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Riemann–Stieltjes operators and multipliers on Q_p spaces in the unit ball of $\mathbf{C}^n$ ☆

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ABSTRACT

This paper is devoted to characterizing the Riemann–Stieltjes operators and pointwise multipliers acting on Möbius invariant spaces Q_p, which unify BMOA and Bloch space in the scale of p. The boundedness and compactness of these operators on Q_p spaces are determined by means of an embedding theorem, i.e. Q_p spaces boundedly embedded in the non-isotropic tent type spaces T_q^∞.

1. Introduction

Let $B = \{z \in \mathbf{C}^n\colon\ |z| < 1\}$ be the unit ball of $\mathbf{C}^n$ $(n > 1)$, $S = \{z \in \mathbf{C}^n\colon\ |z| = 1\}$ be its boundary. $d\upsilon$ denotes the normalized Lebesgue measure of B, i.e. $\upsilon(B) = 1$, and $d\sigma$ denotes the normalized rotation invariant Lebesgue measure of S satisfying $\sigma(S) = 1$. Let $d\lambda(z) = (1 - |z|^2)^{-n-1}\,d\upsilon(z)$, then $d\lambda(z)$ is automorphism invariant, that is for any $\psi \in \mathrm{Aut}(B)$, $f \in L^1(B)$, we have

$$\int_B f(z)\,d\lambda(z) = \int_B f \circ \psi(z)\,d\lambda(z),$$

where $\mathrm{Aut}(B)$ is the group of biholomorphic automorphisms of B.

We denote the class of all holomorphic functions in B by $H(B)$. For $f \in H(B)$, $z \in B$, its complex gradient and invariant gradient are defined as

$$\nabla f(z) = \nabla_z f = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z)\right), \qquad \widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0),$$

where φ_z is the Möbius transformation for $z \in B$, which satisfies $\varphi_z(0) = z$, $\varphi_z(z) = 0$ and $\varphi_z \circ \varphi_z = I$, and its radial derivative $Rf(z) = \langle \nabla f(z), \bar{z} \rangle = \sum_{j=1}^n \frac{\partial f}{\partial z_j}(z) z_j$.

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We say that $f \in H(B)$ is a Bloch function if

$$\|f\|_\beta = |f(0)| + \sup_{z\in B} |\nabla f(z)| (1-|z|^2) < \infty.$$

The collection of Bloch functions is denoted by β. Correspondingly, f is a little Bloch function, denoted as $f \in \beta_0$ if $\lim_{|z|\to 1} |\nabla f(z)|(1-|z|^2) = 0$.

Based on [13] and referring to [3] (see also [22]), the so-called Q_p and $Q_{p,0}$ spaces in [14] are defined as

$$Q_p = \left\{ f \in H(B)\colon \sup_{a\in B} \int_B |\widetilde{\nabla} f(z)|^2 G^p(z,a)\, d\lambda(z) < \infty \right\}, \tag{1.1}$$

it can be also written by (see Lemma 3.2 in [7])

$$Q_p = \left\{ f \in H(B)\colon \sup_{a\in B} \int_B |Rf(z)|^2 (1-|z|^2)^2 (1-|\varphi_a(z)|^2)^{np}\, d\lambda(z) < \infty \right\}. \tag{1.2}$$

$$\|f\|_{Q_p} = |f(0)| + \sup_{a\in B} \left(\int_B |\widetilde{\nabla} f(z)|^2 G^p(z,a)\, d\lambda(z) \right)^{\frac{1}{2}}$$

and

$$Q_{p,0} = \left\{ f \in H(B)\colon \lim_{|a|\to 1} \int_B |\widetilde{\nabla} f(z)|^2 G^p(z,a)\, d\lambda(z) = 0 \right\},$$

for $0 < p < \infty$, where $G(z,a) = g(\varphi_a(z))$ and

$$g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1}\, dt.$$

About Q_p and $Q_{p,0}$, the following properties are proved in [14].

(i) When $0 < p \leqslant \frac{n-1}{n}$ or $p \geqslant \frac{n}{n-1}$, Q_p ($Q_{p,0}$) are trivial, i.e. they contain only the constant functions. When $\frac{n-1}{n} < p < \frac{n}{n-1}$, Q_p ($Q_{p,0}$) are nontrivial.

(ii) $Q_{p_1} \subseteq Q_{p_2}$ ($Q_{p_1,0} \subseteq Q_{p_2,0}$) for $0 < p_1 \leqslant p_2 \leqslant 1$.

(iii) $Q_1 = \mathrm{BMOA}$ ($Q_{1,0} = \mathrm{VMOA}$).

(iv) $Q_p =$ Bloch space ($Q_{p,0} =$ little Bloch space), and $\|\cdot\|_{Q_p}$ is equivalent to $\|\cdot\|_\beta$ for $1 < p < \frac{n}{n-1}$.

T_g and L_g denote the Riemann–Stieltjes operators with the holomorphic symbol g on B respectively (see [6] and [21]):

$$T_g f(z) = \int_0^1 f(tz) Rg(tz) \frac{dt}{t}, \qquad L_g f(z) = \int_0^1 g(tz) Rf(tz) \frac{dt}{t}, \quad z \in B.$$

It is easy to see that the pointwise multipliers M_g are determined by

$$M_g f(z) = g(z) f(z) = g(0) f(0) + T_g f(z) + L_g f(z), \quad z \in B.$$

Of course, in the above definition f is assumed to be holomorphic on B. Clearly, $T_g f = L_f g$ and the Riemann–Stieltjes operator can be viewed as a generalization of the well-known Cesáro operator.

$T_p^\infty(\mu)$ denotes the non-isotropic tent type space of all μ-measurable functions f on B obeying

$$\|f\|^2_{T_p^\infty(\mu)} = \sup\left\{ \delta^{-np} \int_{Q_\delta(\xi)} |f|^2\, d\mu;\ \xi \in S,\ \delta > 0 \right\} < \infty,$$

where $Q_\delta(\xi) = \{z \in B\colon |1-\langle z,\xi\rangle| < \delta\}$ for $\xi \in S$ and $\delta > 0$.

As for the Riemann–Stieltjes operators, they can be traced back to C.H. Pommerenke's paper [17] and A. Siskakis's paper [19] for the Cesáro operator and the extended Cesáro operator. Since that time, in the unit disc D of complex plan, there have been a lot of results on the Riemann–Stieltjes operators on distinct holomorphic function spaces, e.g. see [1,2,5,23] and the references therein. For the case of the unit ball of $\mathbf{C}^n$, recently, we can find that the research on the Riemann–Stieltjes operators has been developing, see [6,8,21] etc.

The purpose of this paper is to study the boundedness and compactness of the Riemann–Stieltjes operators and pointwise multipliers on Q_p spaces as an extension of J. Xiao's paper [23] to the complex ball. It not only is motivated by the importance of Q_p spaces which unify BMOA and Bloch space in the scale of p, but also is inspired by the good idea that a space may be boundedly embedded in tent space as in [12] and [23]. The concept of tent space is from real harmonic analysis [4], however, it is indeed quick way to characterize the boundedness of some operators acting on function spaces.

For a positive Borel measure μ on B, if

$$\|\mu\|^2_{LCM_p} = \sup\left\{ \frac{\mu(Q_\delta(\xi))}{\delta^{np}(\log\frac{2}{\delta})^{-2}};\ \xi \in S,\ \delta > 0 \right\} < \infty,$$

we call μ a logarithmic p-Carleson measure; if

$$\lim_{\delta\to 0} \frac{\mu(Q_\delta(\xi))}{\delta^{np}(\log\frac{2}{\delta})^{-2}} = 0, \quad \text{for } \xi \in S \text{ uniformly},$$

we call μ a vanishing logarithmic p-Carleson measure. The logarithmic Carleson measure and vanishing logarithmic Carleson measure were introduced in [24] and [9]. μ is a usual p-Carleson measure if the factor $(\log\frac{2}{\delta})^{-2}$ is deleted, and denoted by $\|\cdot\|_{CM_p}$ simply.

