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The number of the Gabriel–Roiter measures admitting no direct predecessors over a wild quiver

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ABSTRACT

A famous result by Drozd says that a finite-dimensional representation-infinite algebra is of either tame or wild representation type. But one has to make assumption on the ground field. The Gabriel–Roiter measure might be an alternative approach to extend these concepts of tame and wild to arbitrary Artin algebras. In particular, the infiniteness of the number of GR segments, i.e. sequences of Gabriel–Roiter measures which are closed under direct predecessors and successors, might relate to the wildness of Artin algebras. As the first step, we are going to study the wild quiver with three vertices, labeled by 1, 2 and 3, and one arrow from 1 to 2 and two arrows from 2 to 3. The Gabriel–Roiter submodules of the indecomposable preprojective modules and quasi-simple modules $\tau^{-i}M$, $i \geq 0$ are described, where M is a Kronecker module and $\tau = DTr$ is the Auslander–Reiten translation. Based on these calculations, the existence of infinitely many GR segments will be shown. Moreover, it will be proved that there are infinitely many Gabriel–Roiter measures admitting no direct predecessors.

1. Introduction

Throughout, by Artin algebras or finite-dimensional algebras we always mean connected ones. That is, 0 and the identity are the only central idempotents. Let Λ be an Artin algebra and mod Λ the category of finitely generated left Λ-modules. For each $M \in \text{mod}\,\Lambda$, we denote by $|M|$ the length of M. The symbol $\subset$ is used to denote proper inclusion. We first recall the original definition of the Gabriel–Roiter measure [12,13]. Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers and $\mathcal{P}(\mathbb{N})$ be the set of all subsets of $\mathbb{N}$. A total order on $\mathcal{P}(\mathbb{N})$ can be defined as follows: if I,J are two different subsets of $\mathbb{N}$, write $I < J$ if the smallest element in $(I\backslash J) \cup (J\backslash I)$ belongs to J. Also we write $I \ll J$ provided $I \subset J$ and for all elements $a \in I$, $b \in J\backslash I$, we have $a < b$. We say that J **starts with** I if $I = J$ or $I \ll J$. Thus $I < J < I'$ implies that J starts with I, whenever I' does.

Let $M \in \text{mod}\,\Lambda$ and $\mathcal{M}_\bullet$: $M_1 \subset M_2 \subset \cdots \subset M_t$ be a chain of indecomposable submodules of M. Thus the set of the lengths of these indecomposable modules $|\mathcal{M}_\bullet| := \{|M_1|, |M_2|, \ldots, |M_t|\}$ is a subset of $\mathbb{N}$. Let $\mu(M) = \max\{|\mathcal{M}_\bullet|\}$ with the maximum being taken over all possible chains $\mathcal{M}_\bullet$ of indecomposable submodules of M. We call $\mu(M)$ the **Gabriel–Roiter** (**GR** for short) **measure** of M. If M is an indecomposable Λ-module, we call an inclusion $X \subset M$ with X indecomposable a **GR inclusion** provided $\mu(M) = \mu(X) \cup \{|M|\}$, thus if and only if every proper submodule of M has GR measure at most $\mu(X)$. In this case, we call X a **GR submodule** of M. Note that the factor of a GR inclusion is always indecomposable [12]. This provides the first proof of the fact that any non-simple indecomposable module is an extension of two indecomposables.

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Remark. There is a different way [13] to define the Gabriel–Roiter measure for an Λ-module M, which is a rational number, by induction as follows:

$$\mu(M) = \begin{cases} 0 & \text{if } M = 0; \\ \max_{N \subset M}\{\mu(N)\} & \text{if } M \text{ is decomposable;} \\ \max_{N \subset M}\{\mu(N)\} + \frac{1}{2^{|M|}} & \text{if } M \text{ is indecomposable.} \end{cases}$$

These two definitions (orders) can be identified. Namely, for each $I = \{a_i | i\} \in \mathcal{P}(\mathbb{N})$, let $\mu(I) = \sum_i \frac{1}{2^{a_i}}$. Then $I < J$ if and only if $\mu(I) < \mu(J)$.

A subset I of $\mathbb{N}$ is called a GR measure for Λ if there is an indecomposable Λ-module M with GR measure $\mu(M) = I$. Using the GR measure, Ringel obtained a partition of the module category for any Artin algebra of infinite representation type [12]: there are infinitely many GR measures I_i and I^i with i natural numbers, such that

$$I_1 < I_2 < I_3 < \cdots \quad \cdots < I^3 < I^2 < I^1$$

and such that any other GR measure I satisfies $I_i < I < I^j$ for all i, j. The GR measures I_i (resp. I^i) are called take-off (resp. landing) measures. Any other GR measure is called a central measure. An indecomposable module is called a take-off (resp. central, landing) module if its GR measure is a take-off (resp. central, landing) measure. Ringel also showed in [12] that all landing modules are preinjective in the sense of Auslander and Smalø [2]. In [5], it was shown for tame quivers that all indecomposable preprojective modules are take-off modules.

