# On metric spaces and local extrema

Alessandro Fedeli\textsuperscript{a,\,*}, Attilio Le Donne\textsuperscript{b}

\textsuperscript{a} Department of Pure and Applied Mathematics, University of L’Aquila, 67100 L’Aquila, Italy  
\textsuperscript{b} Department of Mathematics, University of Rome “La Sapienza”, 00100 Rome, Italy

## Abstract

The main aim of this paper is to give a positive answer to a question of Behrends, Geschke and Natkaniec regarding the existence of a connected metric space and a non-constant real-valued continuous function on it for which every point is a local extremum. Moreover we show that real-valued continuous functions on connected spaces such that every family of pairwise disjoint non-empty open sets is of size $< |\mathbb{R}|$ are constant provided that every point is a local extremum.

© 2009 Elsevier B.V. All rights reserved.

## 1. Introduction

Recently, Behrends, Geschke and Natkaniec, motivated by a question posed by Wojcik in [4], showed that a real-valued continuous function on a connected separable metric space for which every point is a local extremum, is constant [1, Theorem 2]. In the same paper the authors noted that the projection $f : X \to \mathbb{R}$, $f(x, y) = x$, where $X$ is the unit square with the lexicographic order, is an example of a non-constant continuous map defined on a compact connected space for which every point is a local extremum and left open the question of the existence of a connected metric space $X$ and a function on $X$ with the same properties [1, Question 7].

In this paper we solve affirmatively this problem by constructing a non-constant real-valued continuous function on a metric connected space for which every point is a local extremum.

Moreover we clarify further this topic by solving also the other question posed in [1, Question 5]: let $X$ be a connected space such that every family of pairwise disjoint non-empty open sets is of size $< |\mathbb{R}|$. If $f : X \to \mathbb{R}$ is continuous and such that every $x \in X$ is a local extremum of $f$, does $f$ have to be constant?

Finally, we give a result regarding the behaviour of Darboux functions for which every point is a local extremum.

We refer the reader to [3] for topological terminology not explicitly given.

## 2. The results

**Example 1.** A connected metric space $(X, d)$ and a non-constant continuous function $\pi : (X, d) \to [0, 1]$ for which every point is a local extremum.

Let $T = ([0, 1] \times [0, 1] \times [0, 1]) \sim$, where $\sim$ is the equivalence relation given by $(x, r, i) \sim (y, s, j)$ whenever $x = 0 = y$, or $x = y$ and $(r, i) = (s, j)$. A generic element of $T$ will be denoted by $[x, r, i]$. Let $\rho$ be the metric on $T$ given by...
\[\rho([x, r, i], [y, s, j]) = |x - y| \quad \text{if } (r, i) = (s, j),\]
\[\rho([x, r, i], [y, s, j]) = \min\{1, x + y\} \quad \text{otherwise.}\]

The (non-separable) connected metric space \((T, \rho)\) is called hedgehog of spininess \(c = \text{card}([0, 1] \times [0, 1]).\)

Now set \(X = [0, 1] \times T\) and let \(d_1\) be the metric on \(X\) defined in the following way. Let \(p, q \in X, p = (t, [x, r, i]), q = (t', [y, s, j]).\)

Set \(d_1(p, q) = \rho([x, r, i], [y, s, j])\) whenever \(t = t'\), and \(d_1(p, q) = 1\) otherwise.

Moreover let \(d_2\) be the metric on \(X\) given by
\[d_2(t', [1, t, 1], (t'', [1, t, 1])) = \min\{1, (t' - t) + (t'' - t)\}\]
whenever \(t', t'' \geq t, t' \neq t''\),
\[d_2(t', [1, t, 0], (t'', [1, t, 0])) = \min\{1, (t' - t) + (t - t'')\}\]
whenever \(t', t'' \leq t, t' \neq t''\), and \(d_2(p, q) = 1\) for all other pairs \(p, q\) of distinct points.

Now let \(d\) be the greatest metric on \(X\) which is not greater than \(d_1\) and \(d_2\).

For the sake of completeness we give an explicit definition of this metric.

Let \(p, q \in X, p = (t, [x, r, i]), q = (t', [y, s, j]).\)

If \(t = t', \) set \(d(p, q) = \rho([x, r, i], [y, s, j]).\) If \(t \neq t', \) set \(d(p, q) = 1\) if \(p\) and \(q\) satisfy one of the following conditions:

(i) \((r, i) \neq (s, j);\)
(ii) \((r, i) = (s, j), i = 0 \text{ and at least one among } t \text{ and } t' \text{ is } > r;\)
(iii) \((r, i) = (s, j), i = 1 \text{ and at least one among } t \text{ and } t' \text{ is } < r.\)

In the remaining cases set \(d(p, q) = \min\{1, 2 - (x + y) + |r - t| + |r - t'|\}.

