Null Controllability of Nonlinear Neutral Differential Equations

ROBERT G. UNDERWOOD

Department of Mathematics, Colorado School of Mines,
Golden, Colorado 80401

AND

E. N. CHUKWU

Department of Mathematics, University of Tennessee,
Knoxville, Tennessee 37916

Submitted by S. M. Meerkov

Received January 9, 1986

In this paper we show under very general conditions that the null controllability of a nonlinear neutral differential system is implied by the null controllability of its linear approximation. It has been previously shown that an analogous result is not in general true for retarded systems but that when certain restrictions are made on the retarded system such an inference can be made. For neutral systems, these further restrictions are not necessary. The essential difference lies in the fact that many controllable neutral systems can be "backed out" of the origin.

1. Introduction

In this paper we shall study the null controllability of nonlinear neutral systems. Linear neutral systems have been carefully studied in a variety of settings, e.g., [5, 7]. For autonomous differential equations of the neutral type, we show conditions under which the local null controllability of nonlinear systems can be inferred from the null controllability of their linear approximation. It is shown in [7] that null controllability of linear neutral systems is, in general, dependent on the length of the time interval over which the system is operating. The only restrictions on the time interval in this paper are those required by the controllability of the linear systems. Because we are concerned with nonlinear systems, the state space used in this paper is the space of continuous functions \( C([-h, 0], \mathbb{R}^n) \).
Consider the neutral differential equation

\[
\frac{d}{dt} \{ x(t) - A_{-1} x(t-h) - Bu(t) \} = L(x) + f(x, u(t)), \quad t \in [t_0, t_1]
\]  

(1.1)

where \( x_\theta(t) = x(t + \theta), \theta \in [-h, 0] \), and \( L, B, \) and \( f \) are suitably defined as described in Section 2. Observe that we allow the control to enter through the differentiated terms in (1.1). This is done in order to gain extra controllability since we are using \( C([-h, 0], \mathbb{R}^n) \) as our state space rather than the Sobolev space \( W^{(1)}([-h, 0], \mathbb{R}^n) \) typically used when analyzing linear systems, cf. [1]. The following examples demonstrate why this structure is advantageous.

**Example 1.** Consider the scalar system

\[
\frac{d}{dt} \{ x(t) - x(t-1) \} = x(t-1) + u(t),
\]

where \( t \in [0, T], T > 3 \). If \( x(t) = 0 \) on \([T-1, T]\), then \( x \) must be absolutely continuous on \([T-2, T-1]\). This implies \( x \) must be absolutely continuous on \([T-3, T-2]\). Continuing in this manner, we see that only absolutely continuous functions can be steered to the origin. Thus, the system is null controllable with respect to \( W^{(1)}([-1, 0], \mathbb{R}) \) but is not null controllable with respect to \( C([-1, 0], \mathbb{R}) \).

**Example 2.** Consider the scalar system

\[
\frac{d}{dt} \{ x(t) - x(t-1) - u(t) \} = x(t-1),
\]

where \( t \in [0, 1+\varepsilon] \) for some \( \varepsilon > 0 \). For any initial function \( \phi \in C([-h, 0], \mathbb{R}) \), it is easy to construct a control so that \( x(t) = 0 \) for all \( t \) in \([\varepsilon, 1+\varepsilon]\), i.e., it is null controllable.

To see the nature of our analysis, we will sketch our argument. Further, suppose \( f \) is defined so that the linearization of (1.1) is

\[
\frac{d}{dt} \{ y(t) - A_{-1} y(t-h) - Bu(t) \} = L(y), \quad t \in [t_0, t_1].
\]

(1.2)

Assume that (1.2) has a solution \( y(t) \) corresponding to initial condition \( y_{t_0} = \psi \) and final condition \( y_{t_1} = 0 \) using the control \( \tilde{u}(t) \). Now consider the final value problem

\[
\frac{d}{dt} \{ z(t) - A_{-1} z(t-h) \} = L(z) + f(z, (T\psi), \tilde{u}(t)), \quad t \in [t_0, t_1],
\]

(1.3)

\[ z_{t_1} = 0. \]
Assume that the solution to (1.3) can be "backed out" uniquely from the origin and denote this solution by \( z(t) = (\xi(t)) \). With this construction we have

\[
x(t) = y(t) + z(t) = (T\psi)(t) + (\xi\psi)(t)
\]

satisfying (1.1), where

\[
x_0 - \psi + (\xi\psi)_0 = \omega(\psi)
\]

and the "final value" is \( x_{i_1} = 0 \). Thus (1.1) is null controllable if \( \omega \) in (1.5) is an open map. The thrust of this paper is to establish this fact.

Critical to these arguments is the step where the \( z \) system is "backed out" from the origin. A key difference between analyzing nonlinear neutral control systems and nonlinear retarded control systems (see [13]) is that it is possible to reverse the time orientation for a large class of neutral control systems. For instance, consider

\[
\frac{d}{dt} \{ x(t) - A_-(t-1) - Bu(t) \} = A_0x(t) + A_1x(t-1)
\]

on \([t_0, t_1]\). With \( \hat{x}(s) = x(-s + t_1 + t_0 - 1) \) and \( \hat{u}(s) = (-s + t_1 + t_0 - 1) \), \( s \in [t_0 - 1, t_1 - 1] \), (1.6) becomes

\[
\frac{d}{ds} \{ \hat{x}(s) - A_-(s) - B\hat{u}(s) \} = -A_0\hat{x}(s) - A_1\hat{x}(s+1),
\]

where now the time sense is reversed. If \( A_- \) is invertible, then (1.7) is of the same form as (1.6) and no complications arise in "backing up" the neutral system (1.6). In this paper we will show that when \( A_- \) is invertible, the null controllability of (1.2) implies the local null controllability of (1.1).

2. Notation and a Description of the General Nonlinear Control Systems Considered

Let \( H \) be a Hilbert space with inner product \( \langle x, y \rangle \) and norm \( \| x \| = \langle x, x \rangle^{1/2} \). When appropriate we shall denote them by \( \langle x, y \rangle_H \) and \( \| x \|_H \) to avoid ambiguous notation. All statements concerning measures will refer to Lebesgue measure on the real line. For \( p \) a positive integer, denote by \( R^p \) the space of real \( p \)-tuples with the usual Euclidean norm. For \( E \subset R, E \) measurable, \( L_2(E, R^p) \) denotes the Hilbert space of square-integrable (equivalence classes of) functions from \( E \) to \( R^p \).
We will denote the collection of all $p \times q$ matrices (where $p$ and $q$ are positive integers) by $M_{pq}$. The space of bounded linear transformations from $X$ to $Y$ will be indicated by $\mathcal{B}(X, Y)$, where $X$ and $Y$ are Banach spaces, and if $X = Y$ we shall write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$. The identity operator will be denoted by $I$.

