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A Sturmian sequence related to the uniqueness conjecture for Markoff numbers

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To Juhani Karhumäki for his 60th birthday

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ABSTRACT

Sturmian sequences appear in the work of Markoff on approximations of real numbers and minima of quadratic functions. In particular, Christoffel words, or equivalently pairs of relatively prime nonnegative integers, parametrize the Markoff numbers. It was asked by Frobenius if this parametrization is injective. We answer this conjecture for a particular subclass of these numbers, and show that a special Sturmian sequence of irrational slope determines the order of the Markoff numbers in this subclass.

1. Introduction

Sturmian sequences have been intensively studied in combinatorics on words these last years. For an exposition of this theory, see [10]. They are intimately related to continued fractions and discretization of straight lines: indeed, the continued fraction of the slope of the line gives a way to compute the Sturmian sequence that discretizes the given line. But there is a more subtle relation to continued fractions, discovered by Markoff.

Indeed, in his theory of minima of indefinite binary quadratic forms and Diophantine approximations of real numbers, Markoff [11] introduced his famous integers, now called Markoff numbers. They are parametrized by pairs (p, q) of relatively prime natural numbers (equivalently by nonnegative rational numbers, together with ∞). The uniqueness conjecture for Markoff numbers, first stated as an open problem by Frobenius ([8, p. 614]), is the claim that this parametrization is injective.

If the conjecture is true, then a natural question to ask is what total order on these pairs (p, q) is induced by the natural order of the Markoff numbers. As far as we know, no result is known on this problem. Our main result answers the conjecture and this problem in a particular case. It is the case of the Markoff numbers parametrized by the pairs $(m, 1)$ and $(1, n)$, $n, m \geq 2$. We show that these numbers are all distinct and that their order is determined (in a natural sense explained below; see Fig. 1) by a special Sturmian sequence, whose slope is the irrational number $\log\left(\frac{1+\sqrt{5}}{2}\right) / \log(1+\sqrt{2})$. It is a striking fact that Sturmian sequences, which appear centrally in the definition of Markoff numbers (compare [5, p. 28–30], [13], [10, Th. 2.1.5][15, Th. 3.1]), appear again here.

Our particular case for the conjecture differs from the cases considered before: we take a certain subset of all Markoff numbers based on the parametrization, and prove that the values are distinct. In the literature, the emphasis has been more on proving uniqueness based on the property of the Markoff number itself (as opposed to its parametrization): if it is prime, a prime power, twice a prime power, etc. See [2,19] and the references therein.

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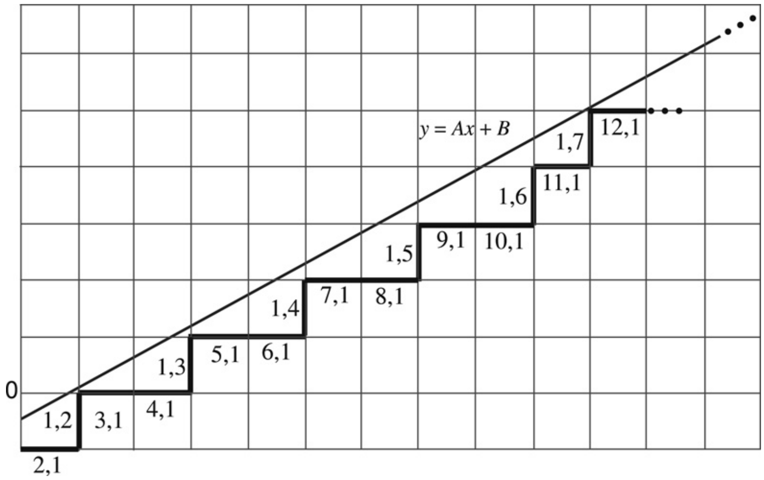

Fig. 1. Ordering the Markoff numbers $M(m, 1)$ and $M(1, n)$, $m, n \geq 2$, through the Sturmian sequences associated with a certain irrational half-line.