In this paper we only need to consider the case $\frac{n-1}{n} < p < \frac{n}{n-1}$, since Q_p spaces are trivial when $0 < p \leqslant \frac{n-1}{n}$ or $p \geqslant \frac{n}{n-1}$. The main results are as follows.

Theorem 2.1. *Let $\frac{n-1}{n} < p \leqslant q < \frac{n}{n-1}$, μ be a positive Borel measure on B. Then the identity operator $I : Q_p \mapsto T^\infty_q(\mu)$ is bounded if and only if μ is a logarithmic q-Carleson measure.*

Theorem 2.2. *Let $\frac{n-1}{n} < p \leqslant q < \frac{n}{n-1}$, g be holomorphic on B, $d\mu_{q,g}(z) = |Rg(z)|^2(1-|z|^2)^{n(q-1)+1}\,dv(z)$. Then*

(i) *$T_g : Q_p \mapsto Q_q$ is bounded if and only if $\mu_{q,g}$ is a logarithmic q-Carleson measure.*
(ii) *$L_g : Q_p \mapsto Q_q$ is bounded if and only if $\|g\|_{H^\infty} < \infty$.*
(iii) *$M_g : Q_p \mapsto Q_q$ is bounded if and only if $\mu_{q,g}$ is a logarithmic q-Carleson measure and $\|g\|_{H^\infty} < \infty$.*

Theorem 3.1. *Let $\frac{n-1}{n} < p \leqslant q < \frac{n}{n-1}$, g be holomorphic on B, $d\mu_{q,g}(z) = |Rg(z)|^2(1-|z|^2)^{n(q-1)+1}\,dv(z)$. Then*

(i) *$T_g : Q_p \mapsto Q_q$ is compact if and only if $\mu_{q,g}$ is a vanishing logarithmic q-Carleson measure, here the part "if" holds except for the case of $\frac{n-1}{n} < p \leqslant q < 1$.*
(ii) *$L_g : Q_p \mapsto Q_q$ is compact if and only if $g = 0$.*
(iii) *$M_g : Q_p \mapsto Q_q$ is compact if and only if $g = 0$.*

Theorem 2.1 is the base of arguments of Theorem 2.2 and Theorem 3.1. These two theorems are extension of Theorem 1.2 of [23] to the case of the unit ball of $\mathbf{C}^n$, especially for the operators T_g, L_g and M_g between Q_p spaces in distinct scale of p and in terms of logarithmic p-Carleson measure defined by the non-isotropic metric $|1-\langle z,\xi\rangle|^{\frac{1}{2}}$ on the ball $\overline{B}$. By Lemma 2.1 below with $s = nq$, it is easy to see that $\|\mu_{q,g}\|_{LCM_q} < \infty$ with $d\mu_{q,g}(z) = |Rg(z)|^2(1-|z|^2)^{n(q-1)+1}\,dv(z)$ is equivalent to

$$\sup_{w\in B}\left\{ \log^2\frac{2}{1-|w|^2} \int_B \left|Rg(z)\right|^2\left(1-|z|^2\right)^2\left(1-\left|\varphi_w(z)\right|^2\right)^{nq} d\lambda(z) \right\} < \infty. \tag{3.3}$$

Thus (iii) of Theorem 2.2 is not only an extension of Theorem 1 of [15] to the unit ball of $\mathbf{C}^n$, but also an extension to the whole range $\frac{n-1}{n} < p \leqslant q < \frac{n}{n-1}$ (the results in [15] is only for $0 < p = q < 1$). Recalling another expression (1.2) of definition of Q_q spaces, the class of all symbol functions g satisfying (1.3) would be smaller than Q_q, which we might call a logarithmic type Q_q spaces, denoted as $\log Q_q$. In other words, the necessary and sufficient condition $\|\mu_{q,g}\|_{LCM_q} < \infty$ in Theorem 2.2 may be alternatively changed into $g \in \log Q_q$, which seems to be more convenient for verifying the boundedness of the operators T_g and M_g.

Among the above theorems, some new and special techniques will be adapted to overcome the difficulty causing by the differences of one and several complex variables or target spaces. The embedding result for the pointwise multipliers on Q_p spaces will prompt us to solve a corona type problem for Q_p spaces in the future.

Throughout this paper, C, M denote positive constants which are not necessarily the same at each appearance. The expression $A \approx B$ means that there exists a positive C such that $C^{-1}B \leqslant A \leqslant CB$.

2. Boundedness

The following lemma is a version of Lemma 3.2 of [12] with $q = 2$, $N = s$ and replacing n by np. We omit its proof.

Lemma 2.1. *Let $0 < p < \infty$, μ be a positive Borel measure. Then the following statements are equivalent*:

(i) *The measure μ satisfies*

$$\sup\{\mu(Q_\delta(\xi));\ \xi \in S\} \leqslant C\frac{\delta^{np}}{\log^2 \frac{2}{\delta}}.$$

(ii) *For every $s > 0$,*

$$\sup\left\{\log^2 \frac{2}{1-|w|^2}\int_B \frac{(1-|w|^2)^s}{|1-\langle z,w\rangle|^{np+s}}\,d\mu(z);\ w \in B\right\} < \infty. \tag{2.1}$$

(iii) *For some $s > 0$,*

$$\sup\left\{\log^2 \frac{2}{1-|w|^2}\int_B \frac{(1-|w|^2)^s}{|1-\langle z,w\rangle|^{np+s}}\,d\mu(z);\ w \in B\right\} < \infty.$$

Lemma 2.2. *Let $n \geqslant 2$, $\frac{n-1}{n} < p \leqslant q < \frac{n}{n-1}$, μ is a logarithmic q-Carleson measure. Then, for $s > n(q-p)+1$,*

$$\int_{Q_\delta(\xi)}\left(\int_B \frac{(1-|w|^2)^s|g(w)|}{|1-\langle z,w\rangle|^{n+1+s}}\,dv(w)\right)^2 d\mu(z) \leqslant C\delta^{nq}\|\mu\|^2_{LCM_q}\left\||g(z)|^2(1-|z|^2)^{n(p-1)-1}\,dv(z)\right\|^2_{CM_p}.$$

Proof. Let $\xi \in S$, $0 < \delta \leqslant 2$ and

$$I_{\xi,\delta} = \left(\int_{Q_\delta(\xi)}\left(\int_B \frac{(1-|w|^2)^s|g(w)|}{|1-\langle z,w\rangle|^{n+1+s}}\,dv(w)\right)^2 d\mu(z)\right)^{\frac{1}{2}}.$$

Fix $Q_\delta(\xi)$, let $\|\cdot\|_{Q_\delta(\xi)}$ denote the usual norm on $L^2(Q_\delta(\xi), d\mu)$. By duality,

$$I_{\xi,\delta} = \sup_{\|\psi\|_{Q_\delta(\xi)}=1}\left\{\int_{Q_\delta(\xi)}\int_B \frac{(1-|w|^2)^s|g(w)|}{|1-\langle z,w\rangle|^{n+1+s}}\,dv(w)\left|\psi(z)\right|d\mu(z)\right\}.$$

For $j \in N$, let $A_1 = Q_{4\delta}(\xi)$ and $A_j = Q_{4^j\delta}(\xi)\backslash Q_{4^{j-1}\delta}(\xi)$, $j \geqslant 2$. Clearly, $B = \bigcup_{j=1}^{J_\delta} A_j$, where J_δ is the integer part of $1 + \log_4 \frac{2}{\delta}$.