If $f$ is any continuous function from a subset of a Banach space $X$ to a Banach space $Y$ and if $f$ has a Frechet derivative at $x \in X$, then this derivative will be denoted by $Df(x)$. If $X$ is the cartesian product of several Banach spaces, then the partial derivative with respect to the $i$th variable will be denoted by $D_i f(x)$.

Throughout the remainder of the paper, $h$ will be a positive real number, $n$ and $m$ will be positive integers, and $t_0$ and $t_1$ will be real numbers, with $t_1 - t_0 > h$. The interval $[t_0, t_1]$ will be denoted by $J$.

For function space controllability, results obtained depend on the particular state space chosen (e.g., $[10, 7]$). As we explained earlier, because we are concerned with nonlinear control systems, we use the space of continuous functions as our state space.

We use the customary notation $x_i$ for the system's "states," where $x_i$ is defined to be an element of $C([-h, 0], R^n)$ given by $x_i(t) = x(t + \theta)$, $-h \leq \theta \leq 0$, $t \in J$. Since the control is allowed to enter through differentiated terms as shown in (1.1), the space of admissible control will always be $C([t_0, t_1], R^n)$. It will be convenient to use the following notations:

(i) $X = C([-h, 0], R^n)$, the state space;
(ii) $T = C([t_0 - h, t_1], R^n)$, the trajectory space;
(iii) $U = C([t_0, t_1], R^n)$, the control space.

The general control systems considered in this paper will now be described. Let $\mathcal{O} \subset R^n$ be an open convex set containing the origin. The form of the control system will be

$$\frac{d}{dt} \{x(t) - A_{-1} x(t-h) - Bu(t)\} = L(x_i) + f(x_i, u(t)),$$

where $f: C([-h, 0], \mathcal{O}) \times R^n \to R^n$, $L: C([-h, 0], R^n) \to R^n$, $A_{-1} \in \mathcal{M}_{nn}$, and $B \in \mathcal{M}_{nm}$ satisfy certain assumptions to be indicated below.

The technical assumptions on the nonlinear term $f$ are the following:

(A1) $f(\cdot, \cdot)$ is continuously differentiable.

(A2) For each compact set $K \subset \mathcal{O}$ there exist constants $M_i \geq 0$, $i = 1, 2, 3$, such that

$$\|D_1 f(\phi, w)\| \leq M_1 + M_2 \|w\|_{R^n},$$

$$\|D_2 f(\phi, \cdot)\| \leq M_3$$

for all $w \in R^n$ and $\phi \in C([-h, 0], K)$. 
(A3) \( f(0, 0) = 0 \).

(A4) \( D_1 f(0, 0) = 0 \) and \( D_2 f(0, 0) = 0 \).

For the linear term \( L \) we assume:

(A5) \( L \in \mathcal{B}(X, \mathbb{R}^n) \).

By the Riesz representation theorem, there exists a unique function \( \eta: [-h, 0] \to \mathcal{M}_{m}, \) where \( \eta \) is of bounded variation on \([-h, 0]\), is left-continuous on \((-h, 0)\), and satisfies \( \eta(0) = 0 \) and such that

\[
I.(\phi) = \int_{-h}^{0} \left[ d\eta(\theta) \right] \phi(\theta) \quad \text{for} \quad \phi \in X.
\]

(A6) There exists a constant \( M_a \geq 0 \) such that

\[
\left\| \int_{-h}^{0} \left[ d\eta(\theta) \right] \right\|_{\mathcal{M}_{m}} \leq M_a,
\]

where \( \| \cdot \|_{\mathcal{M}_{m}} \) is the operator norm for \( n \times n \) matrices.

Observe that \( \| I. (\cdot) \| \leq M_a \), where \( \| I. (\cdot) \| \) is the operator norm of \( L(\cdot) \).

We also will assume that

(A7) \( \text{Rank}[A_{-1}] = n \).

This relationship is utilized in the "backing out" argument (for interpretation of this phrase, cf. [8, pp. 4–11] in the proof of Theorem 3.8). Effectively (A7) allows the difference between (2.1) and its linear approximation to be backed out from the origin in \( C([-h, 0], \mathbb{R}^n) \).

\( x \) is said to be a solution of (2.1) on \( [t_0 - h, t_1] \), corresponding to the control \( u \in U \), if

(i) \( x \in C([t_0 - h, t_1], \mathbb{R}^n) \) and \( (x, u(t)) \) is in the domain of \( f \) for \( t \in [t_0, t_1] \);

(ii) \( \{x(t) - A_{-1} x(t - h) - Bu(t)\} \) is absolutely continuous on \( [t_0, t_1] \);

(iii) \( x \) satisfies (2.1) almost everywhere on \( [t_0, t_1] \).

If for some \( u \in U \), (2.1) has a unique solution on \( [t_0 - h, t_1] \), then we say that \( x \) is a trajectory on \( [t_0 - h, t_1] \) corresponding to the control \( u \) (which is in agreement with the notation \( T \) for the trajectory space as given in (ii) for this section). If \( x \) also satisfies the initial condition \( x_{t_0} = \phi \), then we say that \( x \) is the trajectory corresponding to \( u \) and \( \phi \). If \( x \) is the trajectory on \( [t_0 - h, t_1] \) corresponding to \( u \) and \( \phi \) if \( x_{t_0} = \psi \), then we say \( u \) steers \( \phi \) to \( \psi \) at time \( t_1 \). We shall say that (2.1) is (globally) null controllable on \( [t_0, t_1] \) if for every \( \phi \in X \) there exists a control \( u \) which steers \( \phi \) to \( 0 \in X \) at time \( t_1 \).
Equation (2.1) is \textit{locally null controllable} on \([t_0, t_1]\) if such a \(u\) exists for each \(\phi\) in some open neighborhood of the origin in \(X\).

Finally, the linear approximation to (2.1) is

\[
\frac{d}{dt} \{ y(t) - A_{-1} y(t-h) - Bu(t) \} = L(y_t),
\]

where \(A_{-1}, L, \) and \(B\) are the matrices and operators described in (2.1).

3. Preliminary Arguments

For \(\|u\| \) and \(\|x_{t_0}\|\) sufficiently small, the existence of a solution \(x\) on \([t_0-h, t_1]\) to (2.1) will be established in Theorem 3.3. For ordinary linear control systems, it is well known [6, p. 92] that if the system is null controllable, then a control which "does the job" can be chosen depending on the initial condition in a bounded linear way. In Lemma 3.4 we prove an analogous result pertaining to the system (2.2). The "backing out equation" is analyzed in Lemma 3.6.