In order to prove that the Markoff numbers parametrized by the pairs $(m, 1)$ and $(1, n)$ are distinct, we note, after Frobenius, that they lie in two binary recurring sequences $(u_m)_{m\geq 0}$ and $(v_n)_{n\geq 0}$. The first sequence is the sequence of odd-indexed Fibonacci numbers F_n, and the second of odd-indexed Pell numbers P_n; precisely, normalize these two classical sequences so that $F_0 = P_0 = 1$, $F_1 = P_1 = 1$, with the recursions $F_{n+2} = F_{n+1} + F_n$ and $P_{n+2} = 2P_{n+1} + P_n$ (cf. the Sloane On-Line Encyclopedia of Integer Sequences [16]); then $u_m = F_{2m+3}$ and $v_n = P_{2n+5}$. We have

$$(u_m)_{m\geq 1} = 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657, 75025, 196418, \ldots$$

Furthermore

$$(v_n)_{n\geq 1} = 29, 169, 985, 5741, 33461, 195025, \ldots$$

We show that the only common value to these two sequences is $u_0 = v_0 = 5$; for this, we use Baker's theory of linear forms in the logarithms of algebraic numbers, and more precisely a powerful refinement of his theory due to Matveev [12]. We then reduce the astronomical bounds obtained to something more manageable, using a form of the Baker–Davenport lemma.

We thank the two referees for many suggestions, which helped us to improve the present article.

2. The conjecture and another problem

Markoff numbers $M(i, j)$ are natural integers which are parametrized by two relatively prime natural integers i, j. See [4–6,8,11,18]. One way to define Markoff numbers is as follows: they are the positive integer solutions to the equation $x^2 + y^2 + z^2 = 3xyz$. The parametrization by pairs (i, j) of relatively prime integers was introduced by Frobenius [8], see e.g., [5, p. 24]. This parametrization coincides with the parametrization of the Markoff tree (see [5, p. 19]), which is an infinite binary tree, when this tree is identified with the Stern–Brocot tree (see [9, p. 117]); recall that the latter, which is obtained by a process generalizing Farey sequences, contains in its nodes exactly once each positive rational number. The *uniqueness conjecture for Markoff numbers* states that the function $(i, j) \mapsto M(i, j)$ is injective. If this conjecture is true, a natural question is to determine the total order on the pairs (i, j) induced by the $M(i, j)$. Equivalently, what is the order on the positive rational numbers $\frac{j}{i}$, including ∞, induced by the Markoff numbers?

We answer these questions for the special case where $(i, j) = (m + 1, 1)$ or $(1, n + 1)$, $m, n \geq 0$. For this let $u_m = M(m + 1, 1)$ and $v_n = M(1, n + 1)$ and consider the set $E = \{u_m \mid m \geq 1\} \cup \{v_n \mid n \geq 1\}$.

Let $E = \{e_0 < e_1 < e_2 < \cdots\}$ and define the sequence $(a_k)_{k\geq 0}$ of 0 and 1's as follows: $a_k = 0$ if e_k is among the numbers u_m, $m \geq 1$, and $a_k = 1$ if e_k is among the numbers v_n, $n \geq 1$. Of course, the sequence (a_n) is well defined only if the sequences u_m, $m \geq 1$ and v_n, $n \geq 1$ have no common value; equivalently if the Markoff numbers $M(m + 1, 1)$ and $M(1, n + 1)$ are distinct for $n, m \geq 1$. Note that the sequences (u_n) and (v_m) are strictly increasing, so that the knowledge of the sequence (a_n) completely determines the order of the set E. We have

$$E = \{13, \mathbf{29}, 34, 89, \mathbf{169}, 233, 610, \mathbf{985}, 1597, 4181, \mathbf{5741}, 10946, 28657, \mathbf{33461}, 75025, \mathbf{196025}, 196418, \ldots\},$$

where the elements of the sequence (u_m) are in bold, and therefore

$$(a_n)_{n\geq 0} = 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, \ldots$$

Our main result is the following. For definitions and properties of Sturmian sequences, see [10].

Theorem 1. (a) *The Markoff numbers* $M(m+1, 1)$ *and* $M(1, n+1)$ *are distinct for* $n, m \geq 1$.
(b) *The sequence* $(a_k)_{k\geq 0}$ *is a Sturmian sequence.*

The proof will show that $(a_k)_{k\geq 0}$ is the *lower Christoffel sequence* (see [3]) associated with the half-line $y = Ax + B$, with $A = \frac{\log\phi_1}{\log\phi_2} \approx 0.5459\ldots$ and $B = \frac{\log(\delta_1\phi_1/\delta_2\phi_2)}{\log\phi_2} \approx -0.4557\ldots$, see Fig. 1; here, δ_1, δ_2, ϕ_1, ϕ_2 are the quadratic numbers defined in Section 3.3. It is a consequence of Baker's theory (see e.g., [17, p. 3]) that the slope A is transcendental. With the notation of [10, p. 53], the sequence $(a_k)_{k\geq 0}$ is equivalently equal to the *mechanical word* $s_{\alpha,\rho}$, with $\alpha = \frac{A}{A+1}$, $\rho = \frac{A+B+1}{A+1}$.