Note that \(d(p, q) \geq 1 - x\) whenever \(p = (t, [x, r, i]), q = (t', [y, s, j]), t \neq t'.\)

To show that \(d\) indeed is a metric it is enough to verify that the triangle inequality holds in the most relevant cases.

So let \(p_1, p_2, p_3 \in X, p_1 = (t_1, [x_1, r_1, i_1]), p_2 = (t_2, [x_2, r_2, i_2])\) and \(p_3 = (t_3, [x_3, r_3, i_3]).\) Let us show that \(d(p_1, p_2) \leq d(p_1, p_3) + d(p_2, p_3).\)

Clearly we may assume that \(d(p_1, p_2) \neq 1\) and \(d(p_2, p_3) \neq 1\) (because \(d\) is bounded by 1). This assumption yields that \((r_1, i_1) = (r_2, i_2)\) whenever \(t_1 \neq t_2\), and \((r_2, i_2) = (r_3, i_3)\) whenever \(t_2 \neq t_3\).

(i) \(t_1, t_2, t_3\) are distinct points. In this case we have \(r_1 = r_2 = r_3 = r\) and \(i_1 = i_2 = i_3 = 0\) (the case \(i_1 = i_2 = i_3 = 1\) is similar). Moreover observe that \(t_1, t_2, t_3\) are \(\leq r\).

So \(d(p_1, p_2) + d(p_2, p_3) = (1 - x_1) + (1 - x_2) + (r - t_1) + (r - t_2) + (1 - x_2) + (1 - x_3) + (r - t_2) + (r - t_3) \geq (1 - x_1) + (1 - x_3) + (r - t_1) + (r - t_3) \geq d(p_1, p_3).\)

(ii) \(t_1 = t_2 \neq t_3.\) In this case \((r_2, i_2) = (r_3, i_3) = (r, i),\) moreover it is not restrictive to set \(i = 0.\) Since \(d(p_1, p_3) \neq 1,\) it follows that \(t_1 = t_2 < t_3.\)

If \((r_1, i_1) \neq (r, 0),\) then \(d(p_1, p_3) = 1, d(p_1, p_2) = x_1 + x_2\) (recall that \(d(p_1, p_2) \neq 1\) and \(d(p_2, p_3) \geq 1 - x_2,\) so we are done.

If \((r_1, i_1) = (r, 0),\) then \(d(p_1, p_2) = |x_1 - x_2|\). Since \(d(p_2, p_3) \neq 1,\) it follows that \(t_3 \leq r.\) Therefore \(d(p_1, p_3) = \min\{1, 2 - (x_1 + x_3) + |r - t_1| + |r - t_3|\}.

If \(x_1 \leq x_2,\) then \(d(p_1, p_3) \leq (1 - x_1) + (1 - x_3) + (r - t_1) + (r - t_3) = (x_2 - x_1) + (1 - x_2) + (1 - x_3) + (r - t_2) + (r - t_3) = d(p_1, p_2) + d(p_2, p_3).\)

If \(x_2 > x_1,\) then \(d(p_1, p_3) \leq (1 - x_1) + (1 - x_3) + (r - t_1) + (r - t_3) < (1 - x_2) + (1 - x_3) + (r - t_2) + (r - t_3) = d(p_2, p_3).\)

(iii) \(t_2 = t_3 \neq t_1.\) This case is similar to the previous one.

(iv) \(t_1 = t_3 \neq t_2.\) Observe that \((r_1, i_1) = (r_2, i_2) = (r_3, i_3).\) We have \(d(p_1, p_3) = |x_1 - x_3| \leq (1 - x_1) + (1 - x_3) \leq d(p_1, p_2) + d(p_2, p_3).\)

We claim that the metric space \((X, d)\) is connected.

First observe that for every \(t \in [0, 1],\) the subspace \([t] \times T\) of \((X, d)\) is isometric to the hedgehog \((T, \rho).\)

Let us suppose that \((X, d)\) is disconnected. Then there is a pair of non-empty open subsets \(C\) and \(D\) of \((X, d)\) such that \(C \cap D = \emptyset\) and \(C \cup D = X.\)

Since every \([t] \times T\) is a connected subspace of \((X, d)\) there are some \(A, B \subset [0, 1]\) such that \(C = A \times T, D = B \times T, A \cap B = \emptyset, A \cup B = [0, 1], 0 \in A \text{ and } B \neq \emptyset.\)

Now let \(k = \inf B.\) If \(k \in A,\) let us consider a sequence \((b_n)_{n \in N}\) in \(B\) converging to \(k,\) and set \(p = (k, [1, k, 1]), q_n = (b_n, [1, k, 1])\) for every \(n \in N.\) Then \(p \in A \times T, q_n \in B \times T\) for every \(n \in N\) and the sequence \((q_n)_{n}\) converges to \(p.\) So \(A \times T\) is not closed, a contradiction.