$$\begin{aligned} I_{\xi,\delta} &\leqslant \sup_{\|\psi\|_{Q_\delta(\xi)}=1}\left\{\int_{Q_\delta(\xi)}\int_{Q_{4\delta}(\xi)} \frac{(1-|w|^2)^s|g(w)|}{|1-\langle z,w\rangle|^{n+1+s}}\,dv(w)\left|\psi(z)\right|d\mu(z)\right. \\ &\quad \left. + \sum_{j=2}^{J_\delta}\int_{Q_\delta(\xi)}\int_{A_j} \frac{(1-|w|^2)^s|g(w)|}{|1-\langle z,w\rangle|^{n+1+s}}\,dv(w)\left|\psi(z)\right|d\mu(z)\right\} \\ &= \sup_{\|\psi\|_{Q_\delta(\xi)}=1}\left\{I^{(1)}_{\xi,\delta} + I^{(2)}_{\xi,\delta}\right\}. \end{aligned}$$

At first, to estimate $I^{(1)}_{\xi,\delta}$. By Hölder's inequality and Fubini's theorem, we have

$$I^{(1)}_{\xi,\delta} \leqslant \left(\int\limits_{Q_\delta(\xi)} \int\limits_{Q_{4\delta}(\xi)} \frac{|g(w)|^2 (1-|w|^2)^s \log^2 \frac{2}{1-|w|^2}}{|1-\langle z,w\rangle|^{n+1+s}} \, dv(w)\, d\mu(z) \right)^{\frac{1}{2}}$$

$$\times \left(\int\limits_{Q_\delta(\xi)} \int\limits_{Q_{4\delta}(\xi)} \frac{|\psi(z)|^2 (1-|w|^2)^s}{|1-\langle z,w\rangle|^{n+1+s} \log^2 \frac{2}{1-|w|^2}} \, dv(w)\, d\mu(z) \right)^{\frac{1}{2}}$$

$$= \left(\int\limits_{Q_{4\delta}(\xi)} \int\limits_{Q_\delta(\xi)} \frac{|g(w)|^2 (1-|w|^2)^s \log^2 \frac{2}{1-|w|^2}}{|1-\langle z,w\rangle|^{n+1+s}} \, d\mu(z)\, dv(w) \right)^{\frac{1}{2}}$$

$$\times \left(\int\limits_{Q_\delta(\xi)} \int\limits_{Q_{4\delta}(\xi)} \frac{(1-|w|^2)^s}{|1-\langle z,w\rangle|^{n+1+s} \log^2 \frac{2}{1-|w|^2}} \, dv(w) \left|\psi(z)\right|^2 d\mu(z) \right)^{\frac{1}{2}}.$$

Similar to the proof of Lemma 3.4 in [10], it is clear that the inner integral of the last line above is bounded. And note that $1-|w| \leqslant |1-\langle w,\xi\rangle| < 4\delta$ for $w \in Q_{4\delta}(\xi)$ and so $(1-|w|)^{\frac{nq-np}{2}} < (4\delta)^{\frac{nq-np}{2}}$ for $p \leqslant q$. Therefore

$$I^{(1)}_{\xi,\delta} \leqslant C \left(\int\limits_{Q_{4\delta}(\xi)} \int\limits_{Q_\delta(\xi)} \frac{(1-|w|^2)^s \log^2 \frac{2}{1-|w|^2}}{|1-\langle z,w\rangle|^{np+s-1}} \, d\mu(z) \left|g(w)\right|^2 \left(1-|w|^2\right)^{n(p-1)-2} dv(w) \right)^{\frac{1}{2}}$$

$$\times \left(\int\limits_{Q_\delta(\xi)} \left|\psi(z)\right|^2 d\mu(z) \right)^{\frac{1}{2}}$$

$$\leqslant C\delta^{\frac{nq-np}{2}} \left(\int\limits_{Q_{4\delta}(\xi)} \int\limits_{Q_\delta(\xi)} \frac{(1-|w|^2)^{s-nq+np-1} \log^2 \frac{2}{1-|w|^2}}{|1-\langle z,w\rangle|^{nq+(s-nq+np-1)}} \, d\mu(z) \left|g(w)\right|^2 \left(1-|w|^2\right)^{n(p-1)-1} dv(w) \right)^{\frac{1}{2}}$$

$$\times \left(\int\limits_{Q_\delta(\xi)} \left|\psi(z)\right|^2 d\mu(z) \right)^{\frac{1}{2}}.$$

By Lemma 2.1, we can get

$$\sup_{\|\psi\|_{Q_\delta(\xi)}=1} I^{(1)}_{\xi,\delta} \leqslant C\delta^{\frac{nq-np}{2}} \|\mu\|_{LCM_q} \left(\int\limits_{Q_{4\delta}(\xi)} \left|g(w)\right|^2 \left(1-|w|^2\right)^{n(p-1)-1} dv(w) \right)^{\frac{1}{2}}$$

$$\leqslant C\delta^{\frac{nq}{2}} \|\mu\|_{LCM_q} \left\| \left|g(w)\right|^2 \left(1-|w|^2\right)^{n(p-1)-1} dv(w) \right\|_{CM_p}.$$

Next to consider $I^{(2)}_{\xi,\delta}$. For $j \geqslant 2$, $z \in Q_\delta(\xi)$ and $w \in A_j$, we have

$$\left|1-\langle w,z\rangle\right|^{\frac{1}{2}} \geqslant \left|1-\langle w,\xi\rangle\right|^{\frac{1}{2}} - \left|1-\langle z,\xi\rangle\right|^{\frac{1}{2}} \geqslant \left(4^{j-1}\delta\right)^{\frac{1}{2}} - \delta^{\frac{1}{2}} \geqslant 2^{j-2}\delta^{\frac{1}{2}}.$$

By these estimates, Hölder's inequality and Fubini's theorem, we have

$$I^{(2)}_{\xi,\delta} \leqslant C \sum_{j=2}^{J_\delta} \left(4^{j-2}\delta\right)^{-n-1} \int\limits_{Q_\delta(\xi)} \int\limits_{A_j} \left|g(w)\right| dv(w) \left|\psi(z)\right| d\mu(z)$$

$$\leqslant C \sum_{j=2}^{J_\delta} \left(4^{j-2}\delta\right)^{-n-1} \left(\int\limits_{Q_\delta(\xi)} \int\limits_{A_j} \left|g(w)\right|^2 \left(1-|w|^2\right)^{n(p-1)-1} dv(w)\, d\mu(z) \right)^{\frac{1}{2}}$$

$$\times \left(\int\limits_{Q_\delta(\xi)} \int\limits_{A_j} \left|\psi(z)\right|^2 \left(1-|w|^2\right)^{1-n(p-1)} dv(w)\, d\mu(z) \right)^{\frac{1}{2}}$$

$$\leqslant C\sum_{j=2}^{J_\delta}\left(4^{j-2}\delta\right)^{-n-1}\left(\int\limits_{A_j}\left|g(w)\right|^2\left(1-|w|^2\right)^{n(p-1)-1}dv(w)\right)^{\frac{1}{2}}$$

$$\times\,\mu^{\frac{1}{2}}\left(Q_\delta(\xi)\right)\times\left(\int\limits_{A_j}\left(\int\limits_{Q_\delta(\xi)}\left|\psi(z)\right|^2d\mu(z)\right)\left(1-|w|^2\right)^{1-n(p-1)}dv(w)\right)^{\frac{1}{2}}.$$

