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Chordal embeddings of planar graphs

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Abstract

Robertson and Seymour conjectured that the treewidth of a planar graph and the treewidth of its geometric dual differ by at most one. Lapoire solved the conjecture in the affirmative, using algebraic techniques. We give here a much shorter proof of this result.

Keywords: Planar graphs; Treewidth; Duality

1. Introduction

The notions of *treewidth* and *tree decomposition* of a graph have been introduced by Robertson and Seymour in [14] for their study of minors of graphs. These notions have been intensively investigated for algorithmical purposes since many NP-hard problems become polynomial and even linear when restricted to classes of graphs with bounded treewidth.

Robertson and Seymour conjectured in [13] that the treewidth of a planar graph and the treewidth of its geometric dual differ by at most one. Lapoire [11] solved this conjecture in the affirmative, in fact he proved a more general result. In order to prove his result, Lapoire worked on hypermaps and introduced the notion of splitting of hypermaps, his approach is essentially an algebraic one.

Computing the treewidth of an arbitrary graph is NP-hard. Nevertheless, the treewidth can be computed in polynomial time for several well-known classes of graphs, for example chordal bipartite graphs [9], circle and circular-arc graphs [8,16], permutation graphs [2] and weakly triangulated graphs [3]. Actually all these classes of graphs have a polynomial number of minimal separators. We proved in [4] that, for every class of

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graphs with polynomial number of minimal separators, we can compute the treewidth of these graphs in polynomial time.

We know very few classes of graphs having an exponential number of minimal separators, for which the treewidth is computable in polynomial time. For instance the problem remains NP-hard on AT-free graphs [1] and it is polynomial for rectangular grids. May be the most challenging open problem is the computation of the treewidth for planar graphs. In [15], Seymour and Thomas gave a polynomial time algorithm that approximates the treewidth of planar graph within a factor of $\frac{3}{2}$.

In this paper, we give a new approach to tackle the problem of the treewidth computation for planar graphs. First, we recall how to obtain minimal chordal embeddings of graphs by completing some families of minimal separators. Secondly, we show that we can interpret minimal separators of planar graphs as Jordan curves of the plane. Then, we study the structure of Jordan curves that give a minimal triangulation of the graph. Next, given a family of curves of the plane, we show how to build a minimal triangulation of the geometric dual of the graph. Finally, given an optimal triangulation w.r.t. treewidth of the initial graph, we give a triangulation of the dual graph whose maximal cliquesize is no more than the maximal cliquesize of the original graph plus one. So, we get a new proof of the conjecture of Robertson and Seymour which is much simpler than the proof of Lapoire.

2. Preliminaries

Throughout this paper we consider simple, finite, undirected graphs.

Let $G=(V,E)$ be a graph. A sequence of pairwise distinct vertices $[x_1,\dots,x_p]$ is a *path* if $x_1x_2,\ x_2x_3,\dots,\ x_{p-1}x_p \in E$. A path $[x_1,\dots,x_p]$ is a *cycle* if in addition $x_1x_p \in E$. Given a set of vertices $W \subseteq V$, we denote by $N_G(W)$ (or simply $N(W)$) the *neighborhood* of W: $N_G(W)=\{x\in V\backslash W \mid \exists y\in W, xy\in E\}$.

A graph $G=(V,E)$ is *planar* if it can be drawn in the plane such that no two edges meet in a point other than a common end. The plane will be denoted by Σ. A *plane graph* $G=(V,E)$ is a drawing of a planar graph. That is, each vertex $v\in V$ is a point of Σ, each edge $e\in E$ is a curve between two vertices, distinct edges have distinct sets of endpoints and the interior of an edge contains no point of another edge. A *face* of the plane graph G is a region of $\Sigma\backslash G$. $F(G)$ denotes the set of faces of G. Sometimes we will also use *plane multigraphs*, i.e. we allow loops and multiple edges.

Let $G=(V,E)$ be a plane graph. The *dual* $G^*=(F,E^*)$ of G is a plane multigraph obtained in the following way: for each face of G, we place a point f into the face, and these points form the vertex set of G^*. For each edge e of G, we link the two vertices of G^* corresponding to faces incident to e in G, by an edge e^* crossing e; if e is incident with only one face, then e^* is a loop.

A graph H is *chordal* (or *triangulated*) if every cycle of length at least four has a chord. A *triangulation* of a graph $G=(V,E)$ is a chordal graph $H=(V,E')$ such that $E\subseteq E'$. H is a *minimal triangulation* if for any intermediate set E'' with $E\subseteq E''\subset E'$, the graph (V,E'') is not triangulated. We point out that not every triangulation of a planar graph G is planar.

Let $\omega(H)$ denote the maximum cliquesize of the graph H.

Definition 1. Let G be a graph. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum, over all triangulations H of G, of $\omega(H)-1$. The treewidth of a multigraph is the treewidth of the corresponding simple graph.

The aim of this paper is to prove the following assertion, stated by Robertson and Seymour in [13]:

Claim 2. *For any plane graph* $G=(V,E)$,

$$\mathrm{tw}(G^*)-1 \leqslant \mathrm{tw}(G) \leqslant \mathrm{tw}(G^*)+1.$$

We say that a graph G' is a minor of a graph G if we can obtain G' from G by repeatedly using the following operations: vertex deletion, edge deletion and edge contraction. Kuratowski's theorem states that a graph G is planar if and only if the graphs $K_{3,3}$ and K_5 are not minors of G. It is well-known that if G' is a minor of G, then $\mathrm{tw}(G') \leqslant \mathrm{tw}(G)$. We refer to Diestel [5] for more details on these results.

When we compute the treewidth of a graph G, we are searching for a triangulation of G with smallest cliquesize, so we can restrict our work to minimal triangulations. We need a characterization of the minimal triangulations of a graph, using the notion of minimal separator.

A subset $S \subseteq V$ is an *a,b-separator* for two nonadjacent vertices $a,b \in V$ if the removal of S from the graph separates a and b in different connected components. S is a *minimal a,b-separator* if no proper subset of S separates a and b. We say that S is a *minimal separator* of G if there are two vertices a and b such that S is a minimal a,b-separator. Notice that a minimal separator can be strictly included into another. We denote by Δ_G the set of all minimal separators of G.

Let G be a graph and S be a minimal separator of G. A component C of $G\backslash S$ is called a *full component associated to* S if every vertex of S is adjacent to some vertex of C. For the following lemma, we refer to [7].

Lemma 3. *A set S of vertices of G is a minimal a,b-separator if and only if a and b are in different full components associated to S.*

Definition 4. Two separators S and T *cross*, denoted by $S\sharp T$, if there are some distinct components C and D of $G\backslash T$ such that S intersects both of them. If S and T do not cross, they are called *parallel*, denoted by $S\|T$.

It is easy to prove that the parallel and crossing relations are symmetric for minimal separators.