Let us now consider the initial value problem

\[
\frac{d}{dt} \{ x(t) - A_{-1} x(t-h) - Bu(t) \} = L(x_t) + f(x_t, u(t)), \quad t \in [t_0, t_1]
\]

\(x_{t_0} = \phi \in X\)

on the interval \([t_0-h, t_1]\) and where \(f: C([-h, 0], \mathbb{C}) \times \mathbb{R}^n \to \mathbb{R}^n,\)

\(L: X \to \mathbb{R}^n, A_{-1}, \) and \(B\) all satisfy the assumptions (A1)-(A7).

Note that for any \(u \in U, \phi \in X,\) and there exists at most one solution of (3.1) on \([t_0-h, t_1]\) satisfying the initial condition \(x_{t_0} = \phi.\) This can be proved using (A2) and standard Gronwell-type arguments.

We shall analyze (3.1) in its integral form and consider it as an operator equation. The implicit function theorem will be used to prove the existence of its solution and the differentiability of this solution with respect to its initial condition and its control.

Define \(G: T \to T\) by

\[
G(x)(t) = \begin{cases} 
0 & \text{if } t_0 - h \leq t < t_0 \\
A_{-1} \{ x(t-h) - x(t_0-h) \} & \text{if } t_0 \leq t \leq t_1.
\end{cases}
\]

Define \(H: C([t_0-h, t_1], \mathbb{C}) \times U \to T\) by

\[
H(x, u)(t) = \begin{cases} 
0 & \text{if } t_0 - h \leq t < t_0 \\
B(u(t) - u(t_0)) + \int_{t_0}^{t} L(x_s) + f(x_s, u(s)) \, ds & \text{if } t_0 \leq t \leq t_1.
\end{cases}
\]
Finally, define $\mathcal{F} : X \to T$ by
\[
(\mathcal{F} \phi)(t) = \begin{cases} 
\phi(t - t_0) & \text{if } t_0 - h \leq t < t_0 \\
\phi(0) & \text{if } t_0 \leq t \leq t_1. 
\end{cases}
\] (3.4)

If the transformation given by (3.3) is well defined, i.e., the integrand is integrable, then $x$ is a solution of (3.1) on $[t_0 - h, t_1]$ if and only if it satisfies
\[
x = Gx + H(x, u) + \mathcal{F} \phi
\] (3.5)

for some $x \in T = C([t_0 - h, t_1], R^n)$.

To see that (3.3) is well defined, observe the following. Suppose $f$ in (3.3) satisfies (A1)--(A3) and let $\mathcal{O}, K, M_1,$ and $M_3$ be as stated in these assumptions. It follows from the mean value theorem (which uses the convexity of $K$) that
\[
\|f(\phi, w)\|_{R_n} \leq \|f(0, 0)\|_{R_n} + M_1\|\phi\| + M_3\|w\|_{R_n}
\] (3.6)

for $t \in [t_0, t_1]$, $\phi \in C([-r, 0], K)$, and $w \in R^n$. Using standard techniques, it can be shown for $x \in C([-h, 0], K)$, $\phi \in X$, and $u \in U$ that $f(x_t, u(t))$, $D_1f(x_t, u(t))x_t$, $D_1f(x_t, u(t))\phi$, and $D_2f(x_t, u(t))$ are integrable functions of $t$ on $[t_0, t_1]$.

It is here and in establishing the differentiability of $H$ in (3.3) that the assumptions (A1)--(A3) are required. If the controls are restricted to a compact set, assumption (A2) could be relaxed somewhat.

**Lemma 3.1.** Suppose $H : C([-h, 0], 0) \times U \to T$ is defined by (3.3), where $L$, $B$, and $f$ are given as in (3.1) (and therefore satisfy (A1)--(A6)), and suppose $G : T \to T$ is defined by (3.2). Then $H$ is continuously differentiable, and
\[
(D_1H(0, 0)x)(t) = \begin{cases} 
0 & \text{if } t_0 - h \leq t < t_0 \\
\int_{t_0}^t L(x_s) \, ds & \text{if } t_0 \leq t \leq t_1 
\end{cases}
\] (3.7)

and
\[
(D_2H(0, 0)u)(t) = \begin{cases} 
0 & \text{if } t_0 - h \leq t < t_0 \\
B(u(t) - u(t_0)) & \text{if } t_0 \leq t \leq t_1 
\end{cases}
\] (3.8)
hold for all \( x \in T, u \in U, \) and \( t \in [t_0 - h, t_1]. \) Also \( G \) is continuously differentiable and for any \( \tilde{x} \) in \( T \)

\[
DG(\tilde{x}) x = Gx. \quad (3.9)
\]

The proof of this lemma is essentially the same as the proof described in Lemma 2.3 of [13]. The limits involved in verifying (3.7) and (3.8) and the continuity of these maps are taken "under the integral" using the Lebesgue dominated convergence theorem, cf. [11]. These arguments rely on assumptions (A1), (A2), (A5), and (A6) and the integrability of \( f \) and its derivatives.

**Lemma 3.2.** Define \( K \in \mathcal{B}(T) \) by

\[
K x = DG(0) x + D_1 H(0, 0) x. \quad (3.10)
\]

Then \( (I - K)^{-1} \) exists and is a bounded linear map.

**Proof.** Let \( y \) be an arbitrary function in \( T. \) Consider

\[
\frac{d}{dt} \{x(t) - A \ldots, x(t-h) - y(t)\} = Lx, \quad t \in \left[t_0, t_1\right]. \quad (3.11)
\]

where \( x_{t_0} = y_{t_0} = \phi. \)

From Theorem 8.1 on p. 301 of [4], we know (3.11) has a unique solution. The integrated form for (3.11) is

\[
x(t) = (DG(0) x)(t) + \phi(0) + y(t) - y(t_0) + (D_1 H(0, 0) x)(t), \quad t \in \left[t_0, t_1\right]. \quad (3.12)
\]

Since \( x_{t_0} = y_{t_0} \) and thus \( y(t_0) = \phi(0), \)

\[
x = DG(0) x + D_1 H(0, 0) x + y \quad (3.13)
\]

also has a unique solution for any \( y \in T. \)

We conclude that \( I - K \) is a one-to-one map from \( T \) onto \( T. \) Furthermore, from the definition of \( G \) and assumption (A6) it follows that \( I - K \) is a bounded linear mapping. As a consequence of the open-mapping theorem, \( (I - K)^{-1} \) is a bounded linear map, cf. [12].

Now we shall examine the existence of solutions to the nonlinear problem (3.1) in its integrated form (3.5). Let \( \zeta(\phi, u) \in T \) be the (necessarily unique) solution of (3.5) for any pair \( (\phi, u) \) such that the solution exists.