Therefore

$$\begin{aligned}\sup_{\|\psi\|_{Q_\delta(\xi)}=1} I^{(2)}_{\xi,\delta} &\leqslant C\sum_{j=2}^{J_\delta}\left(4^{j-2}\delta\right)^{-n-1}\left(4^j\delta\right)^{\frac{np}{2}}\left\|\left|g(w)\right|^2\left(1-|w|^2\right)^{n(p-1)-1}dv(w)\right\|_{CM_p}\\ &\quad\times\frac{\delta^{\frac{nq}{2}}}{\log\frac{2}{\delta}}\|\mu\|_{LCM_q}\left(\int\limits_{A_j}\left(1-|w|^2\right)^{1-n(p-1)}dv(w)\right)^{\frac{1}{2}}\\ &\leqslant C\sum_{j=2}^{J_\delta}\left(4^{j-2}\delta\right)^{-n-1}\left(4^j\delta\right)^{\frac{np}{2}}\left\|\left|g(w)\right|^2\left(1-|w|^2\right)^{n(p-1)-1}dv(w)\right\|_{CM_p}\\ &\quad\times\frac{\delta^{\frac{nq}{2}}}{\log\frac{2}{\delta}}\|\mu\|_{LCM_q}\left(4^j\delta\right)^{\frac{2+2n-np}{2}}\\ &\leqslant C\frac{J_\delta}{\log\frac{2}{\delta}}\delta^{\frac{nq}{2}}\|\mu\|_{LCM_q}\left\|\left|g(w)\right|^2\left(1-|w|^2\right)^{n(p-1)-1}dv(w)\right\|_{CM_p}\\ &\leqslant C\delta^{\frac{nq}{2}}\|\mu\|_{LCM_q}\left\|\left|g(w)\right|^2\left(1-|w|^2\right)^{n(p-1)-1}dv(w)\right\|_{CM_p}.\end{aligned}$$

Thus, we have

$$I_{\xi,\delta}\leqslant C\delta^{\frac{nq}{2}}\|\mu\|_{LCM_q}\left\|\left|g(w)\right|^2\left(1-|w|^2\right)^{n(p-1)-1}dv(w)\right\|_{CM_p},$$

which ends the proof. □

Lemma 2.3. *Let $n\geqslant 2$, $\frac{n-1}{n}<p<\frac{n}{n-1}$. For $w\in B$, the functions $f_w(z)=\log\frac{1}{1-\langle z,w\rangle}$ satisfy $\sup_{w\in B}\|f_w\|_{Q_p}<\infty$.*

Proof. By Theorem 3.2 of [7], we have

$$\begin{aligned}\|f_w\|^2_{Q_p}&\leqslant C\sup_{a\in B}\int\limits_B\left|Rf_w(z)\right|^2\left(1-|z|^2\right)^{n(p-1)+1}\left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{np}dv(z)\\ &=C\sup_{a\in B}\int\limits_B\frac{1}{|1-\langle z,w\rangle|^2}\left|\langle z,w\rangle\right|^2\left(1-|z|^2\right)^{n(p-1)+1}\left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{np}dv(z)\\ &\leqslant C\sup_{a\in B}\left(1-|a|^2\right)^{np}\int\limits_B\frac{(1-|z|^2)^{n(p-1)+1}}{|1-\langle z,a\rangle|^{2np}|1-\langle z,w\rangle|^2}dv(z).\end{aligned}\tag{2.2}$$

Let $s=n(p-1)+1$, $r=2np$, $t=2$. It is easy to know $s>-1$, $r,t\geqslant 0$, $r+t-s>n+1$ and $t-s<n+1$. Using Lemma 2.5 of [11], we have

(i) When $n\geqslant 3$, $r-s=np+n-1>(n-1)+(n-1)\geqslant n+1$,

$$(2.2)\leqslant C\sup_{a\in B}\left(1-|a|^2\right)^{np}\frac{1}{(1-|a|^2)^{np-2}|1-\langle a,w\rangle|^2}\leqslant C\sup_{a\in B}\frac{(1-|a|^2)^2}{(1-|a|)^2}\leqslant C.$$

(ii) $n=2$.
If $r-s<n+1$,

$$(2.2)\leqslant C\sup_{a\in B}\left(1-|a|^2\right)^{np}\frac{1}{|1-\langle a,w\rangle|^{np}}\leqslant C\sup_{a\in B}\frac{(1-|a|^2)^{np}}{(1-|a|)^{np}}\leqslant C.$$

If $r - s > n + 1$,

$$(2.2) \leqslant C \sup_{a\in B} \left(1 - |a|^2\right)^{np} \frac{1}{(1 - |a|^2)^{np-2} |1 - \langle a, w\rangle|^2} \leqslant C \sup_{a\in B} \frac{(1 - |a|^2)^2}{(1 - |a|)^2} \leqslant C.$$

If $r - s = n + 1$, i.e $p = 1$, by Lemma 3.1 of [12], we can get $\sup_{w\in B} \|f_w\|_{Q_1} < \infty$. □

Proof of Theorem 2.1. Suppose the identity operator $I: Q_p \mapsto T_q^\infty(\mu)$ is bounded. For any $\xi \in S$ and $0 < \delta < 1$, we consider the function $f_{\xi,\delta}(z) = \log \frac{2}{1-\langle z,(1-\delta)\xi\rangle}$, by Lemma 2.6 of [11], we have

$$\left|f_{\xi,\delta}(z)\right| \approx \log \frac{2}{\delta}, \quad z \in Q_\delta(\xi),$$

and by Lemma 2.3,

$$\delta^{-nq} \int_{Q_\delta(\xi)} |f_{\xi,\delta}|^2 \, d\mu \leqslant C \|f_{\xi,\delta}\|^2_{Q_p} \leqslant C.$$

Accordingly, $\|\mu\|_{LCM_q} \leqslant C$.

Conversely, suppose μ is a logarithmic q-Carleson measure. For a holomorphic function f, we recall the following representation formula

$$Rf(z) = C_\alpha \int_B Rf(w) \frac{\left(1 - |w|^2\right)^\alpha}{\left(1 - \langle z, w\rangle\right)^{n+1+\alpha}} \, dv(w)$$

for α large enough. Acting on the above equation by the inverse operator R^{-1},

$$f(z) = C_\alpha R^{-1} \int_B Rf(w) \frac{\left(1 - |w|^2\right)^\alpha}{\left(1 - \langle z, w\rangle\right)^{n+1+\alpha}} \, dv(w),$$

and consequently, we can get

$$\left|f(z)\right| \leqslant C \int_B \left|Rf(w)\right| \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w\rangle|^{n+\alpha}} \, dv(w). \tag{2.3}$$

Using (2.3) and Lemma 2.2 with $g(w) = |Rf(w)|(1 - |w|^2)$, we have

$$\begin{aligned}
\delta^{-nq} \int_{Q_\delta(\xi)} \left|f(z)\right|^2 d\mu(z) &\leqslant C\delta^{-nq} \int_{Q_\delta(\xi)} \left(\int_B \left|Rf(w)\right| \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w\rangle|^{n+\alpha}} \, dv(w) \right)^2 d\mu(z) \\
&= C\delta^{-nq} \int_{Q_\delta(\xi)} \left(\int_B \frac{|Rf(w)|(1 - |w|^2)(1 - |w|^2)^{\alpha-1}}{|1 - \langle z, w\rangle|^{n+1+(\alpha-1)}} \, dv(w) \right)^2 d\mu(z) \\
&\leqslant C\|\mu\|^2_{LCM_q} \left\| \left|Rf(z)\right|^2 \left(1 - |z|^2\right)^{n(p-1)+1} dv(z) \right\|^2_{CM_p} \\
&\leqslant C\|\mu\|^2_{LCM_q} \|f\|^2_{Q_p},
\end{aligned}$$

the last inequality holds because the norm of $f \in Q_p$ for $\frac{n-1}{n} < p < \frac{n}{n-1}$ is comparably dominated by the geometric quantity

$$\left|f(0)\right| + \sup\left\{ \left(\delta^{-np} \int_{Q_\delta(\xi)} \left|Rf(z)\right|^2 \left(1 - |z|^2\right)^{n(p-1)+1} dv(z) \right)^{\frac{1}{2}};\ \xi \in S,\ \delta > 0 \right\} < \infty$$

by Corollary 3.2 of [7] with $m = 1$. □

Proof of Theorem 2.2. (i) Note that $R(T_g f)(z) = f(z)Rg(z)$. So, Theorem 2.1 implies that T_g maps boundedly Q_p into Q_q is equivalent to $\|\mu_{q,g}\|_{LCM_q} < \infty$.