3. Proof of the main theorem, part 1

In this section we prove the first part of the Theorem. Before we do this we need some tools from Diophantine approximation.

3.1. The Baker–Davenport lemma

To begin, we need a certain version of the celebrated Baker–Davenport Lemma [1]. The one we give here is adapted from [7]. We let $\|x\|$ denote the distance from the real number x to the nearest integer.

Theorem 2. *Let* M *be a positive integer and* κ, μ, A, B *be real numbers satisfying* $\kappa > 0$, $A > 0$, $B > 1$. *Let* p, q *be positive integers satisfying*

$$|q\kappa - p| \leq \alpha, \tag{1}$$

$$\|\mu q\| > M\alpha, \tag{2}$$

for some real $\alpha > 0$. *Write* $\varepsilon = \|\mu q\| - M\alpha$. *Then the inequality*

$$0 < m\kappa - n + \mu < AB^{-m} \tag{3}$$

has no solution in integers m, n *with* $\frac{\log(Aq/\varepsilon)}{\log B} \leq m \leq M$.

Proof. Suppose that (3) holds with $0 \leq m \leq M$. Multiplying by q and rearranging a little we obtain

$$0 < [(mp - nq) + \mu q] + m(q\kappa - p) < qAB^{-m}.$$

Hence

$$\begin{aligned} qAB^{-m} &> |(mp - nq) + \mu q| - m|q\kappa - p| \\ &\geq \|\mu q\| - m\alpha \\ &\geq \|\mu q\| - M\alpha = \varepsilon > 0. \end{aligned}$$

Thus $\log(qA) - m\log B > \log\varepsilon$, which implies $m < \log(qA/\varepsilon)/\log(B)$. □

Now let K_1, K_2 be real quadratic fields (identified with fixed embeddings into $\mathbb{R}$). Let ϕ_1, ϕ_2 be respectively fundamental units of K_1, K_2, both chosen to be greater than 1. Let $\delta_i \in K_i$, $\delta_i > 0$ and let

$$u_m = \delta_1\phi_1^m + \bar{\delta}_1\bar{\phi}_1^m, \qquad v_n = \delta_2\phi_2^n + \bar{\delta}_2\bar{\phi}_2^n,$$

where $\bar{\cdot}$ means conjugation within K_i. We shall use the shorthand

$$\delta = \min(\delta_1, \delta_2), \quad \delta' = |\bar{\delta}_1| + |\bar{\delta}_2|$$

and

$$\phi = \min(\phi_1, \phi_2).$$

Our objective is to solve the equation

$$u_m = v_n, \qquad m,\ n \geq 0,$$

and we assume that this equality holds throughout. We thus have

$$\begin{aligned} |\delta_1\phi_1^m - \delta_2\phi_2^n| &= |\bar{\delta}_1\bar{\phi}_1^m - \bar{\delta}_2\bar{\phi}_2^n| \\ &\leq |\bar{\delta}_1| + |\bar{\delta}_2|, \end{aligned}$$

since $|\bar{\phi}_1|, |\bar{\phi}_2| < 1$, because $\pm 1 = \mathrm{Norm}(\phi_i) = \phi_i\bar{\phi}_i$. Hence

$$|\delta_1\phi_1^m - \delta_2\phi_2^n| \leq \delta'. \tag{4}$$

Lemma 3.1. *If* $\delta_1\phi_1^m \geq \frac{3}{2}\delta_2\phi_2^n$ *(resp.* $\delta_2\phi_2^n \geq \frac{3}{2}\delta_1\phi_1^m$*), then* m *(resp.* n*) is* $\leq \frac{\log(3\delta'/\delta)}{\log\phi}$.