**Theorem 3.3.** Let \( H \) be given by (3.3) and let \( \zeta(\phi, u) \) be defined as above. Then there exist open neighborhoods \( N_X \) and \( N_U \) of the origin in \( X \) and \( U, \) respectively, so that for \( \phi \in N_X \) and each \( u \in N_U, \) (3.5) has a unique solution that is continuously differentiable on \( N_X \times N_U. \)
Proof. Define

\[ T(x, u, \phi) - x - Gx - H(x, u) - J\phi. \]

Then we see from (A3) that \( T(0, 0, 0) = 0 \), and by Lemma 3.1, \( T \) is continuously differentiable on \( C([t_0 - h, t_1], 0) \times U \times X \) with

\[ D_1 T(0, 0, 0) = I - DG(0) - D_1 H(0, 0). \]

It follows from Lemma 3.2 that \( D_1 T(0, 0, 0)^{-1} \) exists. Therefore, the implicit function theorem (cf. [2, p. 265]) implies that there exists a continuously differentiable function \( \xi(\phi, u) \) defined on an open neighborhood \( N \) of the origin in \( X \times U \) such that

\[ T(\xi(\phi, u), u, \phi) = 0 \]

for all \((\phi, u) \in N\). Furthermore, \( x = \xi(\phi, u) \) satisfies (3.5) or equivalently (3.1), and the solution is unique (see the note following (3.1)). By choosing \( N_X \) and \( N_U \) so that \( N_X \times N_U \subset N \), the proof is complete.

We shall now prove that the null controller for the linear system approximating (3.1) can be chosen in a bounded linear manner.

**Lemma 3.4.** Let \( L \in \mathcal{B}(X, \mathbb{R}^n) \) satisfy (A5) and (A6). Let \( A_{-1} \in \mathcal{M}_{nn} \) and \( B \in \mathcal{M}_{nm} \). Suppose the system

\[ \frac{d}{dt} \{ x(t) - A_{-1} x(t - h) - Bu(t) \} = L(x, t) \quad (3.14) \]

is null controllable. For each initial function \( \phi \in X \) and corresponding null controller \( u \in U \) denote the solution of (3.14) by \( \xi(\phi, u) \). Then there exists a bounded linear mapping \( S: X \rightarrow T \) such that for each \( \phi \in X \) and for \( t \in [t_1 - h, t_1] \),

\[ \xi(\phi, S\phi)(t) = 0. \quad (3.15) \]

Furthermore, the mapping \( \Gamma: X \rightarrow T \) defined by

\[ \Gamma \phi = \xi(\phi, S\phi) \quad (3.16) \]

is bounded and linear.

**Proof.** Let \( G \) be defined as in (3.2), let \( H \) be defined as in (3.3) with \( f = 0 \), and let \( J \) be defined as in (3.4). The integrated form of (3.14) with initial condition \( x_{t_0} = \phi \) is

\[ x = DG(0) x + D_1 H(0, 0) x + D_2 H(0, 0) u + J\phi \quad (3.17) \]
or, using the notation of (3.10),
\[ x = Kx + D_2H(0, 0)u + \phi, \]  
(3.18)
where the differential operators are those in Lemma 3.1.

We know from Lemma 3.2 that the operator \((I - K)^{-1}\) exists and is a bounded linear map. Hence the solution to (3.18) can be expressed as
\[ \xi(\phi, u) = (I - K)^{-1}(D_2H(0, 0)u + \phi). \]
(3.19)
Define \(p \in B(X)\) by
\[ (\phi)(\Theta) = ((I - K)^{-1}\phi)(t_1 + \Theta), \quad -h \leq \Theta \leq 0, \]
and \(Q \in B(U, X)\) by
\[ (Qu)(\Theta) = ((I - K)^{-1}D_2H(0, 0)u)(t_1 + \Theta), \quad -h \leq \Theta \leq 0. \]
Using this terminology, the final value of the solution is given by
\[ \xi(\phi, u)(t_1 + \Theta) = (P\phi)(\Theta) + (Qu)(\Theta), \quad -h \leq \Theta \leq 0. \]
(3.20)
Thus (3.14) is null controllable on \([t_0, t_1]\) if for every \(\phi \in X\) there exists a \(u \in U\) such that \(Qu + P\phi = 0\), or in other words \(P(x) \subset Q(u)\).

\(Q\) is a bounded linear map but not necessarily one-to-one. Let \(N'\) be the null space of \(Q\) and let \(U/N'\) be the factor space where \(N' + U/N' = U\). Let \(Q_1 : U/N' \to Q(U)\) be the restriction of \(Q\) to \(U/N'\). \(Q_1\) is necessarily a one-to-one onto map. Define \(S : X \to U\) by \(S\phi = -Q_1^{-1}P\phi\).

Thus,
\[ \xi(\phi, S\phi)(t_1 + \Theta) = (Q_1\phi + P\phi)(\Theta) = 0, \quad -h \leq \Theta \leq 0, \]
(3.21)
and we see that \(u \neq S\phi\) steers to \(0 \in X\), i.e., it is a null controller.

We will now show that \(S\) is a bounded linear mapping. The open mapping theorem, cf. [12], implies \(Q_1^{-1}\) is bounded if \(U/N'\) and \(Q(U)\) are Banach spaces. Clearly \(U/N'\) is closed, but it may be the case that \(Q(U)\) is not closed. To see that \(S\) is bounded, consider the following argument. Let \(\{\phi_n\}\) be a convergent sequence in \(X\) such that \(\{S\phi_n\}\) converges in \(U\), and let \(\phi = \lim_{n \to \infty} \phi_n\) and \(u = \lim_{n \to \infty} S\phi_n\). Since \(U/N'\) is closed, \(u \in U/N'\). Since \(Q\) and \(P\) are continuous, \(Qu + P\phi = \lim_{n \to \infty} Q\phi_n + P\phi_n = 0\). Therefore by our construction of \(Q_1\), \(u = Q_1^{-1}P\phi = S\phi\), and hence \(S\) is closed. The closed graph theorem implies that \(S\) is bounded.

Since \((I - K)^{-1}, D_2H(0, 0), S, \) and \(\phi\) are bounded linear maps, \(\xi(\phi, S\phi)\), where \(\xi\) is defined by (3.19), is a bounded linear map.

We shall now consider the “backing out” equation
\[ \frac{d}{dt}\left\{z(t) - A^{-1}z(t - h)\right\} = L(z, s) + f((I\psi(t), + z, S\psi(t)), \quad t \in [t_0 - h, t_1 - h] \]
(3.22)
together with its final value condition

\[ z_{t_1} = 0. \tag{3.23} \]

Assume \( A_{-1}, L, \) and \( f \) are defined as in (3.1) and satisfy (A1)-(A7), and \( \psi \subset X \) and \( S \) and \( \Gamma \) are defined as in Lemma 3.4. First observe that if (3.22) has a unique solution, then

\[ x = \Gamma \psi + z \]

satisfies (3.1) with \( x_{t_0} = \psi + z_{t_0} \) and \( x_{t_1} = 0. \) In Section 4 we will show that \( \psi + z_{t_0} \) cover an open neighborhood of the origin in \( X. \)

In the following discussion, we will establish that (3.22) has a unique solution which is differentiable with respect to \( \psi. \) To do this, we will reverse the sense of time.