(ii) If $\|g\|_{H^\infty} < \infty$, then

$$\delta^{-nq} \int\limits_{Q_\delta(\xi)} \left|R(L_g f)(z)\right|^2 \left(1-|z|^2\right)^{n(q-1)+1} dv(z) = \delta^{-nq} \int\limits_{Q_\delta(\xi)} \left|g(z)\right|^2 \left|Rf(z)\right|^2 \left(1-|z|^2\right)^{n(q-1)+1} dv(z)$$
$$\leqslant C\|g\|_{H^\infty}^2 \|f\|_{Q_q}^2 \leqslant C\|g\|_{H^\infty}^2 \|f\|_{Q_p}^2,$$

this implies that $\|L_g f\|_{Q_q} \leqslant C\|g\|_{H^\infty}\|f\|_{Q_p}$. So, $L_g : Q_p \mapsto Q_q$ is bounded.

Conversely, suppose $L_g : Q_p \mapsto Q_q$ is bounded. We fix $\xi \in S$ and give a point $w \in B$ near to the boundary with $|w| > \frac{2}{3}$, there exists $0 < \delta < 1$ such that

$$E\left(w, \frac{1}{2}\right) \subset Q_\delta(\xi) \quad \text{and} \quad 1-|w|^2 \approx \delta,$$

where $E(z,r) = \{w \in B\colon\ |\varphi_z(w)| < r\}$ denote the pseudo-hyperbolic metric ball at z. Choosing $f_w(z) = \log \frac{1}{1-\langle z,w\rangle}$. By Lemma 2.3, we know $\sup_{w\in B} \|f_w\|_{Q_p} \leqslant C$. It is well known that

$$v\left(E\left(w, \frac{1}{2}\right)\right) \approx \left(1-|w|^2\right)^{n+1}, \qquad 1-|w|^2 \approx 1-|z|^2 \approx \left|1-\langle z,w\rangle\right| \quad \text{for } z \in E\left(w, \frac{1}{2}\right).$$

Also note that for $z \in E(w, \frac{1}{2})$, we have

$$1-\left|\varphi_w(z)\right|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\langle z,w\rangle|^2} > \frac{3}{4}.$$

Thus

$$1-\left|\langle z,w\rangle\right| \leqslant \left|1-\langle z,w\rangle\right| < \frac{2}{\sqrt{3}}\left(1-|w|^2\right)^{\frac{1}{2}}\left(1-|z|^2\right)^{\frac{1}{2}} \leqslant \frac{2}{\sqrt{3}}\left(1-|w|^2\right)^{\frac{1}{2}} < \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{5}}{3} = \frac{2\sqrt{15}}{9},$$

this implies $|\langle z,w\rangle| > 1-\frac{2\sqrt{15}}{9}$. By the $\mathscr{M}$-subharmonicity of $|g(w)|^2$, we have

$$\begin{aligned}
\left|g(w)\right|^2 &\leqslant C\frac{1}{v(E(w,\frac{1}{2}))} \int\limits_{E(w,\frac{1}{2})} \left|g(z)\right|^2 dv(z) \\
&\leqslant C\frac{1}{(1-|w|^2)^{n+1}} \int\limits_{E(w,\frac{1}{2})} \left|g(z)\right|^2 dv(z) \\
&\leqslant C\delta^{-nq} \int\limits_{E(w,\frac{1}{2})} \frac{|g(z)|^2(1-|z|^2)^{n(q-1)+1}}{|1-\langle z,w\rangle|^2} dv(z) \\
&\leqslant C\delta^{-nq} \int\limits_{E(w,\frac{1}{2})} \frac{|g(z)|^2|\langle z,w\rangle|^2(1-|z|^2)^{n(q-1)+1}}{|1-\langle z,w\rangle|^2} dv(z) \\
&\leqslant C\delta^{-nq} \int\limits_{Q_\delta(\xi)} \left|g(z)\right|^2 \left|Rf_w(z)\right|^2 \left(1-|z|^2\right)^{n(q-1)+1} dv(z) \\
&\leqslant C\left\|L_g(f_w)\right\|_{Q_q}^2 \leqslant C\|L_g\|^2\|f_w\|_{Q_p}^2 \leqslant C,
\end{aligned}$$

and consequently, $|g(w)| \leqslant C$ for $|w| > \frac{2}{3}$. By maximum modulus principle, we have $|g(w)| \leqslant C$ for $w \in B$. Thus $g \in H^\infty$.

(iii) The "if" part follows from the corresponding ones of (i) and (ii). We only need to see the "only if" part. Note that $f_w(z) = \log \frac{2}{1-\langle z,w\rangle}$ belongs to Q_p with $\sup_{w\in B}\|f_w\|_{Q_p} \leqslant C$ and any function $f \in Q_p$ has the growth (see [16])

$$\left|f(z)\right| \leqslant \left|f(0)\right| + C\|f\|_{Q_p} \log\frac{1}{1-|z|^2} \leqslant C\|f\|_{Q_p}\log\frac{2}{1-|z|^2}, \quad \text{for every } z \in B.$$

So, if $M_g : Q_p \mapsto Q_q$ is bounded, then for every $w \in B$,

$$\left|g(z)f_w(z)\right| \leqslant C\|M_g f_w\|_{Q_q} \log\frac{2}{1-|z|^2} \leqslant C\|M_g\|\log\frac{2}{1-|z|^2}, \quad z \in B$$

and hence $|g(w)| \leqslant C\|M_g\|$ (upon taking $z = w$ in the last estimate), that is, $\|g\|_{H^\infty} < \infty$, equivalently, $L_g : Q_p \mapsto Q_q$ is bounded by (ii). Consequently, $T_g f = M_g f - L_g f - f(0)g(0)$ gives the boundedness of $T_g : Q_p \mapsto Q_q$ and then $\|\mu_{q,g}\|_{LCM_q} < \infty$. □

Corollary 2.1. *Let $1 < q < \frac{n}{n-1}$, g be holomorphic on B, $d\mu_{q,g}(z) = |Rg(z)|^2(1-|z|^2)^{n(q-1)+1}\,dv(z)$ and $\|g\|_{H^\infty} = \sup_{z\in B}|g(z)|$. Then*

(i) *$T_g : \mathrm{BMOA} \mapsto \beta$ is bounded if and only if $\mu_{q,g}$ is logarithmic q-Carleson measure.*
(ii) *$L_g : \mathrm{BMOA} \mapsto \beta$ is bounded if and only if $\|g\|_{H^\infty} < \infty$.*
(iii) *$M_g : \mathrm{BMOA} \mapsto \beta$ is bounded if and only if $\mu_{q,g}$ is logarithmic q-Carleson measure and $\|g\|_{H^\infty} < \infty$.*

3. Compactness

Before proving the compactness of T_g, L_g and M_g, we need to give the following lemmas.

Lemma 3.1. *(See Lemma 3.7 of [20].) Let X, Y be two Banach spaces of analytic functions on D. Suppose*

(1) *the point evaluation functionals on Y are continuous*;
(2) *the closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets*;
(3) *$T : X \mapsto Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then T is a compact operator if and only if given a bounded sequence $\{f_j\}$ in X such that $f_j \to 0$ uniformly on compact sets, then the sequence $\{Tf_j\}$ converges to zero in the norm of Y.

Although this lemma is shown for the unit disc D of the complex plane, it is still valid for any complex domain, of course, including the unit ball of $\mathbf{C}^n$. In this section, to prove the compactness of the operators T_g and L_g from Q_p to Q_q, we need to verify the three assumptions of the above lemma.