Let Λ be an Artin algebra and I, I' be two GR measures for Λ. We call I' a **direct successor** of I (or I a **direct predecessor** of I') if, first, $I < I'$ and second, there does not exist a GR measure I'' with $I < I'' < I'$. The so-called **Successor Lemma** in [13] states that any GR measure I different from I^1, the maximal one, has a direct successor. However, there is no 'Predecessor Lemma'. For example, the minimal central measure (if it exists) does not have a direct predecessor.

Definition 1.1. Let Λ be an Artin algebra. A sequence of GR measures for Λ is called a GR segment if it is closed under direct successors and direct predecessors.

As a direct consequence of Ringel's partition theorem and the Successor Lemma, we may easily obtain a characterization for representation-finite Artin algebras.

Lemma 1.2. *Let Λ be an Artin algebra. Then the following are equivalent.*

(1) *Λ is of finite representation type.*
(2) *Λ admits only one GR segment.*
(3) *Λ admits a finite GR segment.*

Let us simply denote by k an algebraically closed field. Drozd's famous theorem [7,8] says that a finite-dimensional representation-infinite k-algebra Λ is of either tame or wild representation type. Note that tameness makes no sense for arbitrary fields.

Motivated by Lemma 1.2, we want to characterize tame and wild algebras using the GR measure and to extend the concepts of tame and wild to arbitrary Artin algebras. Let $\mathrm{ndp}\,(\Lambda)$ denote the number of the GR measures, different from the minimal one I_1, for Λ, which do not admit direct predecessors. It is clear that $\mathrm{ndp}\,(\Lambda) = 0$ for any representation-finite Artin algebra. It was shown in [6] that $1 \leq \mathrm{ndp}\,(\Lambda) < \infty$ for each tame hereditary algebra over an algebraically closed field. An positive answer of the following conjecture will obviously give an alternative characterization of tameness and wildness and will thus provide a method to generalize these notions to arbitrary Artin algebras.

Conjecture 1.3. *Let k be an algebraically closed field and Λ be a finite-dimensional k-algebra of infinite representation type. Then the following are equivalent.*

(1) *Λ is of wild type.*
(2) *Λ admits infinitely many GR segments.*
(3) *There are infinitely many GR measures having no direct predecessors.*

For wild algebras, however, it is difficult to calculate the GR measures of the indecomposable modules or to determine if a GR measure has a direct predecessor or not. In this paper, we will study the following wild quiver

$$1 \longrightarrow 2 \rightrightarrows 3$$

and the category of finite-dimensional representations (simply called modules) over an algebraically closed field k. After some preliminaries, the GR submodules will be calculated for all the indecomposable preprojective modules (Proposition 3.1) and the take-off part will also be described (Proposition 3.3). In particular, in contrast to the case of tame quivers, the take-off part contains only finitely many preprojective modules. The GR measure of the indecomposable projective module P_1 is the minimal central measure, thus is a GR measure which does not admit a direct predecessor (Proposition 3.4). An indecomposable module with dimension vector $(0, a, b)$ is called a Kronecker module. The GR submodules of the modules of the form $\tau^{-i}M$, $i \geq 0$ are described, where M is a Kronecker module and τ is the Auslander–Reiten translation (Propositions 3.7, 3.10 and 3.12). Based on these calculations, the existence of infinitely many GR segments will be shown (Theorem 3.11). Finally, in Section 4 infinitely many GR measures will be constructed such that they do not admit direct predecessors (Theorem 4.1). Thus the above conjecture is shown for this special case.

2. Preliminaries and examples

2.1. Some basic properties of the Gabriel–Roiter measure

We refer to [1,11] for general facts of Auslander–Reiten theory and refer to [12,13] as general references to GR measures. We collect some results concerning GR measures, which will be used later on.

Lemma 2.1 ([12]). *Let Λ be an Artin algebra and X and $Y_1, \ldots, Y_m$ be indecomposable Λ-modules. Assume that $f : X \to \bigoplus Y_i$ is a monomorphism.*

(1) $\mu(X) \leq \max\{\mu(Y_i)\}$.
(2) *If $\mu(X) = \max\{\mu(Y_i)\}$, then f splits.*

Proposition 2.2. *Let Λ be an Artin algebra and $X \subset M$ a GR inclusion.*

(1) *If all irreducible maps $N \to M$ with N indecomposable are monomorphisms, then the GR inclusion is an irreducible map.*
(2) *Every irreducible map to M/X is an epimorphism.*
(3) *There is an irreducible monomorphism $X \to Y$ with Y indecomposable and an epimorphism $Y \to M$.*
(4) *There is an epimorphism $\tau^{-1}X \to M/X$.*

The proof of the above statements can be found for example in [3,4].