Define \( i: [t_0 - h, t_1] \to [t_0 - h, t_1] \) to be

\[ i(t) = -t + t_1 + t_0 - h, \quad t \in [t_0 - h, t_1]. \]

Observe that \( i(t_1) = t_0 - h \) and \( i(t_0 - h) = t_1. \)

Define \( R: T \to T \) by

\[ (Rz)(\tau) = z(i(\tau)), \quad \tau \in [t_0 - h, t_1], \]

where \( z \) is an arbitrary element of \( T. \) Also denote

\[ \hat{z}(\tau) = z(i(\tau)), \quad \tau \in [t_0 - h, t_1] \]

and thus

\[ \hat{z}(\tau + h) = z(i(\tau) - h), \quad \tau \in [t_0 - h_1, t_1 - h]. \]

Define \( \rho: X \to X \) by

\[ (\rho \phi)(\theta) = \phi(-h - \theta). \]

Observe that for \( \tau + h \in [t_0, t_1] \)

\[ \rho \hat{z}_{\tau + h} = z_{i(\tau)}. \]

Finally, define \( \hat{S}: X \to U \) by

\[ (\hat{S}\psi)(\tau + h) = (S\psi)(i(\tau)), \quad \tau \in [t_0 - h, t_1 - h]. \]

Using the above transformations, (3.22) becomes

\[ \frac{d}{d\tau} \{ \hat{z}(\tau) - A_{-1} \hat{z}(\tau + h) \} \]

\[ = -L(\rho \hat{z}_{\tau + h}) - f(\rho (RI' \psi)_{\tau + h} + \hat{z}_{\tau + h}, \hat{S}\psi(\tau + h)), \tag{3.24} \]
where \( \tau \in [t_0 - h, t_1 - h] \) and the final value is now the initial condition

\[
\rho \hat{z}_{t_0} = 0. \quad (3.25)
\]

Multiplying both sides of (3.24) by \(-A^{-1}\) and substituting \( \tau \) for \( \tau + h \) yields

\[
\frac{d}{dt} \{ \hat{z}(\tau) - A^{-1}_\tau \hat{z}(\tau - h) \} = A^{-1}_\tau L(\rho \hat{z}_\tau) + A^{-1}_\tau f(\rho [(RG\hat{\psi})_\tau + \hat{\psi}_\tau], \hat{\psi}(\tau)),
\]

\( \tau \in [t_0, t_1]. \) \( (3.26) \)

The initial condition can be expressed equivalently as

\[
\hat{z}_{t_0} = 0. \quad (3.27)
\]

Since \( R \) is a one-to-one, bounded, linear mapping from \( T \) onto \( T \), \( R^{-1} \) exists and is bounded and linear. In fact, we can compute it. The same applies for \( \rho \) and \( \hat{\psi} \). From the above construction we conclude that if \( \hat{z} \) is a solution of (3.26) with initial condition (3.27), then \( z = R^{-1} \hat{z} \) is a solution of (3.22) with the final value condition (3.23).

We shall examine (3.26) in the same manner as we did (3.1). Define \( \hat{L} : X \to R^n \) by

\[
\hat{L}(\phi) = A^{-1}_\tau L(\rho \phi), \quad \phi \in X.
\]

Thus \( \hat{L} \in \mathcal{B}(X, R^n) \) and satisfies (A5) and (A6). Define \( \hat{f} : C([-h, 0], \emptyset) \times R^m \to R^n \) by

\[
\hat{f}(\phi, w) = A^{-1}_\tau f(\rho \phi, w)
\]

for all \( \phi \in C([-h, 0], \emptyset) \) and \( w \in R^m \). \( \hat{f}(\cdot, \cdot) \) is continuously differentiable,

\[
D_1 \hat{f}(\phi, w) = A^{-1}_\tau D_1 f(\rho \phi, w) \rho,
\]

and

\[
D_2 \hat{f}(\phi, w) = A^{-1}_\tau D_2 f(\rho \phi, w).
\]

Thus \( \hat{f} \) satisfies (A1)–(A4). Using this notation (3.26) becomes

\[
\frac{d}{dt} \{ \hat{z}(\tau) - A^{-1}_\tau \hat{z}(\tau - h) \} = \hat{L}(\hat{z}_\tau) + \hat{f}((RG\hat{\psi})_\tau + \hat{\psi}_\tau, \hat{\psi}(\tau)). \quad (3.28)
\]

Define \( \hat{G} : T \to T \) by

\[
(\hat{G}\hat{z})(\tau) = \begin{cases} 
0 & \text{if } t_0 - h \leq \tau < t_0 \\
A^{-1}_\tau \{ \hat{z}(\tau - h) - \hat{z}(t_0 - h) \} & \text{if } t_0 \leq \tau \leq t_1.
\end{cases} \quad (3.29)
\]
Define $H_1: T \to T$ by
\[
(H_1(\tau))(z) = \begin{cases} 0 & \text{if } t_0 - h \leq \tau < t_0 \\ \int_{t_0}^{\tau} \hat{L}(\tau) \, ds & \text{if } t_0 \leq \tau \leq t_1. \end{cases}
\] (3.30)

Define $H_2: C([-t_0-h, t_1], \mathcal{C}) \times U \to T$ by
\[
H_2(\tau, u)(\tau) = \begin{cases} 0 & \text{if } t_0 - h \leq \tau < t_0 \\ \int_{t_0}^{\tau} \hat{J}(\tau, u(s)) \, ds & \text{if } t_0 \leq \tau \leq t_1. \end{cases}
\] (3.31)

From the integrated viewpoint (3.26) becomes
\[
\dot{z} = \hat{G}\dot{z} + H_1(\dot{z}) + H_2(R\dot{\Gamma}\psi + \dot{z}, \dot{\psi}).
\] (3.32)

In the following theorem we will prove the existence of a solution to (3.32) and its parent equation (3.22).

**Theorem 3.5.** There exists an open neighborhood $\mathcal{N}_X$ about the origin in $X$ and a continuously differentiable function $\zeta: \mathcal{N}_X \to T$ such that $z = \zeta \psi$ satisfies (3.22) and its final value condition (3.23). Furthermore, $\zeta(0) = 0$ and $D\zeta(0) = 0$.

**Proof.** Assume that $A_{-1}$, $L$, and $f$ satisfy assumptions (A1)-(A7) and $S$ and $T$ are defined as in Lemma 3.4. Since $f(0) = 0$, there exist open balls $N_T$ and $N_X$ around the origin in $T$ and $X$ such that $N_T + R\Gamma(N_X) \subset C([-t_0-h, t_1], \mathcal{C})$, where $C([-t_0-h, t_1], \mathcal{C}) \times R^m$ is the domain of $f$.