Let $S \in \Delta_G$ be a minimal separator. We denote by G_S the graph obtained from G by *completing* S, i.e. by adding an edge between every pair of nonadjacent vertices of S. If $\Gamma \subseteq \Delta_G$ is a set of separators of G, G_Γ is the graph obtained by completing all the separators of Γ. The results of Kloks et al. [10], concluded in [12], establish a strong relation between the minimal triangulations of a graph and its minimal separators.

Theorem 5 (Parra and Scheffler [12]). *Let $\Gamma \in \Delta_G$ be a maximal set of pairwise parallel separators of G. Then $H = G_\Gamma$ is a minimal triangulation of G and $\Delta_H = \Gamma$.*

Let H be a minimal triangulation of a graph G. Then Δ_H is a maximal set of pairwise parallel separators of G and $H = G_{\Delta_H}$. Moreover, for each $S \in \Delta_H$, the connected components of $H \backslash S$ are exactly the connected components of $G \backslash S$.

In other terms, every minimal triangulation of a graph G is obtained by considering a maximal set Γ of pairwise parallel separators of G and completing the separators of Γ. The minimal separators of the triangulation are exactly the elements of Γ.

3. Minimal separators as curves

We show in this section that, in plane graphs, we can associate to each minimal separator S a Jordan curve such that, if S separates two vertices of the graph, then the curve separates the corresponding points in the plane.

Definition 6. Let $G = (V, E)$ be a planar graph. We fix a plane embedding of G. Let F be the set of faces of this embedding. The *intermediate graph* $G_I = (V \cup F, E_I)$ has vertex set $V \cup F$. We place an edge in G_I between an original vertex $v \in V$ and a face-vertex $f \in F$ whenever the corresponding vertex and face are incident in G.

Proposition 7. *Let G be a 2-connected plane graph. Then the intermediate graph G_I is also 2-connected.*

Proof. Let us prove that, for any couple of original vertices x and y of G_I and for any face or original vertex a, there is an x, y-path in G_I avoiding a. Let $\mu = [x = v_1, v_2, \ldots, v_p = y]$ an x, y-path of G. If $a \in V(G)$, since $\{a\}$ is not an x, y separator of G, we can choose μ such that $a \notin \mu$. For each edge $e_i = v_i, v_{i+1}, 1 \leqslant i < p$, let f_i be a face incident to e_i in G. If a is a face-vertex, we use the fact that in a 2-connected plane graph each edge is incident to at least two faces and we choose $f_i \neq a$. Then $[v_1, f_1, v_2, f_2, \ldots, v_p]$ is an x, y-path of G_I, avoiding a. It follows that, for any $x, y \in V(G)$ and for any $a \in V(G) \cup F(G)$, $\{a\}$ is not an x, y-separator of G_I. Each face-vertex is adjacent in G_I to at least two original vertices. It follows easily that for any $a \in V \cup F$, $\{a\}$ is not a separator of G_I. $\square$

The following propositions show that a minimal separator of G can be viewed as a cycle in the intermediate graph G_I. This result of Eppstein appears in [6], in a slightly different form.

Proposition 8. *Consider a cycle ν of G_I. Its drawing defines a Jordan curve $\tilde{\nu}$ in the plane. Removing $\tilde{\nu}$ separates the plane into two regions. If both regions* *contain at least one original vertex, then the original vertices of v form a separator of G.*

Proof. Let x and y be two original vertices, separated by the curve $\tilde{v}$ in the plane. Clearly, no edge of G crosses an edge of G_I, and therefore no edge of G crosses the curve $\tilde{v}$. Every path μ connecting x and y in G intersects $\tilde{v}$, so μ has a vertex in v. It follows that $v \cap V$ is an x, y-separator of G. □

Proposition 9. *Let S be a minimal separator of a 2-connected plane graph G and C be a full component associated to S. Then S corresponds to a cycle $v_S(C)$ of G_I, of the same original vertices and of equal number of face-vertices in G_I, such that $G_I \backslash v_S(C)$ has at least two connected components. Moreover, the original vertices of one of these components are exactly the vertices of C.*

Proof. Let C be a full component associated to S, let G^C be formed by contracting C into a supervertex, and let S' be the set of faces and vertices adjacent in G^C to the contracted supervertex. Then S' is neighborhood of the supervertex in G_I^C, so it has the structure of a cycle in G_I^C and therefore in G_I. This cycle will be denoted $v_S(C)$. Since C is a full component associated to S in G, we have that $S = N_G(C)$, so the original vertices of S' are exactly vertices of S. The cycle separates C from $V \backslash \{S \cup C\}$ in G_I. □

The cycle $v_S(C)$ defined in the previous proposition will be called the cycle associated to S and C, *close to* C. Since the graph G has at least two full components associated to S (Lemma 3), we can associate to S two cycles $v_S(C)$ and $v_S(D)$ of G_I, closed to C and D, respectively, and these cycles might be different. In the next two sections we show how to represent each minimal separator S of G by a unique cycle of G_I, when the graph G is 3-connected.

Remark 10. Any cycle v of G_I forms a Jordan curve in the plane. We denote $\tilde{v}$ this curve. Removing $\tilde{v}$ separates the plane into two open regions. Consider the cycle $v_S(C)$ of G_I associated to a minimal separator S and a full component C of $G \backslash S$, close to C. Then one of the regions defined by $\tilde{v}_S(C)$ contains all the vertices of C and the other contains all the vertices of $V \backslash (S \cup C)$.

4. Some technical lemmas

In the next section we show how to associate to each minimal separator S of a 3-connected plane graph G a *unique* cycle of G_I having good separation properties. We group here some technical lemmas that will be used in the next sections.

Lemma 11. *Let G be a 3-connected planar graph and S be a minimal separator of G. Then $G \backslash S$ has exactly two connected components.*

Proof. By Lemma 3, there are two distinct full components C_1 and C_2 associated to S. Suppose there is another component C_3 of $G\backslash S$ and let $S_3 = N(C_3)$. Clearly, S_3 is a separator of G, so $|S_3| \geqslant 3$. Let x_1, x_2, x_3 be three distinct vertices of S_3. Consider the plane graph G' obtained from G by contracting each component C_1, C_2 and C_3 into a supervertex. The three supervertices are adjacent in G' to x_1, x_2, x_3, so G' contains a subgraph isomorphic to $K_{3,3}$—contradicting Kuratowski's theorem. □

Proposition 12. *Let S be a minimal separator of a 3-connected planar graph G. Then S is also an inclusion minimal separator of G.*

Proof. Suppose there is a separator T of G such that $T \subset S$. There is a connected component C of $G\backslash T$ such that $C \cap S = \emptyset$. Indeed, if S intersects each component of $G\backslash T$, then S and T cross, and since the crossing relation is symmetric T must intersect two connected components of $G\backslash S$, contradicting $T \subset S$. Since $S \cap C = \emptyset$ and $T \subset S$, C is also a connected component of $G\backslash S$. By Lemma 3, there are two full components D_1, D_2 associated to S. Notice that C is not a full component associated to S, because $N(C) = T \subset S$. It follows that D_1, D_2 and C are three distinct components associated to S in G, contradicting Lemma 11. □

Lemma 13. *Let G be a plane graph and ν be a cycle of G_I such that $\tilde{\nu}$ separates two original vertices a and b in the plane. Consider two vertices x and y of ν. Suppose there is a path μ from x to y in G_I, such that $a, b \notin \mu$ and μ does not intersect the cycle ν except in x and y.*

The vertices x and y split ν into two x, y-paths of G_I, denoted μ_1 and μ_2. Consider the cycles ν_1 (respectively ν_2) of G_I formed by the paths μ and μ_1 (respectively μ and μ_2). Then $\tilde{\nu}_1$ or $\tilde{\nu}_2$ separates two original vertices in the plane.