Proof. Suppose that $\delta_1\phi_1^m \geq \frac{3}{2}\delta_2\phi_2^n$. Then (4) implies $\delta_1\phi_1^m \leq \delta_2\phi_2^n + \delta'$; thus $\frac{3}{2}\delta_2\phi_2^n \leq \delta_2\phi_2^n + \delta'$, which implies $\frac{1}{2}\delta_2\phi_2^n \leq \delta'$. Moreover $\delta_1\phi_1^m = \delta_2\phi_2^n + (\delta_1\phi_1^m - \delta_2\phi_2^n) \leq 3\delta'$. We conclude that $m\log\phi_1 + \log\delta_1 \leq \log(3\delta')$ and the bound on m follows, since $\phi_1 \geq \phi > 1$ and $\delta_1 \geq \delta$. The other inequality is proved similarly. □

Lemma 3.2. *Suppose*

$$\delta_2\phi_2^n < \delta_1\phi_1^m < \frac{3}{2}\delta_2\phi_2^n. \tag{5}$$

Then m, n satisfy the inequality (3) *with*

$$\kappa = \frac{\log\phi_1}{\log\phi_2}, \qquad \mu = \frac{\log(\delta_1/\delta_2)}{\log\phi_2}, \qquad A = \frac{3\delta'}{2\delta_1\log\phi_2}, \qquad B = \phi_1.$$

Proof. From (5) we deduce

$$0 < \frac{\delta_1\phi_1^m}{\delta_2\phi_2^n} - 1.$$

Then from (4), we deduce

$$0 < \frac{\delta_1\phi_1^m}{\delta_2\phi_2^n} - 1 \leq \frac{\delta'}{\delta_2\phi_2^n}.$$

Now, $\log(1+x) < x$ for $x > 0$; thus

$$\begin{aligned} 0 &< \log\left(\frac{\delta_1\phi_1^m}{\delta_2\phi_2^n}\right) = \log\left(1 + \frac{\delta_1\phi_1^m}{\delta_2\phi_2^n} - 1\right) \\ &< \frac{\delta_1\phi_1^m}{\delta_2\phi_2^n} - 1 \leq \frac{\delta'}{\delta_2\phi_2^n} < \frac{3\delta'}{2\delta_1\phi_1^m}, \end{aligned}$$

where the last inequality follows from the hypothesis. We obtain now

$$0 < m\log\phi_1 - n\log\phi_2 + \log(\delta_1/\delta_2) < \frac{3}{2}\frac{\delta'}{\delta_1\phi_1^m}.$$

Dividing by $\log\phi_2$, we obtain

$$0 < m\kappa - n + \mu < AB^{-m},$$

which was to be proved. □

3.2. The bounds of Matveev

Let $\mathbb{L}$ be a number field of degree D, let $\alpha_1, \ldots, \alpha_k$ be non-zero elements of $\mathbb{L}$ and $b_1, \ldots, b_k$ be rational integers. Set

$$B = \max\{|b_1|, \ldots, |b_k|\},$$

and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_k^{b_k} - 1.$$

For an algebraic integer α whose minimal polynomial over $\mathbb{Z}$ is of the form $P(X) = a\prod_{i=1}^d (X - \alpha^{(i)})$, we write $h(\alpha)$ for its *logarithmic height*, that is,

$$h(\alpha) = \frac{1}{d}\left(\log|a| + \sum_{i=1}^{d}\log(\max\{1, |\alpha^{(i)}|\})\right).$$

Let $A_1, \ldots, A_k$ be real numbers with

$$A_j \geq \max\{D\,h(\alpha_j), |\log\alpha_j|, 0.16\}$$

for $j = 1, \ldots, k$.

Baker's theory of linear forms in logarithms gives a lower bound for $|\Lambda|$, provided that $\Lambda \neq 0$. We shall use the following recent result of Matveev [12].

Theorem 3 (*Matveev*). *If Λ is non-zero and $\mathbb{L}$ a real field, then*

$$\log|\Lambda| > -1.4 \cdot 30^{k+3} k^{4.5} D^2 A_1 \cdots A_k (1 + \log D)(1 + \log B).$$

3.3. Proof of the main theorem, part (a)

With the tools above we are now ready to prove the first part of our main theorem.

Proof. We take

$$\phi_1 = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2, \phi_2 = 3+2\sqrt{2} = (1+\sqrt{2})^2, \delta_1 = \frac{25+11\sqrt{5}}{10}, \delta_2 = \frac{10+7\sqrt{2}}{4}. \quad (6)$$

Then one has, by Frobenius [8, p. 616–617]:

$$u_m = \delta_1\phi_1^m + \bar{\delta}_1\bar{\phi}_1^m, \qquad v_n = \delta_2\phi_2^n + \bar{\delta}_2\bar{\phi}_2^n. \quad (7)$$

Let $\Lambda := (\delta_2/\delta_1)\phi_2^n\phi_1^{-m} - 1$. Assume that $u_m = v_n$ and assume also that $(m, n) \neq (0, 0)$. We have

$$\delta_2\phi_2^n/\delta_1\phi_1^m - 1 = (\bar{\delta}_1\bar{\phi}_1^m - \bar{\delta}_2\bar{\phi}_2^n)/\delta_1\phi_1^m;$$

now, $\bar{\delta}_1, \bar{\phi}_1, \bar{\delta}_2, \bar{\phi}_2$ are all positive and smaller than 1, and δ_i is > 2. Thus we get $|\Lambda| \leq \phi_1^{-m}$, that is $\log|\Lambda| \leq -m\log\phi_1$.