Let $M: N_T \times N_X \to T$ be given by
\[
M(\dot{z}, \psi) = \hat{G}\dot{z} + H_1(\dot{z}) + H_2(R\dot{\Gamma}\psi + \dot{z}, \dot{\psi}),
\] (3.33)

where $\hat{G}$, $H_1$, and $H_2$ are defined by (3.29), (3.30), and (3.31). $\hat{G}$ is the same as $G$ in (3.2) with $A_{-1}$ replacing $A_{-1}$ in (3.2). $H_1$ is of the form $H$ in (3.3) with $B = 0$, $f = 0$, and $u$ suppressed. Similarly, $H_2$ is of the form $H$ in (3.3) with $L = 0$ and $B = 0$. Hence, by Lemma 3.1, $\hat{G}$, $H_1$, and $H_2$ are continuously differentiable over their respective domains. From the chain rule and Lemma 3.1, $D_2M(0, 0) = 0$. Also,
\[
D_1M(0, 0) = D\hat{G}(0) + DH_1(0) + D_1H_2(0, 0) = D\hat{G}(0) + D_1H(0, 0),
\]
where $H: C([-t_0-h, t_1], \mathcal{C}) \times U \to T$ is given by $H(\dot{z}, u) = H_1(\dot{z}) + H_2(\dot{z}, u)$. From Lemma 3.2 we know that $(I - D_1M(0, 0))^{-1}$ exists and is a bounded linear map.

Since $M(0, 0) = 0$, we can apply the implicit function theorem to solve the equation $z = M(z, \psi)$. Thus, there is an open neighborhood $\mathcal{N}_X$ about the origin in $X$ and a continuously differentiable function $\tilde{\zeta}: \mathcal{N}_X \to T$ such that $\tilde{\zeta}(0) = 0$ and $\zeta(\psi) = M(\tilde{\zeta}(\psi), \psi)$ for all $\psi$ in $\mathcal{N}_X$. The chain rule implies
that $D\zeta(0) = D_1 M(0, 0) D_\zeta(0) + D_2 M(0, 0)$. Since $D_2 M(0, 0) = 0$ and $(I - D_1 M(0, 0))^{-1}$ exists, $D\zeta(0) = 0$.

It follows from our formulation of (3.28) and (3.31) that $\dot{z} = \zeta(\psi)$ satisfies (3.26) and (3.27). Therefore, $z = \zeta(\psi) = R^{-1}\zeta(\psi)$ satisfies (3.22) and its final value condition (3.23). Furthermore, $\zeta(0) = 0$ and $D\zeta(0) = 0$.

### 4. The Null Controllability of (3.1) Assuming $A_1^1$ Exists

Consider the nonlinear neutral control system

$$\frac{d}{dt}\{x(t) - A_{-1} x(t-h) - Bu(t)\} = L(x, ) + f(x, , u(t)), \quad t \in [t_0, t_1],$$  \hspace{1cm} (4.1)

where $A_{-1}$, $L$, $B$, and $f$ satisfy assumptions (A1)-(A7). Thus, here we are assuming $A_1^1$ exists. The linear approximation to (4.1) is

$$\frac{d}{dt}\{y(t) - A_{-1} y(t-h) - Bu(t)\} = L(y, ).$$  \hspace{1cm} (4.2)

In Theorem 4.1 below, the null controllability of (4.2) is shown to imply the local null controllability of (4.1). As outlined in Section 1, this result is based on the backing out equation

$$\frac{d}{dt}\{z(t) - A_{-1} z(t-h)\} = L(z, ) + f(y, + z, , u(t)), \quad t \in [t_0-h, t_1-h],$$  \hspace{1cm} (4.3)

where $y$ is a solution to the null control problem (4.2) satisfying $y_{t_1} = 0$, $u$ is chosen as in Lemma 3.4, and $z_{t_1} = 0$.

**Theorem 4.1.** The null controllability of (4.1) on $[t_0, t_1]$ implies the local null controllability of (4.2).

**Proof.** Assume that $A_{-1}$, $L$, $B$, and $f$ satisfy the hypothesis (A1)-(A7), and assume that (4.2) is null controllable on $[t_0, t_1]$. Let $y = \zeta(\psi, u)$ denote the solution of (4.2) corresponding to the initial function $\psi$ and control $u$. From Lemma 3.4 there exists an operator $S \in \mathcal{B}(X, U)$ such that for all $\psi \in X$, $y = \Gamma \psi = \zeta(\psi, S \psi)$ satisfies $y_{t_1} = (\Gamma \psi)_{t_1} = 0$.

Consider (4.3), where $y = \Gamma \psi$ and $u = S \psi$. Theorem 3.5 implies that there exists an open neighborhood $\mathcal{N}_x$ about the origin in $X$ and a mapping $\zeta: \mathcal{N}_x \to T$ such that $z = \zeta \psi$ satisfies (4.3) and the final condition $z_{t_1} = 0$.

Letting $x = y + z = \Gamma \psi + \zeta \psi$ we see that $x$ satisfies (4.1), where the control is $u = S \psi$. Furthermore, $x_{t_0} = \psi + (\zeta \psi)_{t_0}$ and $x_{t_1} = y_{t_1} + z_{t_1} = 0.$
Define $\omega: \mathcal{N}_X \to X$ by
\[
\omega(\psi) = \psi + (\zeta \psi)_{t_0}.
\]

The mapping from $T$ to $X$ given by $x \to x_{t_0}$, $x \in T$, is a bounded linear map. From Theorem 3.5 we know that $\zeta(0) = 0$ and $D\zeta(0) = 0$. Consequently, $\omega(0) = 0$ and $D\omega(0) = I$. By the implicit function theorem, the range of $\omega$ covers an open neighborhood of the origin in $X$. For each $\psi \in \mathcal{N}_X$, $u = S\psi$ steers $x_{t_0} = \omega(\psi)$ to $0 \in X$ at time $t_1$ for (4.1). The proof is complete.

5. **Techniques for Analyzing the Null Controllability of Nonlinear Neutral Systems where $A_{-1}$ is Not Invertible**

In this section we will show how the null controllability of nonlinear neutral systems can be achieved when the inverse of $A_{-1}$ fails to exist. To do this we shall use a feedback control. The following examples will help to illustrate our approach.