At first, it is clear that the assumption (1) holds by setting $e_z(f) = f(z) : Q_q \to \mathbf{C}$ because any function $f \in Q_q$ has the growth

$$\left|f(z)\right| \leqslant \left|f(0)\right| + C\|f\|_{Q_q}\log\frac{1}{1-|z|^2}, \quad z \in B. \tag{3.1}$$

Let $\{f_j\}$ be a sequence in the closed unit ball $\mathscr{B}$ of Q_p. Since the functions in $\mathscr{B}$ are bounded uniformly on compact sets of B, by Montel's theorem we can pick out a subsequence $f_{j_k} \to h$ uniformly on compact sets of B, for some $h \in H(B)$. To verify the assumption (2), we show that $h \in Q_p$. Indeed,

$$\begin{aligned}\int_B \left|\widetilde{\nabla} h(z)\right|^2 G^p(z,a)\,d\lambda(z) &= \int_B \lim_{k\to\infty}\left|\widetilde{\nabla} f_{j_k}(z)\right|^2 G^p(z,a)\,d\lambda(z)\\ &\leqslant \liminf_{k\to\infty}\int_B \left|\widetilde{\nabla} f_{j_k}(z)\right|^2 G^p(z,a)\,d\lambda(z)\\ &\leqslant \liminf_{k\to\infty}\|f_{j_k}\|_{Q_p}^2 \leqslant 1\end{aligned}$$

by Fatou's lemma for every $a \in B$, so $h \in \mathscr{B}$.

The assumption (3) means that if bounded sequence $\{f_j\}$ in Q_p converges uniformly to zero on compact sets of B, then $\{T_g f_j\}$ (and $\{L_g f_j\}$) converges uniformly to zero on compact sets of B. Now we verify it. Let $f_j(z) \to 0$ uniformly on compact sets G of B, then $\{D^\alpha f_j\}$ converges uniformly to zero on compact sets K of B and $\sup_{z\in K}|D^\alpha f_j| \leqslant C_\alpha \sup_{z\in G}|f_j|$ by the well-known Weierstrass theorem. Therefore

$$\begin{aligned}\lim_{j\to\infty}\left|L_g f_j(z)\right| &= \lim_{j\to\infty}\left|\int_0^1 g(tz)Rf_j(tz)\frac{dt}{t}\right|\\ &\leqslant \lim_{j\to\infty}\int_0^1 \left|g(tz)\right|\left|\nabla f_j(tz)\right|dt\end{aligned}$$

$$= \int_0^1 \lim_{j\to\infty} \left|g(tz)\right|\left|\nabla f_j(tz)\right| dt$$
$$= 0, \quad \text{uniformly on } K$$

by Lebesgue's dominated convergence theorem, since the sequence $\{f_j(tz)\}$ is bounded uniformly for j and $t \in [0,1]$ from (3.1). Similarly, the assumption (3) can be verified for T_g.

Summarizing the above arguments, we can get a criterion of the compactness of the T_g and L_g as follows.

Lemma 3.2. *For the Riemann–Stieltjes operators T_g and L_g with the holomorphic symbol g, the following statements are equivalent*

(i) *T_g (resp. L_g) is a compact operator from Q_p to Q_q.*
(ii) *For every bounded sequence $\{f_j\}$ in Q_p such that $f_j \to 0$ uniformly on compact sets of B, then the sequence $\{T_g f_j\}$ (and $\{L_g f_j\}$) converges to zero in the norm of Q_q.*

For $\xi \in S$ and $\delta > 0$, set

$$Q'_\delta(\xi) = \left\{\eta \in S\colon \left|1 - \langle \eta, \xi\rangle\right| < \delta\right\}.$$

Q'_δ is a ball of radius $\delta^{\frac{1}{2}}$ on S in the non-isotropic metric $|1-\langle\eta,\xi\rangle|^{\frac{1}{2}}$ and note that $Q'_\delta = S$ when $\delta > 2$. We have the following covering lemma, which is a version of non-isotropic ball of Lemma 2.22 in [25], and will be used in the proof of Theorem 3.1 and elsewhere later on.

Lemma 3.3. *Given any natural number m, there exists a natural number N such that every non-isotropic ball of "radius" $\delta \leqslant 2$, can be covered by N non-isotropic balls of "radius" δ/m.*

Proof. The first half of the proof is the same process as that of Lemma 2.22 of [25] for Bergman metric ball. We can get a covering $\{Q'_{\delta/m}(\xi'_k)\}$ of $Q'_\delta(\xi)$ with $|1-\langle\xi'_i,\xi'_j\rangle| \geqslant \delta/2m$ for $i \neq j$ where each $\xi'_k \in Q'_\delta(\xi)$. We omit its details.

Since the sets $\{Q'_{\delta/4m}(\xi'_k)\}$ are disjoint and contained in $Q'_{\delta+(\delta/4m)}(\xi)$, we can get

$$\sum_k \sigma\left(Q'_{\delta/4m}(\xi'_k)\right) \leqslant \sigma\left(Q'_{\delta+(\delta/4m)}(\xi)\right).$$

Moreover, there is a positive constant C, independent of δ but dependent on m such that

$$\sigma\left(Q'_{\delta+(\delta/4m)}(\xi)\right) \leqslant C\sigma\left(Q'_{\delta/4m}(\xi'_k)\right)$$

for each k. This inequality is true because Proposition 5.1.4 of [18] implies that

$$C = \sup_{Q'} \frac{\sigma(Q'_{\delta+(\delta/4m)})}{\sigma(Q'_{\delta/4m})} \leqslant (4m+1)^n A_0/2^{-n}.$$

Thus we see that $k \leqslant C$ and so the natural number $N = [C]+1$ as desired. □

In the proof of Theorem 3.1(ii), we need to use the lemma below, which is also of independent interest.

Lemma 3.4. *Let $f(z)$ be bounded holomorphic function on B, $\|f\|_{H^\infty} = \sup_{z\in B}|f(z)|$. Then*

$$\left|f(z_1) - f(z_2)\right| \leqslant 2\|f\|_{H^\infty}\left|\varphi_{z_1}(z_2)\right|$$

holds for $z_1, z_2 \in B$, where φ_z is the Möbius transformation of B.

Proof. The conclusion for the unit disc D was pointed out in [23]. In fact, using the invariant form of Schwarz's lemma, i.e. Schwarz–Pick lemma for $f(z)/\|f\|_{H^\infty}$, we have

$$\frac{\frac{1}{\|f\|_{H^\infty}}|f(z_1) - f(z_2)|}{|1 - \frac{1}{\|f\|^2_{H^\infty}}\overline{f(z_2)}f(z_1)|} \leqslant \left|\frac{z_1 - z_2}{1 - \overline{z_2}z_1}\right| = \left|\varphi_{z_2}(z_1)\right|.$$

Therefore

$$\left|f(z_1) - f(z_2)\right| \leqslant \left|\|f\|_{H^\infty} - \|f\|^{-1}_{H^\infty}\overline{f(z_2)}f(z_1)\right|\left|\varphi_{z_1}(z_2)\right| \leqslant 2\|f\|_{H^\infty}\left|\varphi_{z_1}(z_2)\right|. \tag{3.2}$$

In the case of the unit ball of $\mathbf{C}^n$, $n \geqslant 2$, we consider the function $f(z)/\|f\|_{H^\infty}$ and $z_1, z_2 \in B$. Given a unitary map U so that $z_3 = U^{-1}(\varphi_{z_1}(z_2)) \in (\mathbf{C}, 0, \dots, 0) \in \mathbf{C}^n$ and let $F(z) = f \circ \varphi_{z_1} \circ U/\|f\|_{H^\infty}$ restricted to $(D, 0, \dots, 0)$. Then $|z_3| = |\varphi_{z_1}(z_2)|$ and $F(z_3) = f(z_2)/\|f\|_{H^\infty}$, $F(0) = f(z_1)/\|f\|_{H^\infty}$. Since $F(z)$ is a function from the unit disc to itself, applying Schwarz–Pick lemma to $F(z)$, we get

$$\frac{\frac{1}{\|f\|_{H^\infty}}|f(z_1) - f(z_2)|}{|1 - \frac{1}{\|f\|^2_{H^\infty}}\overline{f(z_2)}f(z_1)|} = \frac{|F(0) - F(z_3)|}{|1 - \overline{F(z_3)}F(0)|} \leqslant |z_3| = \left|\varphi_{z_1}(z_2)\right|.$$

This means that (3.2) is still true for the unit ball of $\mathbf{C}^n$. □

Proof of Theorem 3.1. (i) Suppose $\mu_{q,g}$ is a vanishing logarithmic q-Carleson measure. Let $\{f_j\}$ be any bounded sequence in Q_p and $f_j \to 0$ uniformly on compact sets of B. For the compactness of T_g, it suffices to prove $\lim_{j\to\infty}\|T_g f_j\|_{Q_q} = 0$ by Lemma 3.2.