2.2. Kronecker quiver

Let Q be the Kronecker quiver

$$1 \rightrightarrows 2 .$$

Now we describe the GR measures, which will be very useful in our later discussion. The finite-dimensional representations over an algebraically closed field k are simply called modules.

The GR measure of the indecomposable module with dimension vector $(n, n+1)$ is $\{1, 3, 5, \ldots, 2n+1\}$. The take-off modules are these preprojective modules as well as the simple injective module.

Every indecomposable regular module with dimension vector (n, n) has GR measure $\{1, 2, 4, 6, \ldots, 2n\}$. An indecomposable module is in the central part if and only if it is a regular module.

The GR measure of the indecomposable module with dimension vector $(n+1, n)$ is $\{1, 2, 4, \ldots, 2n, 2n+1\}$. The landing part consists of all the non-simple indecomposable preinjective modules.

2.3. A wild quiver

We refer to, for example, [9,10] for some basic results on wild hereditary algebras. From now on, we fix an algebraically closed field k and consider the following wild quiver:

$$1 \longrightarrow 2 \rightrightarrows 3 .$$

The Cartan matrix C and the Coxeter transformation Φ are the following:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}, \qquad \Phi = -C^{-t}C = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 3 & 2 \\ -2 & -2 & -1 \end{pmatrix}, \qquad \Phi^{-1} = \begin{pmatrix} -1 & -1 & -2 \\ 1 & 0 & 0 \\ 0 & 2 & 3 \end{pmatrix}.$$

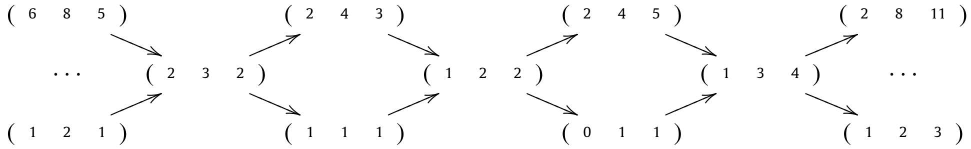
One may calculate the dimension vectors of indecomposable modules using $\underline{\dim}\, \tau M = (\underline{\dim}\, M)\Phi$ if M is not projective and $\underline{\dim}\, \tau^{-1}N = (\underline{\dim}\, N)\Phi^{-1}$ if N is not injective. The Euler form is $\langle \underline{x}, \underline{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_1y_2 - 2x_2y_3$. Then

$$\dim \mathrm{Hom}\,(X, Y) - \dim \mathrm{Ext}^1(X, Y) = \langle \underline{\dim}\, X, \underline{\dim}\, Y \rangle$$

for any two indecomposable modules X and Y.

The Auslander–Reiten quiver consists of one preprojective component, one preinjective component and infinitely many regular ones. An indecomposable regular module X is called quasi-simple if the Auslander–Reiten sequence ending with X has an indecomposable middle term. For each indecomposable regular module M, there is a unique quasi-simple module X and a unique natural number $r \geq 1$ (called quasi-length of M and denoted by $\mathrm{ql}\,(M) = r$) such that there is a sequence of irreducible monomorphisms $X = X[1] \to X[2] \to \cdots \to X[r] = M$.

Let P_i and Q_i, $i = 1, 2, 3$, denote the indecomposable projective and injective modules corresponding to the vertices 1, 2, 3, respectively. We also denote by $H(1)$ an indecomposable module with dimension vector (0, 1, 1). Note that the indecomposable modules with dimension vector (0, 1, 1) are actually parameterized by the projective line $\mathbb{P}^1_k$. We are not going to specify the parameters when our consideration does not depend on them. The following is part of a regular component of the Auslander–Reiten quiver containing some $H(1)$:



Lemma 2.3. (1) *For each $i \geq 0$, $\tau^{-i}H(1)$ contains no proper regular submodules. In particular, a GR submodule of $\tau^{-i}H(1)$ is preprojective.*
(2) *For each $i \geq 0$, $\tau^{i}H(1)$ has no proper regular factors. In particular, a GR factor of $\tau^{i}H(1)$ is preinjective.*

Proof. We show (1) and (2) follows similarly. If X is a proper regular submodule of $\tau^{-i}H(1)$, then the inclusion $X \subset \tau^{-i}H(1)$ induces a proper monomorphism $\tau^{i}X \to H(1)$. This is impossible because $H(1)$ has no proper regular submodules. □

Lemma 2.4. (1) *There is a sequence of monomorphisms*

$$H(1) \to \tau H(1) \to \cdots \to \tau^{i}H(1) \to \tau^{i+1}H(1) \to \cdots.$$