**Example 3.** Consider the system
\[
\begin{align*}
\frac{d}{dt} \{x_1(t) - x_2(t-1)\} &= x_2(t-1) \\
\frac{d}{dt} \{x_2(t) - u(t)\} &= 0.
\end{align*}
\]

This system is null controllable on $[0, 2 + \varepsilon]$ for any $\varepsilon > 0$. A control $u$ steering $x_0 = \phi \in C([-1, 0], R^2)$ to $0 \in C([-1, 0], R^2)$ at time $2 + \varepsilon$ can be found by setting
\[
\begin{align*}
u(t) &= at^2 + bt, \quad 0 \leq t \leq \varepsilon \\
&= ae^2 + be, \quad \varepsilon < t \leq 2 + \varepsilon,
\end{align*}
\]
where $a$ and $b$ are chosen so that $x_2(\varepsilon) = 0$ and $x_1(1 + \varepsilon) = 0$. Notice that here
\[
A_{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
\]
which is not invertible.

**Example 4.** Consider the system
\[
\begin{align*}
\frac{d}{dt} \{x_1(t) - x_2(t-1)\} &= x_2(t-1) \\
\frac{d}{dt} \{x_2(t) - x_1(t-1) - u(t)\} &= x_1(t-1).
\end{align*}
\]
Here

\[ A_{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (5.4) \]

which is invertible. This system is also null controllable on \([0, 2 + \varepsilon]\) as can be seen by the following construction. For any \(\phi \in C([-1, 0], R^2)\) let \(\alpha\) be a control that steers system (5.1) to \(0 \in C([-1, 0], R^2)\) at time \(2 + \varepsilon\), and denote the corresponding trajectory by \(y(t)\). For system (5.3) choose the control to be

\[ u(t) = -y_1(t-1) + \alpha(t) - \int_0^t y_1(s-1) \, ds \]

for \(t \in [0, 2 + \varepsilon]\). Using this feedback control for (5.3), the trajectory \(y(t)\) for system (5.1) also satisfies (5.3) on \([0, 2 + \varepsilon]\), and thus we have shown that (5.3) is null controllable.

**Example 5.** Consider the system

\[ \begin{align*}
\frac{d}{dt} \{x_1(t) - x_2(-1)\} &= \frac{1}{2} x_2^3(t-1) + x_1^3(t) \\
\frac{d}{dt} \{x_2(t) - u(t)\} &= 0.
\end{align*} \]

The linear approximation to this system is (5.1), which is null controllable. However, Theorem 4.1 does not apply since \(A_{-1}\) given by (5.2) is not invertible. However, the nonlinear system

\[ \begin{align*}
\frac{d}{dt} \{x_1(t) - x_2(t-1)\} &= \frac{1}{2} x_2^3(t-1) + x_1^3(t) \\
\frac{d}{dt} \{x_2(t) - x_1(t-1) - u(t)\} &= x_1(t-1)
\end{align*} \]

is locally null controllable since (5.3) is null controllable and \(A_{-1}\) given by (5.4) is invertible. For system (5.6) suppose \(\hat{u}\) steers \(\phi \in C([-1, 0], R^2)\) to the origin at time \(2 + \varepsilon\), and denote the corresponding trajectory of (5.6) by \(y(t)\). For system (5.5) let the control be

\[ u(t) = y_1(t-1) + \hat{u}(t) + \int_0^t y_1(s-1) \, ds, \quad (5.7) \]

where \(t \in [0, 2 + \varepsilon]\). As with Examples 3 and 4, we see that \(y\) also satisfies
(5.5) when the control is given by (5.7). We conclude that (5.5) is locally null controllable.

We need the following definition. For any \( n \times n \) matrix \( A \) and \( n \times m \) matrix \( B \) define the \( n \times v \) matrix

\[
K_v[A, B] = [B, AB, \ldots, A^{v-1}B]
\]

for integers \( v, 1 \leq v \leq n \). In the discussion below, \( \perp \) will denote orthogonal complement. We have the following theorem.

**Theorem 5.1.** Let \( A_{-1}, B, I \) and \( f \) satisfy (A1)-(A6). Assume \( \text{Rank} \ K_v[A_{-1}, B] = n \) for some \( v, 1 \leq v \leq n \). Assume

\[
f(\cdot, w + v) = f(\cdot, w)
\]

for all \( w \in \mathbb{R}^m \), \( v \in \{v: Bv \in (\text{Range} A_{-1})^\perp\} \). If the linear neutral control system

\[
\frac{d}{dt} \{x(t) - A_{-1}x(t-h) - Bu(t)\} = L(x_i), \quad t \in [t_0, t_1]
\]

(5.9) is null controllable on \([t_0, t_1]\), then the nonlinear neutral control system

\[
\frac{d}{dt} \{x(t) - A_{-1}x(t-h) - Bu(t)\} = L(x_i) + f(x_i, u(t)), \quad t \in [t_0, t_1]
\]

(5.10)

is locally null controllable on \([t_0, t_1]\).

Before proceeding with the proof of this result, consider the following companion system to (5.10):

\[
\frac{d}{dt} \{y(t) - A_{-1}y(t-h) - Bu(t)\} = L(y_i) + f(y_i, u(t)), \quad t \in [t_0, t_1],
\]

(5.11)

where \( A_{-1} = A_{-1} + BC \) for some \( C \in \mathcal{M}_{mn} \) and \( y_{t_0} = \phi, \phi \in X \). Assume that \( f(\cdot, w + Cv) = f(\cdot, w) \) for all \( w \in \mathbb{R}^m \) and \( v \in \mathbb{R}^n \). Suppose \( y \) is a solution of (5.11) on \([t_0, t_1]\) corresponding to the control \( u \in U \). If for system (5.10) we choose the control to be

\[
u(t) = Cy(t-h) + \int_{t_0}^{t} u(s) \, ds, \quad t \in [t_0, t_1],
\]

then \( x = y \) is the solution to (5.10) corresponding to this control. These results are summarized in the following lemma.
LEMMA 5.2. Consider the control systems (5.10) and (5.11), where $A_{-1}, L, B, \text{ and } f$ satisfy (A1)-(A6). Suppose $C \in \mathcal{M}_{nn}$ and $\mathcal{A}_{-1} = A_{-1} + BC$. Further assume $f(\cdot, w + Cv) = f(\cdot, w)$ for all $w \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. If $y$ is the trajectory corresponding to $u$ and $\phi$ for (5.11), then $y$ is also the trajectory corresponding to $u$ and $\phi$ for (5.10) when the control

$$\alpha(t) = Cy(t - h) + \int_{t_0}^{t} u(s) \, ds, \quad t \in [t_0, t_1].$$

Notice that if $C$ in Lemma 5.2 can be chosen so that $A_{-1}$ has full rank, system (5.2) can be made to “track” a system of the form (4.1) with $A_{-1}$ replacing $A_{-1}$. Our final observation, Remark 5.3, indicates how to make an appropriate choice of $C$ in order to “fill in” the Rank $A_{-1}$ when Rank $A_{-1} < n$. The construction in this remark can be used to convert control systems into a canonical form (cf. [8, p. 90]).