Proof. Let R_1, R_2 be the two regions obtained by removing $\tilde{\nu}$ from the plane. By hypothesis, both R_1 and R_2 contain original vertices, say $a \in R_1$ and $b \in R_2$. Suppose w.l.o.g. that the path μ is contained in $R_1 \cup \{x, y\}$. Then the drawing of μ splits R_1 into two regions: R_1', bordered by the curve $\tilde{\nu}_1$, and R_1'', bordered by $\tilde{\nu}_2$. If $a \in R_1'$ then $\tilde{\nu}_1$ separates a and b in the plane, otherwise $a \in R_1''$ so $\tilde{\nu}_2$ separates a and b in the plane. □

Lemma 14. *Let S be a minimal separator of a 3-connected plane graph G. Consider a cycle ν_S of G_I such that the original vertices of ν_S are the elements of S. Suppose that $\tilde{\nu}_S$ separates in the plane two original vertices of G.*

If two original vertices of S are at distance two in G_I (i.e. they are incident to the same face of G), these vertices are also at distance two on the cycle ν_S.

If two vertices of ν_S are adjacent in G_I, they are also at distance one on the cycle ν_S.

Proof. Let $\nu_S = [v_1, f_1, \ldots, v_p, f_p]$, where v_i (respectively f_i) are the original (respectively face) vertices of ν_S.

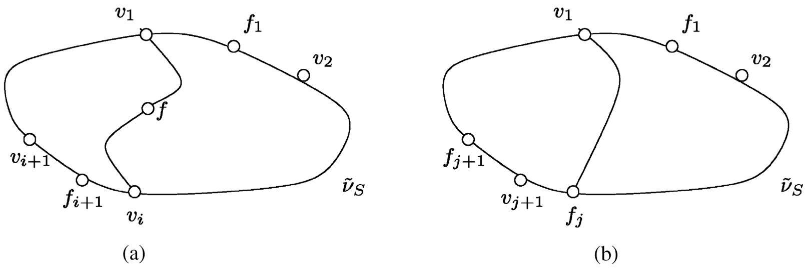


Fig. 1. Proof of Lemma 14.

We begin by proving the first statement. The conclusion is obvious if $p \leqslant 3$. Suppose there are two vertices $x, y \in S$ at distance two in G_I, but not in v_S. W.l.o.g., we suppose $x = v_1$ and $y = v_i$, $3 \leqslant i \leqslant p-2$. Let f be a face vertex adjacent to v_1 and v_i in G_I.

If $f \notin v_S(C)$, we apply Lemma 13 with cycle v_S and path $[v_1, f, v_i]$, so one of the cycles $v_1 = [v_1, f_1, v_2, f_2, \ldots, v_i, f]$ or $v_2 = [v_1, f, v_i, f_i, v_{i+1}, f_{i+1}, \ldots, v_p, f_p]$ separates two original vertices in the plane (see Fig. 1a). By Proposition 8, the original vertices of v_1 or v_2 form a separator T in G. But T is strictly contained in S, contradicting Proposition 12.

The case $f \in v_S$ is very similar. There is some j, $1 \leqslant j \leqslant p$ such that $f = f_j$. Since v_1 and v_i are not at distance two on v_S, we have that $j \notin \{1, p\}$ or $j \notin \{i-1, i+1\}$. Suppose w.l.o.g. that f is not consecutive to v_1 on the cycle v_S. We apply Lemma 13 with cycle v_S and path $[v_1, f]$. We obtain that one of the cycles $v_1 = [v_1, f_1, \ldots, v_j, f_j]$ or $v_2 = [v_1, f_j, v_{j+1}, f_{j+1}, \ldots, v_p, f_p]$ (see Fig. 1b) separates two original vertices of G, so the original vertices of v_1 or v_2 form a separator T of G. In both cases, $T \subset S$, contradicting Proposition 12.

For the second statement, suppose w.l.o.g. there is an edge $v_1 f_j$ of G_I, $2 \leqslant j \leqslant p-1$. We apply Lemma 13 with cycle v_S and path $[v_1, f_j]$. Hence one of the cycles $v_1 = [v_1, f_1, \ldots, v_j, f_j]$ or $v_2 = [v_1, f_j, v_{j+1}, f_{j+1}, \ldots, v_p, f_p]$ separates two original vertices of G. We conclude that one of these cycles contains a minimal separator T of G, and $T \subset S$ contradicting Proposition 12. □

Lemma 15. *Let $G = (V, E)$ be a plane graph and $x, y \in V$ such that at least three faces are incident to both x and y. Then $\{x, y\}$ is a separator of G.*

Proof. Let f_1, f_2, f_3 be three faces incident to both x and y. Consider the three paths $\mu_i = [x, f_i, y]$, $1 \leqslant i \leqslant 3$ of the intermediate graph G_I. The drawings of these paths split the plane into three regions: R_1 bordered by the cycle $v_1 = [x, f_2, y, f_3]$, R_2 bordered by $v_2 = [x, f_1, y, f_3]$ and R_3 bordered by $v_3 = [x, f_1, y, f_2]$. We show that at least two of the three regions contain one or more original vertices.

Suppose that R_2 and R_3 do not contain original vertices. In the graph G, each face is incident to at least three vertices, so f_1 has a neighbor z, different from x and y. The edge $f_1 z$ of G_i does not cross any of the paths μ_1, μ_2, μ_3, so z is in one of the regions R_2 or R_3, incident to f_1. This contradicts our assumption that R_2 and R_3 do not contain original vertices.

We proved that at least two of the three regions R_1, R_2, R_3—say R_1 and R_2—contain original vertices. Then $\tilde{v}_1$, the Jordan curve bordering R_1, separates two original vertices of G. By Proposition 8, the original vertices of μ_1, namely $\{x, y\}$, form a separator of G. $\square$

Lemma 16. *Let $G=(V,E)$ be a 3-connected plane graph and $x, y \in V$.*

(1) *If $xy \in E$, there are exactly two faces incident to both x and y.*
(2) *If $xy \notin E$, there is at most one face of G incident to both x and y.*

Proof. The graph G is 3-connected, so by Lemma 15 there are at most two faces incident to both x and y.