We now apply Theorem 3 to bound $|\Lambda|$ from below: we take $k = 3$, $\mathbb{L} = \mathbb{Q}[\sqrt{2}, \sqrt{5}]$, $D = 4$, $\alpha_1 = \delta_2/\delta_1$, $\alpha_2 = \phi_2$, $\alpha_3 = \phi_1$, $b_1 = 1$, $b_2 = n$, $b_3 = -m$. Note that by what has been shown above, we have $\delta_2\phi_2^n/\delta_1\phi_1^m < 2$. Then a numerical computation, using the fact that ϕ_2 is greater than ϕ_1, implies that $n \leq m$. Thus, with the choices made, we have $B = \max\{|b_1|, |b_2|, |b_3|\} = m$.

Hence Theorem 3 implies that

$$\log|\Lambda| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 16\,A_1A_2A_3(1+\log 4)(1+\log m),$$

with $A_1 = \log(320\delta_2^2) \leq 9$, $A_2 = 2\log\phi_2 \leq 3.6$ and $A_3 = 2\log\phi_1 \leq 2$. Using the upper bound for $\log|\Lambda|$ obtained above, we get

$$-0.96m > -3.55 \cdot 10^{14}(1+\log m),$$

an inequality that is not satisfied when m is greater than 10^{20}. Thus, we have proved thanks to Baker's theory that $m \leq 10^{20}$.

We now take $K_1 = \mathbb{Q}[\sqrt{5}]$, $K_2 = \mathbb{Q}[\sqrt{2}]$. Then ϕ_1, ϕ_2 are respectively units in K_1, K_2. If one of the two alternative hypotheses of Lemma 3.1 holds, then we must have m or $n \leq \frac{\log(3\delta'/\delta)}{\log\phi}$.

Numerical computation shows that this latter number is ≤ -3, which is impossible. Thus we must have $\delta_1\phi_1^m < \frac{3}{2}\delta_2\phi_2^n$ and $\delta_2\phi_2^n < \frac{3}{2}\delta_1\phi_1^m$.

Note that we cannot have $\delta_1\phi_1^m = \delta_2\phi_2^n$: indeed, it is easy to see that this equality would imply an equality of the form $a + b\sqrt{5} = c + d\sqrt{2}$, with a, b, c, d positive rational numbers, which is impossible. Thus we have either $\delta_2\phi_2^n < \delta_1\phi_1^m$ or $\delta_1\phi_1^m < \delta_2\phi_2^n$.

Suppose that $\delta_2\phi_2^n < \delta_1\phi_1^m$. Then Lemma 3.2 implies that (3) in Theorem 2 is satisfied. We may take $M = 10^{20}$ and we would like to choose values for p, q and α so that the remaining hypotheses of Theorem 2 are satisfied. This requires some computations for which we use gp [14]. Here $\kappa = \log\phi_1/\log\phi_2$. Working to 1000 decimal places, we write down a floating point approximation κ_0 to κ. Thus certainly

$$|\kappa - \kappa_0| \leq 10^{-900}.$$

Now let p/q be any convergent of the continued fraction expansion of κ_0. We take

$$\alpha = \frac{1}{q} + \frac{q}{10^{900}}$$

and note that (1) is satisfied since $|\kappa_0 - p/q| \leq 1/q^2$. Finally, to apply Theorem 2, we need only choose p/q so that $\varepsilon = \|\mu q\| - M\alpha$ is positive. This turns out to be the case if we take p/q to be the 43rd convergent of the continued fraction of κ_0:

$$p = 387952129646429739199, \qquad q = 710561840528321688446.$$

We deduce from Theorem 2 that we must have $m < \frac{\log(Aq/\varepsilon)}{\log B}$. This number is ≤ 48, by numerical computation. Thus, $m \leq 47$ and by (5), we must have $n \leq 25$.

Suppose on the contrary that $\delta_1\phi_1^n < \delta_2\phi_2^m$. Then, arguing similarly, we obtain $n \leq 26$, $m \leq 48$.