For $r \in (0,1)$, define the cut-off measure $d\mu_{q,g,r}(z) = \chi_{\{z\in B:\ |z|>r\}}\, d\mu_{q,g}(z)$, where χ_E denotes the characteristic function of a set E of B,

$$\begin{aligned}
&\int\limits_{Q_\delta(\xi)} \left|R(T_g f_j)(z)\right|^2 \left(1 - |z|^2\right)^{n(q-1)+1} dv(z) \\
&\quad = \int\limits_{Q_\delta(\xi)} \left|f_j(z)\right|^2 \left|Rg(z)\right|^2 \left(1 - |z|^2\right)^{n(q-1)+1} dv(z) \\
&\quad = \int\limits_{Q_\delta(\xi)} \left|f_j(z)\right|^2 d\mu_{q,g}(z) \\
&\quad = \int\limits_{Q_\delta(\xi)} |f_j|^2 \chi_{\{z\in B:\ |z|\leqslant r\}}\, d\mu_{q,g}(z) + \int\limits_{Q_\delta(\xi)} |f_j|^2 \chi_{\{z\in B:\ |z|>r\}}\, d\mu_{q,g}(z) \\
&\quad \leqslant \int\limits_{Q_\delta(\xi)} |f_j|^2 \chi_{\{z\in B:\ |z|\leqslant r\}}\, d\mu_{q,g}(z) + C\delta^{nq}\|f_j\|^2_{Q_p}\|\mu_{q,g,r}\|^2_{LCM_q}.
\end{aligned} \tag{3.3}$$

The second term of the end of (3.3) follows from the proof of the "if" part of Theorem 2.1.

We claim that $\|\mu_{q,g,r}\|_{LCM_q} \to 0$ when $r \to 1$ for the cut-off measure in the case of $1 \leqslant q < \frac{n}{n-1}$. In the proof of Theorem 4.1 of [7], we know $Q_\delta(\xi) \subset \widehat{Q}_{4\delta}(\xi) \subset Q_{16\delta}(\xi)$, where

$$\widehat{Q}_\delta(\xi) = \left\{z \in B:\ \frac{z}{|z|} \in Q'_\delta(\xi),\ 1 - \delta < |z| < 1\right\}.$$

Hence we can use Q_δ or alternatively $\widehat{Q}_\delta$ in the definition of (vanishing) Carleson type measure. For any $\varepsilon > 0$, there is $\delta_0 > 0$ such that

$$\mu_{q,g}\left(\widehat{Q}_\delta(\xi)\right) < \varepsilon\delta^{nq}\left(\log\frac{2}{\delta}\right)^{-2}$$

for all $\delta \leqslant \delta_0$ and for $\xi \in S$ uniformly, since $\mu_{q,g}$ is vanishing logarithmic q-Carleson measure. If $\delta > \delta_0$, given a natural number $m = [\frac{\delta}{\delta_0}] + 1$ $(< \frac{2\delta}{\delta_0})$ so that $\frac{\delta}{m} < \delta_0$ for all $\delta \leqslant 2$, Q'_δ can be covered by N balls $Q'_{\delta/m}$ on S by Lemma 3.3. Further, it follows that

$$\widehat{Q}_\delta \cap \left\{z \in B:\ |z| > r_0\right\} \subset \bigcup_N \widehat{Q}_{\delta/m}$$

with $r_0 = 1 - \frac{\delta_0}{m}$ from the definition of $\widehat{Q}_\delta$. Therefore

$$\begin{aligned}
\mu_{q,g,r_0}(\widehat{Q}_\delta) &\leqslant \mu_{q,g,r_0}\left(\bigcup_N \widehat{Q}_{\delta/m}\right) \leqslant \mu_{q,g,r_0}\left(\bigcup_N \widehat{Q}_{\delta_0}\right) \leqslant \sum_N \mu_{q,g}(\widehat{Q}_{\delta_0}) \leqslant N\varepsilon\delta_0^{nq}\left(\log\frac{2}{\delta_0}\right)^{-2} \\
&< C\frac{2^n\delta^n}{\delta_0^n}\varepsilon\delta_0^{nq}\left(\log\frac{2}{\delta_0}\right)^{-2} < C\varepsilon\delta^n\delta_0^{nq-n}\left(\log\frac{2}{\delta_0}\right)^{-2} < C\varepsilon\delta^{nq}\left(\log\frac{2}{\delta}\right)^{-2},
\end{aligned}$$

where we use $N \leqslant Cm^n$, $C > 1$ (see Lemma 3.3) and $nq - n \geqslant 0$ when $1 \leqslant q < \frac{n}{n-1}$. It is clear that $\mu_{q,g,r_0}(\widehat{Q}_\delta) < \varepsilon \delta^{nq} (\log \frac{2}{\delta})^{-2}$ holds for $\delta \leqslant \delta_0$. As shown above, for any $\varepsilon > 0$ we may find $r_0 = 1 - \frac{\delta_0}{m}$, so that

$$\frac{\mu_{q,g,r}(\widehat{Q}_\delta)}{\delta^{nq} (\log \frac{2}{\delta})^{-2}} < C\varepsilon \tag{3.4}$$

provided $r > r_0$ and for all $\delta \leqslant 2$. This is as desired.

Note that the integral of the end of (3.3)

$$\int_{Q_\delta(\xi)} |f_j|^2 \chi_{\{z \in B:\ |z| \leqslant r\}}\, d\mu_{q,g}(z) \to 0,$$

since $f_j \to 0$ uniformly on $\{z \in B:\ |z| \leqslant r\}$. Therefore,

$$\delta^{-nq} \int_{Q_\delta(\xi)} \left| R(T_g f_j)(z) \right|^2 \left(1 - |z|^2\right)^{n(q-1)+1} dv(z) \to 0$$

as $j \to \infty$ by (3.4). Noting that $T_g f_j(0) = 0$, we have $\lim_{j\to\infty} \|T_g f_j\|_{Q_q} = 0$.

However, at present, we are not sure the compactness of the operator T_g for the case of $\frac{n-1}{n} < p \leqslant q < 1$.