The first statement is obvious since the edge xy is incident to two faces of G. For the second statement, suppose there are two faces f_1 and f_2 incident to x and y. Consider the plane graph G' obtained from G by adding the edge xy, drawn in the face f_1. Then the face f_1 of G is split into two faces f_1' and f_1'', both incident to x and y in G'. So, in G', the three faces f_1', f_1'' and f_2 are incident to both x and y. But G' is clearly a 3-connected planar graph, and by Lemma 15 we have that $\{x, y\}$ is a separator of G'—a contradiction. $\square$

Proposition 17. *Let G be a 3-connected plane graph. Consider two cycles v and v' of G_I, such that v and v' only differ by their face vertices. Suppose that any face of G is incident to at most two original vertices of v. Then $\tilde{v}$ separates two original vertices a and b in the plane if and only if $\tilde{v}'$ also separates a and b in the plane.*

Proof. The cycles v and v' only differ by their face vertices, so we can write $v=[v_1, f_1, v_2, f_2, \ldots, v_p, f_p]$ and $v'=[v_1, f_1', v_2, f_2', \ldots, v_p, f_p']$. Since each face of G is incident to at most two original vertices of v, for any $i, 1 \leqslant i \leqslant p$, the sequence of vertices $v_i'=[v_1, f_1', \ldots, v_i, f_i', v_{i+1}, f_{i+1}, \ldots, v_p, f_p]$ (the first i face vertices of v are replaced by the first i face vertices of v') is also a cycle of G_I. Thus it is sufficient to prove our statement for two cycles that only differ by one face vertex, say $v=[v_1, f_1, v_2, f_2, \ldots, v_p, f_p]$ and $v'=[v_1, f_1', v_2, f_2, \ldots, v_p, f_p]$, such that $f_1 \neq f_1'$. Since v_1 and v_2 are incident to both f_1 and f_1' in G, it comes by Lemma 16 that v_1 and v_2 are adjacent in G. Thus, f_1 and f_1' are the faces incident in G to the edge $e=v_1v_2$.

Consider the cycle $v''=[v_1, f_1, v_2, f_1']$ of G_I and let R be the region bordered by $\tilde{v}''$ and containing the interior of the edge e. Clearly, the region R contains no original or face vertex of G_I.

Let R_1, R_2 be the two regions obtained by removing $\tilde{v}$ from the plane. Suppose that the edge e, and thus the face f_1', is in R_2. Then the regions obtained by removing $\tilde{v}'$ from the plane are exactly $R_1'=R_1 \cup R \cup [v_1, f_1, v_2]$ and $R_2'=R_2 \backslash R \backslash [v_1, f_1', v_2]$. Since R contains no original vertices, the original vertices of R_1' (respectively R_2') are the original vertices of R_1 (respectively R_2). $\square$

Lemma 18 (Diestel [5], Proposition 4.2.10). *Let G be a 3-connected plane graph. For any face f, the set of vertices incident to f does not form a separator of G.*

Lemma 19. *Let G be a 3-connected plane graph and S be a minimal separator of G. Then each face of G is incident to at most two vertices of S.*

Proof. Suppose there are three vertices x, y, z of S incident to the same face f. Let C be a full component associated to S in G and $v_S(C)$ be the cycle associated to S in G_I, close to C. Consider first the case $|S| \geqslant 4$, so there is some vertex $t \in S \backslash \{x, y, z\}$. Suppose w.l.o.g. that $v_S(C)$ encounters x, y, z and t in this order. Then x and z are not at distance 2 on the cycle $v_S(C)$, contradicting Lemma 14.

If $|S| = 3$, let T be the set of vertices incident to f, so $S \subseteq T$. Thus, T is a separator of G, contradicting Lemma 18. □

5. Minimal separators in 3-connected planar graphs

Consider a minimal separator S of G and two full components C and D associated to S. We can associate to S two cycles of G_I, namely $v_S(C)$ and $v_S(D)$, close to C and to D, respectively. In general, the two cycles are distinct, although they represent for us the same minimal separator S. In the case of 3-connected planar graphs, we slightly modify the construction of Proposition 9 in order to obtain the *unique* cycle representing S in G_I.

Let G be a 3-connected plane graph. Consider two original vertices x and y situated at distance two in G_I. We know that x and y are incident to the same face in G, but this face is not necessarily unique. For each pair of vertices $x, y \in V$ at distance two in G_I, we fix a unique face $f(x, y)$ of G incident to both x and y. Let $v = [v_i, f_1, v_2, f_2, \ldots, v_p, f_p]$ be a cycle of G_I, where v_i are the original vertices and f_i are the face vertices of v. We say that a cycle v is *well-formed* if, for each pair of consecutive original vertices v_i, v_{i+1} of v we have $f_i = f(v_i, v_{i+1})$ ($1 \leqslant i \leqslant p$, $v_{p+1} = v_1$).

Given a minimal separator S of G we construct the unique well-formed cycle v_S associated to S as follows. Let C, D be the full components associated to S in G and let $v_S(C) = [v_1, f_1, v_2, f_2, \ldots, v_p, f_p]$ be the cycle associated to S in G_I, close to C. We denote $v'_S(C) = [v_1, f'_1, v_2, f'_2, \ldots, v_p, f'_p]$ where $f'_i = f(v_i, v_{i+1}) \forall i,\ 1 \leqslant i \leqslant p$. Notice that $\forall i, j,\ 1 \leqslant i < j \leqslant p$ we have $f'_i \neq f'_j$ by Lemma 19, so $v'_S(C)$ is a cycle of G_I.

The cycle $v_S(D)$, associated to S and close to D, has the same original vertices as $v_S(C)$, encountered in the same order, as we can observe by simultaneously contracting each of the components C and D into a supervertex. Thus, $v'_S(D) = v'_S(C)$ and from now on this cycle will be denoted v_S. By Lemma 19 and Proposition 17, $\tilde{v}_S$ separates C and D in the plane:

Proposition 20. *Let G be a 3-connected planar graph and S be a minimal separator of G. Let C, D be the two connected components of $G \backslash S$. Then $\tilde{v}_S$ separates C and D in the plane.*

Definition 21. Two Jordan curves $\tilde{v}_1$ and $\tilde{v}_2$ *cross* if $\tilde{v}_1$ intersects the two regions of $\Sigma \backslash \tilde{v}_2$. Otherwise, they are *parallel*. Two cycles v_1 and v_2 of G_I *cross* if and only if $\tilde{v}_1$ and $\tilde{v}_2$ cross.

Notice that the parallel and crossing relation between curves and cycles are symmetric.

Proposition 22. *Two minimal separators S and T of a* 3*-connected plane graph G are parallel if and only if the corresponding cycles v_S and v_T of G_I are parallel.*

Proof. We prove that if S and T cross, then v_S and v_T cross. Let C and D be the two connected components of $G \backslash S$. By definition of crossing separators, T intersects two connected components of $G \backslash S$, so T intersects C and D. The curve $\tilde{v}_S$ separates C and D in the plane, by Proposition 20. Thus, $\tilde{v}_T$ intersects two different regions of $\Sigma \backslash \tilde{v}_S$, so v_T crosses v_S.