The case \(k \in B\) can be treated in a similar way. First observe that \(k > 0.\) Now let \((a_n)_{n}\) be a sequence in \(A\) converging to \(k\) such that \(a_n < k\) for every \(n.\) Then \(p = (k, [1, k, 0]) \in B \times T, q_n = (a_n, [1, k, 0]) \in A \times T\) for every \(n \in N\) and the sequence \((q_n)_{n}\) converges to \(p.\) So \(A \times T\) is not closed, a contradiction. Hence \((X, d)\) is connected.
Now let \( \pi : (X, d) \rightarrow [0, 1] \) be the projection given by \( \pi ((t, \{(x, r, i)\})) = t \).
To show the continuity of \( \pi \), it is enough to verify that \( |\pi(p) - \pi(q)| \leq d(p, q) \) for every \( p, q \in X \).
So let \( p, q \in X \), \( p = (t, \{x, r, i\}) \), \( q = (t', \{y, s, j\}) \). If either \( d(p, q) = 1 \) or \( t = t' \) we are done. Otherwise we have
\[
|\pi(p) - \pi(q)| = |t - t'| \leq |r - t| + |r - t'| \leq (1 - y) + (1 - y) + |r - t| + |r - t'| = d(p, q).
\]
It remains to show that every point of \( X \) is a local extremum.
Let \( p \in X \), \( p = (t, \{x, r, i\}) \). If

(a) \( x \neq 1 \), then \( \{t\} \times T \) is a neighbourhood of \( p \), in fact \( d(p, q) \geq 1 - x \) for every \( q = (t', \{y, s, j\}) \) with \( t' \neq t \). So \( \pi \) is locally constant at \( p \);
(b) \( x = 1 \) and \( i = 1 \), then \( \{1\} \times T \) is a neighbourhood of \( p \) (it is enough to observe that for every \( q \in \{1\} \times T \) such that \( t' < t \), we have \( d(p, q) = 1 \) whenever \( t = r \), and \( d(p, q) > |t - r| \) whenever \( t \neq r \) such that \( f(p) \leq f(q) \) for every \( q \in \{1\} \times T \), so \( p \) is a local minimum;
(c) \( x = 1 \) and \( i = 0 \), then as in the previous case one can see that now \( [0, 1] \times T \) is a neighbourhood of \( p \) such that \( f(q) \leq f(p) \) for every \( q \in [0, 1] \times T \), so \( p \) is a local maximum.

The proof is complete.

**Proposition 2.** Let \( X \) be a connected space such that every family of pairwise disjoint non-empty open sets is of size \( < |R| \), and let \( f : X \rightarrow R \) be a continuous function for which every point of \( X \) is a local extremum. Then \( f \) is constant.

**Proof.** Suppose that \( f \) is not constant. Then there are \( a, b \in R \), \( a < b \), such that \( [a, b] \subset f(X) \). Let us set, for every \( c \in ]a, b[ \),
\[
S(c) = \{ x \in X : f(x) = c, \ f \text{ is locally constant at } x \}.
\]
We claim that \( S(c) \neq \emptyset \) for every \( c \in ]a, b[ \).

If not, let us take some \( c \in ]a, b[ \) for which \( S(c) = \emptyset \) and set:

(i) \( \min(c) = \{ x \in X : f(x) = c, \ x \text{ is a local minimum} \}, \)
(ii) \( \max(c) = \{ x \in X : f(x) = c, \ x \text{ is a local maximum} \}, \)
(iii) \( A = f^{-1}((c, +\infty)) \cup \min(c), \)
(iv) \( B = f^{-1}((\infty, c]) \cup \max(c). \)

Now let us show that \( A \) is an open subset of \( X \). Since \( f^{-1}((c, +\infty)) \) is open in \( X \), it is enough to show that for every \( x \in \min(c) \) there is an open neighbourhood \( U \) of \( x \) in \( X \) such that \( U \subset A \). So let us take some \( x \in \min(c) \) and let \( U \) be an open neighbourhood of \( x \) such that \( f(x) > f(y) \) for every \( y \in U \). Therefore for every \( y \in U \) we have \( f(y) > f(x) = c \), i.e., \( y \in f^{-1}((c, +\infty)) \) or \( f(y) = c \), in which case \( y \in \min(c) \). Hence \( U \subset A \) and \( A \) is open in \( X \). In a similar way it can be shown that \( B \) is also open in \( X \).