Conversely, suppose $T_g : Q_p \mapsto Q_q$ is compact. $\forall \xi \in S$, $\delta_j \to 0$, we consider the functions

$$f_j(z) = \left(\log \frac{2}{\delta_j}\right)^{-1} \left(\log \frac{2}{1 - \langle z, (1-\delta_j)\xi \rangle}\right)^2.$$

Note that $|1 - \langle z, (1-\delta_j)\xi \rangle| \geqslant \delta_j$. We have

$$\begin{aligned}
\|f_j\|_{Q_p}^2 &\approx \left|f_j(0)\right|^2 + \sup_{a \in B} \int_B \left|Rf_j(z)\right|^2 \left(1 - |z|^2\right)^{n(p-1)+1} \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2}\right)^{np} dv(z) \\
&\leqslant C + C \sup_{a \in B} \left(\log \frac{2}{\delta_j}\right)^{-2} \int_B \left|\log \frac{2}{1 - \langle z, (1-\delta_j)\xi \rangle}\right|^2 \frac{|\langle z, (1-\delta_j)\xi \rangle|^2}{|1 - \langle z, (1-\delta_j)\xi \rangle|^2} \\
&\quad \times \left(1 - |z|^2\right)^{n(p-1)+1} \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2}\right)^{np} dv(z) \\
&\leqslant C + C \sup_{a \in B} \left(\log \frac{2}{\delta_j}\right)^{-2} \int_B \left(\log \frac{2}{\delta_j}\right)^2 \frac{(1 - |z|^2)^{n(p-1)+1}}{|1 - \langle z, (1-\delta_j)\xi \rangle|^2} \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2}\right)^{np} dv(z) \\
&= C + C \sup_{a \in B} \left(1 - |a|^2\right)^{np} \int_B \frac{(1 - |z|^2)^{n(p-1)+1}}{|1 - \langle z, a \rangle|^{2np} |1 - \langle z, (1-\delta_j)\xi \rangle|^2} dv(z).
\end{aligned}$$

Similar to the proof of (2.2), we can get $\|f_j\|_{Q_p}^2 \leqslant C$ for all j. It is clear that $f_j \to 0$ uniformly on compact sets of B as $\delta_j \to 0$. Using Lemma 2.6 of [11], we have $|\log \frac{2}{1 - \langle z, (1-\delta_j)\xi \rangle}| \approx \log \frac{2}{\delta_j}$, $z \in Q_{\delta_j}(\xi)$. If T_g is compact, by Lemma 3.2, we know that for any $\xi \in S$,

$$\begin{aligned}
\frac{\mu_{q,g}(Q_{\delta_j}(\xi))}{\delta_j^{nq} (\log \frac{2}{\delta_j})^{-2}} &\leqslant C \delta_j^{-nq} \int_{Q_{\delta_j}(\xi)} |f_j|^2\, d\mu_{q,g} \\
&= C \delta_j^{-nq} \int_{Q_{\delta_j}(\xi)} |f_j|^2 \left|Rg(z)\right|^2 \left(1 - |z|^2\right)^{n(q-1)+1} dv(z) \\
&= C \delta_j^{-nq} \int_{Q_{\delta_j}(\xi)} \left|R(T_g f_j)(z)\right|^2 \left(1 - |z|^2\right)^{n(q-1)+1} dv(z) \\
&\leqslant C \|T_g f_j\|_{Q_q}^2 \to 0, \quad \text{as } j \to \infty.
\end{aligned}$$

(ii) It is enough to verify that if $L_g : Q_p \to Q_q$ is compact then $g = 0$. By Theorem 2.2(ii), the compactness of L_g implies $g \in H^\infty$. Now, assume g is not identically equal to 0. According to the maximum principle, the boundary value function $g|_S$

cannot be identically the zero function. Accordingly, there are a positive constant ε and a sequence $\{w_j\}$ in B near to the boundary with $|w_j| > \frac{2}{3}$ such that $|g(w_j)| > \varepsilon$. By Lemma 3.4, we have

$$\left|g(z_1) - g(z_2)\right| \leqslant 2\|g\|_{H^\infty}\left|\varphi_{z_1}(z_2)\right|, \quad z_1, z_2 \in B.$$

This inequality implies that there is a sufficiently small number $r > 0$ such that $|g(z)| \geqslant \frac{\varepsilon}{2}$ for all j and z obeying $|\varphi_{w_j}(z)| < r$. Note that each pseudo-hyperbolic ball $E(w_j, r) = \{z \in B\colon\ |\varphi_{w_j}(z)| < r\}$ is contained in $Q_{\delta_j}(\xi)$ with $1 - |w_j|^2 \approx \delta_j$. We consider the functions

$$f_j(z) = \left(\log \frac{2}{1 - |w_j|^2}\right)^{-1}\left(\log \frac{2}{1 - \langle z, w_j\rangle}\right)^2.$$

Assume $|w_j| \to 1$. It is clear that $\|f_j\|_{Q_p} \leqslant C$ and $f_j \to 0$ uniformly on compact sets of B. Note that $|\log \frac{2}{1-\langle z, w_j\rangle}| \approx \log \frac{2}{\delta_j}$ for $z \in Q_{\delta_j}(\xi)$ and $|\langle z, w_j\rangle| \geqslant C$ for $z \in E(w_j, r)$. Thus

$$\begin{aligned}
\|L_g f_j\|_{Q_q}^2 &\geqslant C\delta_j^{-nq} \int\limits_{Q_{\delta_j}(\xi)} \left|Rf_j(z)\right|^2\left|g(z)\right|^2\left(1 - |z|^2\right)^{n(q-1)+1}\,dv(z) \\
&\geqslant C\delta_j^{-nq} \int\limits_{Q_{\delta_j}(\xi)} \left(\log \frac{2}{\delta_j}\right)^{-2}\left|\log \frac{2}{1 - \langle z, w_j\rangle}\right|^2 \frac{|\langle z, w_j\rangle|^2}{|1 - \langle z, w_j\rangle|^2}\left|g(z)\right|^2\left(1 - |z|^2\right)^{n(q-1)+1}\,dv(z) \\
&\geqslant C\varepsilon^2\delta_j^{-nq}\left(\log \frac{2}{\delta_j}\right)^{-2}\left(\log \frac{2}{\delta_j}\right)^2 \int\limits_{|\varphi_{w_j}(z)|<r} \frac{|\langle z, w_j\rangle|^2}{|1 - \langle z, w_j\rangle|^2}\left(1 - |z|^2\right)^{n(q-1)+1}\,dv(z) \\
&\geqslant C\varepsilon^2\delta_j^{-nq} \int\limits_{|\varphi_{w_j}(z)|<r} \frac{(1 - |z|^2)^{n(q-1)+1}}{|1 - \langle z, w_j\rangle|^2}\,dv(z) \\
&\geqslant C\varepsilon^2\delta_j^{-nq}\left(1 - |w_j|^2\right)^{nq-n-1} \int\limits_{|\varphi_{w_j}(z)|<r} dv(z) \\
&\geqslant C\varepsilon^2\delta_j^{-nq}\left(1 - |w_j|^2\right)^{nq-n-1}\left(1 - |w_j|^2\right)^{n+1} \\
&\geqslant C\varepsilon^2.
\end{aligned}$$

However, the compactness of L_g forces $\|L_g f_j\|_{Q_q}^2 \to 0$, and consequently, $\varepsilon = 0$, contradicting $\varepsilon > 0$. Therefore, g must be the zero function.

(iii) Suppose now $M_g : Q_p \mapsto Q_q$ is compact. Then this operator is bounded and hence $\|g\|_{H_\infty} < \infty$. Let $\{w_j\}$ be a sequence in B such that $|w_j| \to 1$, and

$$f_j(z) = \left(\log \frac{2}{1 - |w_j|^2}\right)^{-1}\left(\log \frac{2}{1 - \langle z, w_j\rangle}\right)^2.$$

Then $\|f_j\|_{Q_p} \leqslant C$ and $f_j \to 0$ uniformly on any compact sets of B. So, $\|M_g(f_j)\|_{Q_q} \to 0$. Since

$$\left|g(z)f_j(z)\right| = \left|M_g(f_j)(z)\right| \leqslant C\left\|M_g(f_j)\right\|_{Q_q} \log \frac{2}{1 - |z|^2}, \quad z \in B,$$

we get (by letting $z = w_j$)

$$\left|g(w_j)\right| \log \frac{2}{1 - |w_j|^2} \leqslant C\left\|M_g(f_j)\right\|_{Q_q} \log \frac{2}{1 - |w_j|^2},$$

hence $g(w_j) \to 0$. Since g is bounded holomorphic function on B, it follows that $g = 0$. $\square$

Remark. The compactness result corresponding to Corollary 2.1 can be obtained. We do not go into details.

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