We prove that if v_S crosses v_T, then S crosses T. Let R and R' be the regions of $\Sigma \backslash \tilde{v}_S$. We show that at least one original vertex of v_T is in R. Since v_S crosses v_T, $\tilde{v}_T$ intersects R. Suppose first that $\tilde{v}_T \cap R$ contains no vertex of v_T, so there is an edge vf of $\tilde{v}_T$ whose interior is in R. The endpoints v and f of this edge are in v_S, but the edge vf is not on the curve $\tilde{v}_S$. Therefore v and f are not at distance one in the cycle v_S, contradicting Lemma 14. Thus, an original vertex or a face-vertex of v_T is in R. Suppose that v_T has no original vertex in R and let f be a face-vertex of $v_T \cap R$. On the cycle v_T, the face-vertex f is between two original vertices x and x'. Notice that x is also a vertex of v_S. Indeed, $x \notin R$, and x cannot be in R', because the edge xf of G_I cannot cross the drawing of the cycle v_S. It follows that $x \in \tilde{v}_S$. So x and x' are both vertices of v_S. Since x and x' are adjacent to the same face-vertex of G_I, they are on the same face of G. By Lemma 14, x and x' are at distance two on the cycle v_S, and let f' be the face-vertex of v_S between x and x'. Since v_S and v_T are well-formed cycles, we have $f' = f(x, y) = f$, so $f \in v_S$. This contradicts the fact that f is in one of the regions of $\Sigma \backslash \tilde{v}_S$.

We showed that v_S has original vertices in region R, and for similar reasons it has original vertices in R'. So $\tilde{v}_S$ separates two original vertices of v_T in the plane, and by Proposition 20 S separates these vertices in G. Thus, S crosses T. $\square$

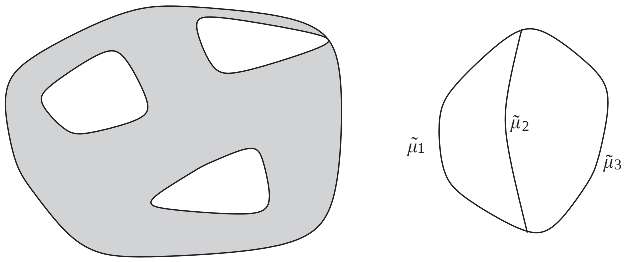
6. Block regions

Let $\tilde{v}$ be a Jordan curve in the plane. Let R be one of the regions of $\Sigma \backslash \tilde{v}$. We say that $(\tilde{v}, R) = \tilde{v} \cup R$ is a one-block region of the plane, bordered by $\tilde{v}$.

Definition 23. Let $\tilde{\mathscr{C}}$ be a set of curves such that for each $\tilde{v} \in \tilde{\mathscr{C}}$, there is a one-block region $(\tilde{v}, R(\tilde{v}))$ containing all the curves of $\tilde{\mathscr{C}}$. We define the *region between* the elements of $\tilde{\mathscr{C}}$ as

$$\mathrm{RegBetween}(\tilde{\mathscr{C}}) = \bigcap_{\tilde{v} \in \tilde{\mathscr{C}}} (\tilde{v}, R(\tilde{v})).$$

We say that the region between the curves of $\tilde{\mathscr{C}}$ is *bordered* by $\tilde{\mathscr{C}}$.



(a) (b)

Fig. 2. Block regions.

Definition 24. A subset $BR \subseteq \Sigma$ of the plane is a *block region* if one of the following holds:

- $BR = \Sigma$.
- There is a curve $\tilde{v}$ such that BR is a one-block region $(\tilde{v}, R)$.
- There is a set of curves $\tilde{\mathscr{C}}$ such that $BR = \mathrm{RegBetween}(\tilde{\mathscr{C}})$.

Remark 25. According to our definition, block regions are always closed sets.

Example 26. In Fig. 2a, we have a block region (in gray) bordered by four Jordan curves. Fig. 2b presents three interior-disjoint paths $\tilde{\mu}_1, \tilde{\mu}_2$ and $\tilde{\mu}_3$ having the same endpoints. Consider the three curves $\tilde{v}_1 = \tilde{\mu}_2 \cup \tilde{\mu}_3$, $\tilde{v}_2 = \tilde{\mu}_1 \cup \tilde{\mu}_3$ and $\tilde{v}_3 = \tilde{\mu}_1 \cup \tilde{\mu}_2$. Notice that the block-region between $\tilde{v}_1, \tilde{v}_2$ and $\tilde{v}_3$ is exactly the union of the three paths.

Consider a set $\tilde{\mathscr{C}}$ of pairwise parallel Jordan curves of the plane. These curves split the plane into several block regions. Consider the set of all the block regions bordered by some elements of $\tilde{\mathscr{C}}$. We are interested in the inclusion-minimal elements of this set, that we call minimal block regions formed by $\tilde{\mathscr{C}}$. The following proposition comes directly from the definition of the minimal block-regions:

Proposition 27. *Let $\tilde{\mathscr{C}}$ be a set of pairwise parallel curves in the plane Σ. A set A of points of the plane is contained in the same minimal block-region formed by $\tilde{\mathscr{C}}$ if and only if for any $\tilde{v} \in \tilde{\mathscr{C}}$, there is a one-block region $(\tilde{v}, R(\tilde{v}))$ containing A.*

7. Minimal triangulations of G

Let G be a 3-connected planar graph and let H be a minimal triangulation of G. According to Theorem 5, there is a maximal set of pairwise parallel separators $\Gamma \subseteq \Delta_G$ such that $H = G_\Gamma$. Let $\mathscr{C}(\Gamma) = \{v_S | S \in \Gamma\}$ be the cycles associated to the minimal separators of Γ and let $\tilde{\mathscr{C}}(\Gamma) = \{\tilde{v}_S | S \in \Gamma\}$ be the curves corresponding to the drawings of these cycles. According to Proposition 22, the cycles of $\mathscr{C}(\Gamma)$ are pairwise parallel.

Thus, the curves of $\tilde{\mathscr{C}}(\Gamma)$ split the plane into block regions. We show that any maximal clique Ω of H corresponds to the original vertices contained in a minimal block region formed by $\tilde{\mathscr{C}}(\Gamma)$.

If BR is a block region, we denote by BR_G the vertices of G contained in BR.

Theorem 28. *Let $H = G_\Gamma$ be a minimal triangulation of a 3-connected planar graph G. $\Omega \subseteq V$ is a maximal clique of H if and only if there is a minimal block region* BR *formed by $\tilde{\mathscr{C}}(\Gamma)$ such that $\Omega = \mathrm{BR}_G$.*

Proof. Let BR be a minimal block region formed by $\tilde{\mathscr{C}}(\Gamma)$, we show that $\Omega = \mathrm{BR}_G$ is a clique of H. Suppose there are two vertices $x, y \in \Omega$, not adjacent in H. Thus, there is a minimal separator S of H separating x and y in H. Then S is also a minimal separator of G, separating x and y in G (cf. Theorem 5). Therefore, $\tilde{\nu}_S \in \tilde{\mathscr{C}}(\Gamma)$ separates x and y in the plane, contradicting Proposition 27.