Since \( a, b \in f(X) \), there are some \( p, q \in X \) such that \( f(p) = a < c \) and \( f(q) = b > c \). Therefore \( p \in B \) and \( q \in A \). Hence \( A \) and \( B \) are two non-empty open subsets of \( X \) such that \( A \cup B = X \) and \( A \cap B = \emptyset \). Since \( X \) is connected, this is a contradiction. Hence \( S(c) \neq \emptyset \) for every \( c \in ]a, b[ \).

Now it is enough to observe that:

(i) every \( S(c) \) is open in \( X \);
(ii) \( S(c) \cap S(c') = \emptyset \) whenever \( c \neq c' \).

Therefore \( \{S(c) : c \in ]a, b[ \} \) is a family of pairwise disjoint non-empty open subsets of \( X \) of size \( |R| \). Hence \( f \) must be constant. \( \Box \)

Recall that a map \( f \) between two topological spaces \( X \) and \( Y \) is called Darboux if \( f(C) \) is a connected subspace of \( Y \) for every connected subspace \( C \) of \( X \) (see, e.g., [2]). The proof of Theorem 2 in [1] shows that the continuity hypothesis can be weakened to the assumption that the map be Darboux.

Our last example will shed some light on the class of Darboux functions for which every point is a local extremum.

**Example 3.** A connected separable Hausdorff space \( (R, \tau) \) and a non-constant Darboux function \( f : (R, \tau) \rightarrow [0, 1] \) for which every point is a local extremum.

Let \( \tau \) be the topology on the set of real numbers \( R \) generated by the following base:
\[
B = \{(]a, b[ \cap Q) \cup K : a, b \in R, \ a < b, \ K \subset ]a, b[\},
\]
where \( Q \) is the set of rational numbers.
\((R, \tau)\) is a separable Hausdorff space, moreover it can be shown, as in the case of the real line endowed with the euclidean topology, that the connected subsets of \((R, \tau)\) are precisely the intervals.

Now let \(\{I_n: n \in \mathbb{N}\}\) be an enumeration of all non-empty intervals \([a, b]\) with rational endpoints, and let \([x] = x + Q\) for every \(x \in R\).

We claim that there is some \(S \subset R\) such that \(|S \cap \{x\}| = 1\) for every \(x \in R\) and \(|S \cap I_n| = 2^\aleph_0\) for every \(n \in \mathbb{N}\).

Let \(\{[x_\alpha]\}: \alpha \in 2^{\aleph_0}\) be an enumeration of \(R/Q\) and let \(h: 2^{\aleph_0} \times \mathbb{N} \rightarrow 2^{\aleph_0}\) be a bijection. For each \(\alpha \in 2^{\aleph_0}\) and \(n \in \mathbb{N}\) let \(x(\alpha, n) \in [x_\alpha \cap I_n]\), and set \(S = \{x(\alpha, n): \alpha \in 2^{\aleph_0}, n \in \mathbb{N}\}\). Then \(|S \cap \{x\}| = 1\) for every \(x \in R\) and \(|S \cap I_n| = |\{x(\alpha, n): \alpha \in 2^{\aleph_0}\}| = 2^{\aleph_0}\) for every \(n \in \mathbb{N}\).

Clearly \(S + q\) satisfies the same properties as \(S\) for every \(q \in Q\). In fact, let \(\{q_n: n \in \mathbb{N}\}\) be an enumeration of \(Q\) and set \(S_n = S + q_n\). Then \(\{S_n: n \in \mathbb{N}\}\) is a partition of \(R\) and \(|S_n \cap I_n| = |(S + q_n) \cap I_n| = |S \cap (I_n - q_n)| = 2^{\aleph_0}\) for every \(n \in \mathbb{N}\).

Now let \(g_n: S_n \cap I_n \setminus Q \rightarrow [0, 1]\) be a bijection for every \(n \in \mathbb{N}\), and let us consider the map \(f: (R, \tau) \rightarrow [0, 1]\) defined in the following way: \(f|_{S_n \cap I_n \setminus Q} = g_n\) for every \(n \in \mathbb{N}\), \(f(Q) = \{1\}\), in the remaining points \(f\) is defined arbitrarily. It is clear that every point is a local minimum for \(f\). Moreover \(f(I_n) = f(S_n \cap I_n) = [0, 1]\) for each \(n\), therefore \(f\) is a Darboux function.

Acknowledgement

The authors are grateful to the referee for several helpful suggestions which improved the exposition of the paper.

References