4$ is a nontrivial connected sum $N_1 \# N_2$ of orientable 4-manifolds with residually finite fundamental groups.

- If $\pi_1(N^4)$ is finite, $N^4$ is a codimension-5 PL o-fibrator.
- If $\pi_1(N_1) \cong 1$ and $\pi_1(N_2)$ has Property NCSA, $N^4$ is a codimension-4 PL o-fibrator.
- If $\pi_1(N_e) \neq 1$ ($e = 1, 2$) and either $\beta_2(N^4) > 0$ or $H_2(N^4; \mathbb{Z}) \cong 0$, $N^4$ is a codimension-4 PL o-fibrator.

Furthermore, if $\chi(N^4) \neq 0$, then in any of the above cases $N^4$ is a codimension-5 PL o-fibrator.

8. Structural matters

Structural issues within this topic have been addressed rather infrequently. Early on, Liem [43] proved the following:

**Theorem 8.1** [43]. If $p : M^{n+1} \rightarrow S^1$ is an $S^n$-shaped map, $n \geq 5$, then $p$ can be approximated by a locally trivial $S^n$-bundle map.

Im found improvements to Theorem 8.1.

**Theorem 8.2** [40]. Suppose $N^n$ is an closed orientable $n$-manifold, $n \geq 5$, such that either (1) $\pi_1(N^n) = 1$, (2) $N^n$ is aspherical and $\pi_1(N^n)$ is poly $\mathbb{Z}$-cyclic or (3) $N^n$ is a Riemannian manifold with nonpositive sectional curvature and $\pi_1(N^n)$ is Hopfian. Then every $N^n$-shaped map $p : M^{n+1} \rightarrow B$ defined on an orientable $(n + 1)$-manifold $M^{n+1}$ can be approximated by a locally trivial $N^n$-bundle map.
Given an $N^n$-shaped approximate fibration $p : M^{n+1} \to S^1$ such that some $p^{-1}(z)$ is a copy of $N^n$ locally flatly embedded in $M^{n+1}$, one can split the domain along $p^{-1}(z)$ to produce an $s$-cobordism $(W^{n+1}, N_0, N_1)$, where $N_0, N_1$ are copies of $N^n$, and a homeomorphism $h : N_0 \to N_1$ such that $M^{n+1} = W^{n+1}/h$. Of course, when $n \geq 5$, vanishing of the obstruction to $W^{n+1}$ being a product $N_0 \times [0, 1]$ (e.g., when the Whitehead group $Wh(\pi_1(N^n))$ itself vanishes) implies $M^{n+1}$ is a locally trivial $N^n$-bundle over $S^1$. Husch’s example [39] of an approximate fibration over $S^1$ not homotopic to a fibration arises in this fashion, as an $W^{n+1}/h$ stemming from a nonproduct cobordism $(W^{n+1}, N_0, N_1)$.

**Theorem 8.3.** Suppose $p : N^3 \to B$ is an $S^1$-shaped map defined on an orientable $3$-manifold $N^3$ that contains no fake $3$-cell. Then $p$ can be approximated by (generalized) Seifert fibrations; in particular, $N^3$ itself is a (generalized) Seifert-fibered $3$-manifold.

This follows from a combination of Theorem 3.1 here and work of Coram–Duvall [9]. A (generalized) Seifert fibration is an $S^1$-bundle with singularities; specifically, it is a map $p : N^3 \to M^2$ between manifolds such that each $z \in M^2$ has a disk neighborhood $D$ whose preimage under $p$ is equivalent to $D \times S^1$, with $p^{-1}(z)$ corresponding to $z \times S^1$, and with $p| : (D - \{x\}) \times S^1 \to D - \{x\}$ a trivial $S^1$-bundle map. Allowed here, unlike with Seifert fibrations, is for the composite

$$\text{typical fiber} \hookrightarrow (D - \{x\}) \times S^1 \hookrightarrow D \times S^1 \to S^1$$

to be an inessential map.

**Theorem 8.4** [40]. If $p : M^{n+2} \to B$ is an $S^n$-shaped map, $n \geq 5$, then $p$ can be approximated by a locally trivial $S^n$-bundle map.

Analogs of 8.4 cannot hold in all codimensions, at least not without restrictions upon the image. The exotic (nonresolvable) generalized manifolds $X^n$ of [1], which cannot be Cartesian factors of manifolds—not even locally so—admit $S^n$-shaped surjections $M^{2n} \to X^n$ that cannot be approximated by locally trivial $S^n$-bundle maps.

**Theorem 8.5** [31]. If $p : M^{2+k} \to B^k$ is an $(S^1 \times S^1)$-shaped map between manifolds, $M^{2+k}$ orientable and $k \geq 6$, then $p$ can be approximated by a locally trivial $(S^1 \times S^1)$-bundle map.

**Theorem 8.6** [40]. Let $N^2$ denote a closed, orientable surface for which $\chi(N^2) < 0$. If $p : M^{2+k} \to B^k$ is an $N^2$-shaped approximate fibration between manifolds, $M^{2+k}$ orientable and $k \geq 3$, then $p$ can be approximated by a locally trivial $N^2$-bundle map.

**Corollary 8.7.** Let $N^2$ denote a closed, orientable surface for which $\chi(N^2) < 0$. If $p : M^{2+k} \to B^k$ is a PL $N^2$-like map, $M^{2+k}$ orientable and $k \geq 3$, then $p$ can be approximated by a locally trivial $N^2$-bundle map.