Let Ω be a clique of H. For any minimal separator S of H there is a connected component $C(S)$ of $H \backslash S$ such that $\Omega \subseteq S \cup C(S)$. By Theorem 5, $S \in \Gamma$ and $C(S)$ is a connected component of $G \backslash S$, so we deduce that the points of Ω are contained in a same one-block region $(\tilde{\nu}_S, R(\tilde{\nu}_S))$ defined by $\tilde{\nu}_S$. This holds for each $S \in \Gamma$, because the minimal separators of H are exactly the elements of Γ. We conclude by Proposition 27 that Ω is contained in some minimal block BR formed by $\tilde{\mathscr{C}}(\Gamma)$. $\square$

8. Triangulations of the dual graph G^*

Let G be a plane graph and $\mathscr{C}$ be a set of pairwise parallel cycles of G_I. The family $\tilde{\mathscr{C}}$ of curves associated to these cycles splits the plane into block regions. Let G^* be the dual of G. We show in this section how to associate to $\mathscr{C}$ a triangulation $H(\mathscr{C})$ of G^* such that each clique of $H(\mathscr{C})$ corresponds to the face-vertices contained in some minimal block-region defined by $\tilde{\mathscr{C}}$.

Definition 29. Consider a planar embedding of the graph $G=(V,E)$ and let $G^*=(F,E^*)$ be the dual of G. Let $\mathscr{C}$ be a set of pairwise parallel cycles of G_I. We define the graph $H(\mathscr{C})=(F,E_H)$ in which two face-vertices f and f' are adjacent if and only if f and f' are in the same minimal block region defined by $\tilde{\mathscr{C}}$.

Theorem 30. *$H(\mathscr{C})$ is a triangulation of G^*. Moreover, for any clique Ω^* of $H(\mathscr{C})$ there is some minimal block region* BR *defined by $\tilde{\mathscr{C}}$ such that Ω^* is formed by the face-vertices contained in* BR.

Proof. We show that H is a supergraph of G^*. If ff' is an edge of G^*, then no cycle of G_I crosses the edge ff' in the plane. Thus, for any cycle $\tilde{\nu}$ of $\tilde{\mathscr{C}}$, $\tilde{\nu}$ does not separate the points f and f'. By Proposition 27, f and f' are in the same block region formed by $\tilde{\mathscr{C}}$, so ff' is an edge of $H(\mathscr{C})$.

We prove now that $H(\mathscr{C})$ is chordal. Suppose there is a chordless cycle ν_H of $H(\mathscr{C})$, having at least four vertices. Let f, f' be two nonadjacent vertices of ν_H. By

Proposition 27, there is a curve $\tilde{v} \in \tilde{\mathscr{C}}$ separating the points f and f' in the plane. Consider the two interior-disjoint paths μ_1 and μ_2 from f to f' in v_H. We show that at least one interior vertex in each of these paths belongs to $\tilde{v}$. Let $\mu_1 = [f = f_1, f_2, \dots, f_p = f']$. Let R and R' be the regions of $\Sigma \backslash \tilde{v}$ containing f, respectively f'. Let f_j the last point of μ_1 contained in R, so $1 \leqslant j < p$. We prove that $f_{j+1} \in \tilde{v}$. Indeed, if $f_{j+1} \notin \tilde{v}$, then $f_{j+1} \in R'$, so $\tilde{v}$ separates in the plane the points f_j and f_{j+1}. By Proposition 27, f_j and f_{j+1} are not in the same minimal block region formed by $\tilde{\mathscr{C}}$, contradicting the fact that $f_j f_{j+1}$ is an edge of $H(\mathscr{C})$. We conclude that μ_1 has a face-vertex on $\tilde{v}$, and in a similar way μ_2 has a face-vertex on $\tilde{v}$. We denote f^1, respectively f^2 these face-vertices. Clearly f^1, f^2 are nonconsecutive vertices of the cycle v_H. Since we assumed that v_H is chordless, f^1 and f^2 are not adjacent in $H(\mathscr{C})$. Therefore, by Proposition 27, there is a cycle $\tilde{v}' \in \tilde{\mathscr{C}}$ separating f^1 and f^2 in the plane. So $\tilde{v}'$ separates two vertices of $\tilde{v}$, contradicting the fact that the curves of $\tilde{\mathscr{C}}$ are pairwise parallel.

We show that any clique Ω^* of $H(\mathscr{C})$ is contained in some minimal block region defined by $\tilde{\mathscr{C}}$. By Proposition 27, for any cycle $v \in \mathscr{C}$, Ω^* is contained in some one-block region $(\tilde{v}, R(\tilde{v}))$. It follows directly that Ω^* is contained in some minimal block defined by $\tilde{\mathscr{C}}$.

Finally, for any minimal block-region BR formed by $\tilde{\mathscr{C}}$, the face-vertices of BR induce a clique in $H(\mathscr{C})$, by definition of $H(\mathscr{C})$. □

9. Main theorem

In this section, we investigate more deeply the structure of the block regions defined by pairwise parallel cycles of G_I which will allow us to compare the number of vertices in G and G^* for all block regions. Before this, we need to state two technical lemmas. These lemmas are stated on an arbitrary 2-connected plane graph, but they will be used on the intermediate graph G_I.

Lemma 31. *Let G be a 2-connected plane graph and let $\mathscr{C}$ be a set of pairwise parallel cycles of G. For any block region* BR *formed by $\tilde{\mathscr{C}}$, the vertices contained in* BR *induce in G a 2-connected subgraph.*

Proof. Let $\mathrm{BR} = \bigcap_{i=1}^{k} (\tilde{v_i}, R(\tilde{v_i}))$ be a block-region of G, we proceed by induction on k, the number of cycles bordering the block region.

If $k = 0$, then $\mathrm{BR}_G = G$ and the result is obvious.

Suppose now $k > 0$. Let x and y be two vertices of BR_G, by induction hypothesis there exists a cycle v' containing x and y inside $\bigcap_{i=2}^{k} (\tilde{v_i}, R(\tilde{v_i}))$. If v' intersects v_1 in at most one vertex then the cycle v' is inside BR.

Otherwise, let μ_x (resp. μ_y) be the path of v' that contains x (resp. y) and whose the only vertices in common with v_1 are its ends x_1 and x_2 (resp. y_1 and y_2). If $\mu_x = \mu_y$ then we can complete μ_x in a simple cycle that belongs to $(\tilde{v_1}, R(\tilde{v_1}))$ by following v_1 from x_1 to x_2. If $\mu_x \neq \mu_y$, on the cycle v_1, the vertices x_1 and x_2 and the vertices y_1 and y_2 are juxtaposed (see Fig. 3). There are two disjoint paths μ_1 and μ_2 of v_1

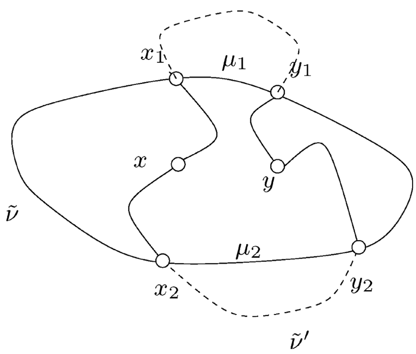


Fig. 3. Cycle inside the block-region.

whose ends are x_1 and x_2, respectively y_1 and y_2. The four paths μ_1, μ_2, μ_x and μ_y form a simple cycle that lies inside $(\tilde{v}_1, R(\tilde{v}_1))$.