Thus we are reduced to solve $u_m = v_n$ for $0 \leq m, n \leq 50$, which is done quickly using a computer. The only possibility is $m = n = 0$, which was excluded. □

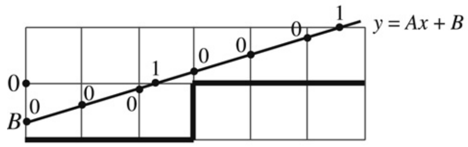

Fig. 2. Discretizing vs. cutting.

4. Proof of the main theorem, part (b)

Consider two disjoint subsets U, V of $\mathbb{R}$ such that $U \cup V$ is isomorphic to $\mathbb{N}$ as ordered set. Then we may write

$$U \cup V = \{x_0 < x_1 < x_2 < \cdots\}$$

and we define a sequence $(a_n)_{n \geq 0}$ over $\{0, 1\}$ by:

$$a_n = \begin{cases} 0, & \text{if } x_n \in U; \\ 1, & \text{if } x_n \in V. \end{cases}$$

This construction will be used several times below. The lemma which follows is well known; we give the proof for completeness.

Lemma 4.1. *Let $U = a\,\mathbb{N} + b$, $V = c\,\mathbb{N} + d$, with $a, c > 0$, and $d - c < b < d$. Then the sequence (a_n) defined above is a Sturmian sequence, obtained by discretizing from below the half-line $ax + b = cy + d$, $x \geq 0$.*

Proof. We may replace U and V by $A\,\mathbb{N} + B$ and $\mathbb{N}$, with $A = \frac{a}{c}$, $B = \frac{b-d}{c}$. We then have the inequalities $-1 < B < 0$. Consider the half-line $y = Ax + B$, $x \geq 0$, and its intersection points with the lattice lines, that is, the lines $x = k$, $y = l$, $k, l \in \mathbb{Z}$. As in [10, p. 55], we define the cutting sequence (b_n) of this half-line by labelling these points by 0 or 1, according to whether the lattice line is of the form $y = l$ or $x = k$. Then $(a_n) = (b_n)$: indeed, if we project these intersection points onto the y-axis, we obtain the real numbers $Al + B$ or k, according to the two cases.

Now, there is a well-known correspondence between cutting sequences and discretization sequences. See Fig. 2. To conclude, observe that $y = Ax + B$ is equivalent to $ax + b = cy + d$. $\square$

Note that, by a change of variables, the sequence of the lemma is the *mechanical sequence*, in the sense of [10, p. 53], corresponding to the half-line $y = \frac{A}{A+1}x + \frac{B}{A+1}$, $x \geq 0$. The verification is left to the reader.

Corollary 4. *Let $U = \{p^m q \mid m \geq 0\}$ and $V = \{r^n s \mid n \geq 0\}$ with $p, r > 1$, $q, s > 0$ and $r^{-1}s < q < s$. If U, V are disjoint, then the sequence (a_n) obtained as above is a Sturmian sequence.*

Proof. Taking logarithms, we define $U' = (\log p)\,\mathbb{N} + \log q$, $V' = (\log r)\,\mathbb{N} + \log s$. Then we apply Lemma 4.1. $\square$

We now set $U = \{u_m \mid m \geq 1\}$ and $V = \{v_n \mid n \geq 1\}$ and construct the sequence (a_n) as at the beginning of the section. This makes sense since we know by part (a) of Theorem 1 that $U \cap V$ is empty.

Let $u'_m = \delta_1 \phi_1^m$ and $v_n = \delta_2 \phi_2^n$. We apply the corollary to the sets $\{u'_m \mid m \geq 1\}$ and $\{v'_n \mid n \geq 1\}$ and obtain a Sturmian sequence. By (7) follows easily that $u_n - 1 < u'_n \leq u_n$ and $v_n - 1 < v'_n \leq v_n$. Hence the sets $\{u_m \mid m \geq 1\}$ and $\{v_n \mid n \geq 1\}$ define the same sequence (a_n), which is therefore Sturmian.

A closer look at the proof of the previous results shows that the half-line to consider is $x(\log \phi_1) + \log(\delta_1 \phi_1) = y \log(\phi_2) + \log(\delta_2 \phi_2)$, since we disregard the values u'_0, v'_0. This implies the remarks following the theorem.

The previous proof raises the following natural problem: given two linearly recursive sequences (u_n), and (v_n), what can be said about the complexity of the sequence of 0's and 1's, constructed as before?

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