Remark 5.3. Suppose $A \in \mathcal{M}_m, B \in \mathcal{M}_{mn}$ and $\text{Rank } K_v[A, B] = n$ for some integer $v$, $1 \leq v \leq n$. Assume Rank $B = m$. Then there exists an $m \times n$ matrix $C$ such that $A + BC$ has rank equal to $n$ and Range $BC = \{\text{Range } A\}^\perp$.

Proof. Suppose Rank $K_v[A, B] = n$ for some $v$, $1 \leq v \leq n$. If $v = 1$, we are done. Denote $\text{Rank } B = m_0$ and assume that $m_0 = m$. Define $P_0 = B$. Now define inductively $P_i$ to be the first $m_i$ independent vectors in $AP_{i-1}$ not in the column span of $[P_{i-1}, \ldots, P_0]$, where $m_i = \text{Rank}[AP_{i-1}, P_{i-1}, \ldots, P_0] - \text{Rank}[P_{i-1}, \ldots, P_0]$. Denote $\mathcal{P} = [P_1, \ldots, P_0]$. It can be shown that $A\mathcal{P}_{-1}$ is in the column span of $\mathcal{P}$. Therefore, if for some column $b_k$ of $B$ we have $A'b_k$ in the column span of $\mathcal{P}_{-1}$, then $A'b_k$, $i < j$, is in the column span of $\mathcal{P}_{j-1}$. It follows that $m_{i+1} \leq m_i$. Define the integer $\alpha$ by the condition $m_{\alpha} > 0$ and $m_{\alpha+1} = 0$. Since the Rank $K_v[A, B] = n$, $\sum_{i=0}^{\alpha} m_i = n$. We can enumerate and reorder the columns in $B$, $b_1, \ldots, b_m$, so that $P_i = A'[b_1, \ldots, b_{m_i}]$, $i = 1, \ldots, \alpha$. Denote $P = \mathcal{P}_\alpha$. By our construction $P$ is an $n \times n$ matrix with Rank $P = n$.

We will now compute $P^{-1}AP$ and $P^{-1}B$, which will yield a canonical form. Let $J = \{i : m_{j+1} = m_j, \quad 0 \leq j \leq \alpha - 1\}$ and $J^c = \{j : m_{j+1} < m_j, \quad 0 \leq j \leq \alpha - 1\} \cup \{\alpha\}$. If $j \in J$ then $AP_j = P_{j+1}$. If $j \in J^c$ then

$$AP_j = A'^{j+1}[b_1, \ldots, b_{m_j}] = [P_{j+1}, D_{j+1}],$$

where $D_{j+1} = A'^{j+1}[b_{m_j+1}, \ldots, b_{m_j}]$, i.e., $D_{j+1}$ is the $n \times (m_j - m_{j+1})$ matrix which consists of the columns in $AP_j$ that are discarded to form $P_{j+1}$. Using these definitions we have

$$AP = [D_{\alpha+1}, P_\alpha, D_{\alpha}, \ldots, P_1, D_1], \quad (5.12)$$

where the $D_j$ matrices are omitted when $j \in J$.
By our construction, \( D_j, j \in J' \), is in the column span of \( \mathcal{P}_{j-1} \), and consequently there exists a \( (\sum_{i=0}^{j-1} m_i) \times (m_{j-1} - m_j) \) matrix \( \Delta_j \) such that

\[
D_j = \mathcal{P}_{j-1} \Delta_j, \quad j \in J'.
\]  

(5.13)

Let \( I_k \) be the \( k \times k \) identity matrix, and define \( \mathcal{A} \in \mathcal{M}_{mn} \) to be

\[
\mathcal{A} = \begin{bmatrix}
\Delta_{x+1} & I_{ml} & 0 & 0 & \cdots & 0 & 0 \\
0 & \Delta_x & I_{m_{x-1}} & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & I_{ml} & 0 \\
0 & 0 & \cdots & 0 & 0 & \Delta_1
\end{bmatrix},
\]

where the \( \Delta_j \) matrices are omitted when \( j \in J \) and 0's represent zero matrices of the appropriate dimensions. From (5.12), (5.13), and (5.14) we have \( AP = P\mathcal{A} \), and therefore \( \mathcal{A} = P^{-1}AP \) is our canonical representation of \( \mathcal{A} \).

Define \( \mathcal{B} \in \mathcal{M}_{nm} \) by \( \mathcal{B} = P\mathcal{A} \), and hence

\[
\mathcal{B} = \begin{bmatrix}
0 \\
I_m
\end{bmatrix},
\]

where 0 represents the \( n - m \times m \) zero matrix. We can now conclude that there exists an \( m \times n \) matrix \( \mathcal{C} \) such that \( \mathcal{A} + \mathcal{B} \mathcal{C} \) is invertible, and \( \text{Range } \mathcal{BC} = \{ \text{Range } \mathcal{A} \}^\perp \). Let \( \mathcal{C} = \mathcal{C} P^{-1} \), then the \( \text{Range } \mathcal{BC} = \{ \text{Range } A \}^\perp \) and the proof is complete.

**Proof of Theorem 5.1.** Let \( A_{-1}, B, L, \) and \( f \) satisfy (A1)-(A6). Assume \( K_v[A_{-1}, B] = n \) for some integer \( v, 1 \leq v \leq n \). Assume \( \text{Rank } B < m \), for otherwise Theorem 4.1 implies the wanted result directly. Choose an \( m \times n \) matrix \( C \) as in Remark 5.2 so that \( \text{Range } BC = \{ \text{Range } A_{-1} \}^\perp \). Thus \( \text{Rank}(A_{-1} + BC) = n \). Consider the control system

\[
\frac{d}{dt} \begin{bmatrix} y(t) - (A_{-1} + BC) y(t - h) - Bu(t) \end{bmatrix} = L(y,).
\]

(5.15)

Assume that (5.9) is null controllable on \([t_0, t_1] \). Using the techniques of Lemma 5.2, we can show that the null controllability of (5.9) implies the null controllability of (5.15).

By Theorem 4.1, the nonlinear system

\[
\frac{d}{dt} \begin{bmatrix} y(t) - (A_{-1} + BC) y(t - h) - Bu(t) \end{bmatrix} = L(y, f(y, u(t)))
\]

(5.15)
is locally null controllable on \([t_0, t_1]\). Suppose \(f\) satisfies the condition given by (5.8). By our choice of \(C\), \(f(\cdot, w + Cz) = f(\cdot, w)\) for all \(w \in \mathbb{R}^m\) and \(z \in \mathbb{R}^n\). Using Lemma 5.2 we see that (5.10) can be steered anywhere that (5.15) can. We conclude that the control system (5.10) is locally null controllable.

REFERENCES