Moreover in both cases, since v_1 also lies inside $\bigcap_{i=2}^{n}(\tilde{v_i}, R(\tilde{v_i}))$ the new cycle lies inside $\bigcap_{i=2}^{n}(\tilde{v_i}, R(\tilde{v_i}))$ and so inside BR.

In any case, we can exhibit a cycle inside BR passing through x and y so BR_G is 2-connected. □

Lemma 32. *Let $G=(V,E)$ be a 2-connected planar graph, consider a family $\mathscr{C}$ of pairwise parallel cycles. Let* BR *be a block-region defined by a subfamily of $\tilde{\mathscr{C}}$ and v be one of the cycles bordering* BR.

Either all vertices BR_G are on v or there exists a path $\mu \subseteq \mathrm{BR}_G$ which intersects v only in its endpoints.

Proof. Suppose there exists a vertex $x \in \mathrm{BR}_G$ which is not on v. We know by Lemma 31 that BR_G is 2-connected. Take two vertices y and z on v. Dirac's Fan Lemma states that in any k connected graph, for any $k+1$ vertices $x, y_1, y_2, \ldots, y_k$ there are k paths μ_i joining x and y_i, respectively, for all $1 \leqslant i \leqslant k$, such that any two of them share only the vertex x. Applying the lemma to x, y, z, we get two disjoint paths except in x, μ_y and μ_z in BR_G, connecting respectively x to y and x to z. Cutting μ_y (resp. μ_z) at the first vertex y' (resp. z') on v, we obtain two paths $\mu_{y'}$ and $\mu_{z'}$ which intersect v only in y' and z'. Then the concatenation of $\mu_{y'}$ and $\mu_{z'}$ is a suitable path μ. □

Theorem 33. *Let $G=(V,E)$ be a 3-connected planar graph. Let $\mathscr{C}$ be an inclusion maximal family of pairwise parallel cycles of G_I. Consider a minimal block-region* BR *of G_I defined by $\tilde{\mathscr{C}}$. Then either $\mathrm{BR}_{G_I} = v$ or $\mathrm{BR}_{G_I} = v \cup \mu$, where v is in $\mathscr{C}$ and μ is a path which intersects v only in its endpoints.*

Proof. If BR_{G_I} is not a cycle, we know by Lemma 32 that there exists a path μ inside BR_{G_I} that intersects v only on its endpoints x and y. The vertices x and y define two subpaths v_1 and v_2 of v, so we have three cycles contained in BR_{G_I}, namely v, μv_1 and μv_2. These cycles are pairwise parallel and parallel to the cycles of $\mathscr{C}$, so by maximality of $\mathscr{C}$ they are in $\mathscr{C}$. These three cycles define a block-region BR$'$ which is exactly $\tilde{v} \cup \tilde{\mu}$. Since $\mathrm{BR}' = \tilde{v} \cup \tilde{\mu} \subseteq \mathrm{BR}$ and BR is a minimal block-region we can conclude that $\mathrm{BR}' = \mathrm{BR}$. $\square$

Theorem 34. *Let $G = (V, E)$ be a 3-connected planar graph without loops. Then*

$$\mathrm{tw}(G^*) \leqslant \mathrm{tw}(G) + 1.$$

Proof. Since G is 3-connected without loops, G, G^* and, by Proposition 7, G_I are 2-connected without loops. Let $\mathscr{C}$ be a family of cycles of G_I that gives a triangulation H of G with $\omega(H) - 1 = \mathrm{tw}(G)$. We complete $\mathscr{C}$ into a maximal family $\mathscr{C}'$ of pairwise parallel cycles of G_I. According to Theorem 30, the family $\mathscr{C}'$ defines a triangulation H^* of G^*. Let BR be a minimal block-region with respect to $\tilde{\mathscr{C}}'$. By Theorem 33, either $\mathrm{BR}_{G_I} = v$ or $\mathrm{BR}_{G_I} = v \cup \mu$. In the first case, since G_I is bipartite we have $|\mathrm{BR}_{G_I} \cap V| = |\mathrm{BR}_{G_I} \cap V^*|$. In the later case, the difference between the number of vertices of G and G^* of BR_{G_I} comes from μ. Once again, since G_I is bipartite the difference can be at most one. But each minimal block-region formed by $\tilde{\mathscr{C}}'$ is contained in a minimal block-region formed by $\tilde{\mathscr{C}}$, so, by Theorem 28, $\mathrm{BR}_{G_I} \cap V$ is a clique of H. Therefore the maximal cardinality of a clique in H^* is the maximal cardinality of a clique in H plus one and the second inequality is proved. $\square$

10. Planar graphs which are not 3-connected

We have proved that, for any 3-connected planar graph G, the treewidth of its dual is at most the treewidth of G plus one. We extend this result to arbitrary planar graphs.

The following lemma is a well-known result, see for example [12] for a proof:

Lemma 35. *Let $G = (V, E)$ be a graph (not necessarily planar) and S be a separator of G such that $G[S]$ is a complete graph. Let $V_1, V_2 \subseteq V$ be such that S, V_1 and V_2 form a partition of V and S separates each vertex of V_1 from each vertex of V_2. Then* $\mathrm{tw}(G) = \max(\mathrm{tw}(G[S \cup V_1]), \mathrm{tw}(G[S \cup V_2]))$.

Lemma 36. *Let $G = (V, E)$ be a graph, not necessarily planar. Suppose that G has a minimal separator $S = \{x, y\}$ of size two. Let $G_{xy} = (V, E \cup \{xy\})$ be the graph obtained from G by adding the edge xy. Then* $\mathrm{tw}(G_{xy}) = \mathrm{tw}(G)$.

Proof. If xy is an edge of G, then $G_{xy} = G$. Suppose that xy is not an edge of G. Since G is a minor of G_{xy}, we have $\mathrm{tw}(G) \leqslant \mathrm{tw}(G_{xy})$, so it remains to show that $\mathrm{tw}(G) \geqslant \mathrm{tw}(G_{xy})$.

S is also a minimal separator of G', so let C be a full component associated to S in G and let $V_2 = V \backslash (C \cup S)$. Let $G_1 = G_{xy}[S \cup C]$ and $G_2 = G[S \cup V_2]$. By Lemma 36, we have $\mathrm{tw}(G_{xy}) = \max(\mathrm{tw}(G_1), \mathrm{tw}(G_2))$. It is sufficient to prove that $\mathrm{tw}(G_1) \leqslant \mathrm{tw}(G)$ and $\mathrm{tw}(G_2) \leqslant \mathrm{tw}(G)$. We show that G_1 and G_2 are minors of G.