**Theorem 8.8** [40]. Let $N^n$ be a closed, Riemannian $n$-manifold with nonpositive sectional curvature, and let $B^k$ denote a contractible manifold. Then any $N^n$-shaped approximate...
fibration \( p : M^{n+k} \to B^k \), \( n + k \geq 5 \), can be approximated by a trivial \( N^n \)-bundle map. In particular, \( M^{n+k} \approx N^n \times B^k \).

**Corollary 8.9.** Let \( N^n \) be a closed, orientable, Riemannian \( n \)-manifold, \( n > 3 \), with non-positive sectional curvature. Then any \( N^n \)-shaped map \( p : M^{n+1} \to \mathbb{R}^1 \) defined on a non-compact, orientable \( (n+1) \)-manifold \( M \) can be approximated by a trivial \( N^n \)-bundle map. In particular, \( M^{n+1} \approx N^n \times \mathbb{R}^1 \).

In the compact setting one encounters an elementary obstruction to the existence of \( N \)-like PL maps.

**Theorem 8.10** [13]. For any \( N \)-like PL map \( p : M \to B \) defined on a closed PL manifold \( M \), \( \chi(M) = \chi(N) \cdot \chi(B) \).

Hughes [36] has studied the space of approximate fibrations \( \{ p : M \to B \} \) between specific \( M, B \).

**Theorem 8.11.** Let \( M^m, m \geq 5 \), and \( B \) be closed manifolds. Then an approximate fibration \( p : M^m \to B \) is homotopic to a locally trivial bundle projection if and only if \( p \) is homotopic through approximate fibrations to one.

Hughes [37] also developed an obstruction theory which answers the question: given a locally trivial fiber bundle \( q : E \to B \) between closed manifolds, \( \dim E = m \geq 5 \), when is a map \( f : M \to E \) from another closed \( m \)-manifold \( M^m \) to \( E \) homotopic to a homeomorphism? Hughes, Taylor and Williams have explored related matters in a series of far-reaching papers. Here is an easily stated sample result.

**Theorem 8.12** [38]. Suppose \( B \) is a manifold of nonnegative Riemannian curvature and the \( N \)-shaped approximate fibrations \( p_1 : M_1 \to B \) and \( p_2 : M_2 \to B \) are controlled homotopy equivalent. Then \( M_1 \) and \( M_2 \) are homeomorphic.

Two maps \( p_1 : E_1 \to B \) and \( p_2 : E_2 \to B \) with the same image space, \( B \), are controlled homotopy equivalent if there exist maps (homotopy equivalences) \( \psi : E_1 \to E_2 \) and \( \psi_2 : E_2 \to E_1 \) such that \( \psi_2 \psi_1, \psi_1 \psi_2 \) are homotopic to the appropriate identity maps via homotopies whose tracks have small images, under \( p_1 \) and \( p_2 \), respectively, in \( B \).

9. Unsolved problems

**Problem 9.1.** Is every orientable closed manifold Hopfian?

**Problem 9.2.** If \( \pi_1(N) \) is a 2-group, is \( N \) a codimension-2 fibrator?

**Problem 9.3.** If \( N \) fails to be a codimension-2 fibrator, does it admit a \( S^1 \)-shaped map \( p : N \to B \)?
Problem 9.4. Which manifolds \( N \) admit a regular, cyclic, self-covering map \( \theta : N \to N \)?

Problem 9.5. Among orientable manifolds is there any difference between the fibrators and the o-fibrators in codimension-2? I claimed otherwise in [12], but the construction given there simply does not work.

Problem 9.6. Is there a finite, noncyclic group \( \Gamma \) which acts freely on \( S^{k-1} \) and also acts freely on some codimension-2 fibrator \( N \) such that \( N/\Gamma \approx N \)? If so, \( N \) cannot be a codimension-\( k \) fibrator.

Problem 9.7. Are all rational homology \( n \)-spheres except \( \mathbb{RP}^n \# \mathbb{RP}^n \) (up to homotopy type) codimension-\( n \) fibrators? What about those covered by \( S^n \)?

Problem 9.8. Are there any fibrators? o-fibrators?

Problem 9.9. If \( N^n \) is a codimension-\((n + 1)\) PL fibrator, is it a PL fibrator?

Problem 9.10. If \( N \) is an orientable PL manifold with \( \chi(N) \) odd, is \( N \) a PL fibrator?

Problem 9.11. If \( \pi_1(N) \) is a nontrivial free product other than \( \mathbb{Z}_2 \ast \mathbb{Z}_2 \), is \( N \) a PL fibrator?

Problem 9.12. If the PL manifold \( N \) is a nontrivial connected sum, \( N \not\cong \mathbb{RP}^n \# \mathbb{RP}^n \), must \( N \) be a PL fibrator?

Problem 9.13. Is the product of codimension-\( k \) fibrators always a codimension-\( k \) fibrator? (Surely not).

Problem 9.14. Are all Lens spaces \( L^n \) codimension-2 fibrators? Example 3.9 assures that often \( L^n \times S^n \) is not a codimension-2 o-fibrator.

Problem 9.15. For \( n > 0 \) does \( \mathbb{R}^{n+k} \) admit any \( N^n \)-shaped maps? Are there (closed?) manifolds \( M \) which admit no \( N^n \)-shaped maps, \( n > 0 \)?

Problem 9.16. If \( p : M \to B \) is \( N \)-shaped and \( M \) is compact, is \( \chi(B) \) defined? If so, does \( \chi(M) = \chi(N) \cdot \chi(B) \)?

References


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