By Lemma 3, there is a full component D associated to S different from C, so $D \subseteq V_2$. There is a path μ from x to y in $G[D \cup \{x, y\}]$, and the interior of μ avoids the vertices of G_1. Therefore, G_1 is a minor of $G[S \cup C \cup \mu]$, so G_1 is a minor of S. In a similar way, there is a path μ' from x to y in $G[C \cup \{x, y\}]$, so G_2 is a minor of $G[V_2 \cup S \cup \mu']$ and thus a minor of G. We conclude that $\mathrm{tw}(G) \geqslant \max(\mathrm{tw}(G_1), \mathrm{tw}(G_2)) = \mathrm{tw}(G_{xy})$. $\square$

Lemma 37. *Suppose there is a plane graph G not satisfying* $\mathrm{tw}(G^*) \leqslant \mathrm{tw}(G) + 1$ *and there is a separator* $S = \{x, y\}$ *of* G*. Then* G_S *also contradicts* $\mathrm{tw}(G_S^*) \leqslant \mathrm{tw}(G_S) + 1$.

Proof. By Lemma 36, $\mathrm{tw}(G_S) = \mathrm{tw}(G)$. We also have that G_S is planar and G^* is a minor of G_S^*. Indeed, if xy is not an edge of G, let C be a full component of $G \backslash S$ and $v_S(C)$ the cycle associated to S and C, close to C. Then $v_S(C) = [x, f, y, f']$, so x, y are incident to the same face f. We obtain a plane drawing of G_S by adding the edge xy in the face f. The new edge will split the face f into two faces f_1 and f_2, and clearly the dual of G is obtained from the dual of G_S by contracting the edge $f_1 f_2$ into a single vertex f. Therefore, $\mathrm{tw}(G^*) \leqslant \mathrm{tw}(G_S^*)$. Consequently, if $\mathrm{tw}(G_S^*) \leqslant \mathrm{tw}(G_S) + 1$, then $\mathrm{tw}(G^*) \leqslant \mathrm{tw}(G) + 1$. $\square$

Theorem 38. *For any plane graph G,*

$$\mathrm{tw}(G^*) \leqslant \mathrm{tw}(G) + 1.$$

Proof. Suppose there is a graph G such that $\mathrm{tw}(G^*) > \mathrm{tw}(G) + 1$. We take G with the minimum number of vertices. It is easy to check that G must have at least four vertices.

By Theorem 34, G is not 3-connected, so let S be a minimal separator of G with at most two vertices. According to Lemma 37, we can consider that S is a clique in G. Let C be a connected component of $G \backslash S$, we denote $G_1 = G[S \cup C]$ and $G_2 = G[V \backslash C]$ (if G is not connected, then $S = \emptyset$ and C is a connected component of G). By Lemma 35, $\mathrm{tw}(G) = \max(\mathrm{tw}(G_1), \mathrm{tw}(G_2))$.

The graphs G_1 and G_2 are clearly planar and they have less vertices that G, so $\mathrm{tw}(G_1^*) \leqslant \mathrm{tw}(G_1) + 1$ and $\mathrm{tw}(G_2^*) \leqslant \mathrm{tw}(G_2) + 1$. It remains to prove that $\mathrm{tw}(G^*) \leqslant \max(\mathrm{tw}(G_1^*), \mathrm{tw}(G_2^*))$.

Consider the case when G is 2-connected. By Proposition 9 there is a cycle $v_S(C)$ of G_I associated to S and C, close to C. The cycle contains four vertices, $\tilde{v}_S(C) = [x, f, y, f']$. Let R_1 (respectively R_2) be the region of $\Sigma \backslash \tilde{v}_S(C)$ containing C (respectively $V \backslash (S \cup C)$). Notice that the vertices of G_1 (respectively G_2) are exactly the original vertices of $(\tilde{v}_S(C), R_1)$ (respectively $(\tilde{v}_S(C), R_2)$). Let F_1 and F_2 be the face-vertices of G_I contained in R_1, respectively R_2. We denote $S_f = \{f, f'\}$. Let G_1^f be the graph obtained from $G^*[S_f \cup F_1]$ by adding the edge ff'. Consider the plane drawing of G_1 obtained by restricting the drawing of G at the one-block region $(\tilde{v}_S(C), R_1)$ and by adding the edge xy through the region R_2. It is easy to see that G_1^f is exactly the dual of G_1. In a similar way, we define the graph G_2^f obtained from $G^*[S_f \cup F_2]$ by adding the edge ff'. If we consider the plane drawing of G_2 obtained by restricting the drawing of G to the one-block region $(\tilde{v}_S(C), R_2)$ and by adding the edge xy through R_2, then G_2^f is the dual of G_2. By the minimality of G, we have $\mathrm{tw}(G_1^f) \leqslant \mathrm{tw}(G_1) + 1$ and $\mathrm{tw}(G_2^f) \leqslant \mathrm{tw}(G_2) + 1$. Observe now that in the graph $G^*_{S_f}$ obtained from G^* by adding the edge ff', S_f separates F_1 from F_2. By Lemma 35, $\mathrm{tw}(G^*_{S_f}) = \max(\mathrm{tw}(G_1^f), \mathrm{tw}(G_2^f))$, so $\mathrm{tw}(G^*_{S_f}) \leqslant \max(\mathrm{tw}(G_1), \mathrm{tw}(G_2)) + 1 = \mathrm{tw}(G) + 1$. We conclude that $\mathrm{tw}(G^*) \leqslant \mathrm{tw}(G) + 1$.

The case when G is not 2-connected is similar. Suppose that G is connected but no 2-connected, so S has a unique vertex x. There is a face f of G_I such that we can draw a Jordan curve $\tilde{v}_S$ passing through x and f, contained in the face f (except the point x), and the curve separates C from $V \backslash (C \cup \{x\})$. If G is not connected, we can take a connected component C and a face f such that a Jordan curve $\tilde{v}_S$ contained in the face f, passing through f, separates C from $V \backslash C$. As in the case of 2-connected graphs, we consider the regions R_1 (respectively R_2) of $\Sigma \backslash \tilde{v}_S$ containing C (respectively $V \backslash (C \cup S)$). We take $S_f = \{f\}$ and we denote F_1 (respectively F_2) the face-vertices of G_I contained in R_1 (respectively R_2). Then $G_1^f = G^*[S_f \cup F_1]$ is the dual of G_1 and $G_2^f = G^*[S_f \cup F_2]$ is the dual of G_2. We conclude that $\mathrm{tw}(G^*) = \max(\mathrm{tw}(G_1^f), \mathrm{tw}(G_2^f)) \leqslant \max(\mathrm{tw}(G_1), \mathrm{tw}(G_2)) + 1$, so $\mathrm{tw}(G^*) \leqslant \mathrm{tw}(G) + 1$. $\square$

We conclude that, for any plane graph $G = (V, E)$,

$$\mathrm{tw}(G^*) - 1 \leqslant \mathrm{tw}(G) \leqslant \mathrm{tw}(G^*) + 1.$$

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