(2) *There is a sequence of epimorphisms*

$$\ldots \to \tau^{-(i+1)}H(1) \to \tau^{-i}H(1) \to \cdots \to \tau^{-1}H(1) \to H(1).$$

Proof. (1) By above lemma, a non-zero homomorphism from $\tau^{i}H(1)$, $i \geq 0$, to a regular module is a monomorphism. On the other hand,

$$\begin{aligned} \dim \operatorname{Hom}(H(1), \tau H(1)) - \dim \operatorname{Ext}^1(H(1), \tau H(1)) &= \langle \underline{\dim}\, H(1), \underline{\dim}\, \tau H(1) \rangle \\ &= \langle (0, 1, 1), (1, 1, 1) \rangle \\ &= 0. \end{aligned}$$

Since $\operatorname{Ext}^1(H(1), \tau H(1)) \neq 0$, $\operatorname{Hom}(H(1), \tau H(1)) \neq 0$ and thus there is a monomorphism $H(1) \to \tau H(1)$. Therefore, there is a sequence of monomorphisms

$$H(1) \to \tau H(1) \to \cdots \to \tau^{i}H(1) \to \tau^{i+1}H(1) \to \cdots.$$

(2) follows dually. □

It is easily seen that the inclusions $H(1) \to \tau H(1) \to \tau^{2}H(1)$ are both GR inclusions. The GR measures of $\tau^{i}H(1)$ are

$$\mu(H(1)) = \{1, 2\}, \mu(\tau H(1)) = \{1, 2, 3\}, \mu(\tau^2 H(1)) = \{1, 2, 3, 4\}$$

and

$$\{1, 2, 3, 4\} < \mu(\tau^{i}H(1)) < \{1, 2, 3, 4, 5\} = \mu(Q_3)$$

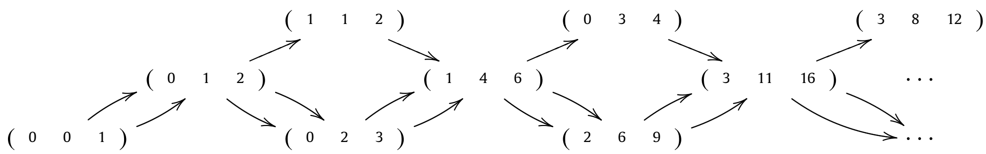
for all $i \geq 3$, where Q_3 is the indecomposable injective module with maximal length.

3. The Gabriel–Roiter submodules

In this section, the take-off part and the minimal central measure will be described. We will also characterize the GR submodules of the indecomposable preprojective modules and the regular modules of the form $\tau^{-i}M$, $i \geq 0$ for Kronecker modules M (meaning the indecomposable modules with $(\underline{\dim}\, M)_1 = 0$). As a consequence of these calculations, we may obtain infinitely many GR segments. This partially answers our conjecture positively in this special case.

3.1. The Gabriel–Roiter submodules of the preprojective modules

The dimension vectors of the indecomposable projective modules are $\underline{\dim}\, P_3 = (0, 0, 1)$, $\underline{\dim}\, P_2 = (0, 1, 2)$ and $\underline{\dim}\, P_1 = (1, 1, 2)$. The beginning part of the preprojective component is the following:



Proposition 3.1. (1) *For each $i \geq 0$, $\tau^{-i}P_2$ is, up to isomorphism, the unique GR submodule of $\tau^{-(i+1)}P_3$.*
(2) *Up to isomorphism, $\tau^{-i}P_3$ is the unique GR submodule of $\tau^{-i}P_1$ for each $i \geq 1$.*
(3) *A GR submodule of $\tau^{-i}P_2$ is isomorphic to $\tau^{-(i-1)}P_1$ if i is odd, or $\tau^{-i}P_3$ if i is even.*

Proof. (1) Since $\tau^{-i}P_2 \bigoplus \tau^{-i}P_2 \to \tau^{-(i+1)}P_3 \to 0$ is a right minimal almost split morphism and the irreducible maps between the involved indecomposable modules are monomorphisms, a GR submodule of $\tau^{-(i+1)}P_3$ is isomorphic to $\tau^{-i}P_2$ (Proposition 2.2(1)).

(2) Let X and Y be indecomposable preprojective modules. Then X is called a predecessors of Y if there is path of irreducible maps $X = X_1 \to X_2 \to \cdots \to X_n = Y$ for some $n > 1$. We first show that $\mu(\tau^{-r}P_3) > \mu(X)$ for all predecessors X of $\tau^{-r}P_3$. This is obvious for $r = 1$. Since the irreducible maps $\tau^{-(r-1)}P_1 \to \tau^{-r}P_2 \to \tau^{-(r+1)}P_3$ are both monomorphisms, $\mu(\tau^{-(r+1)}P_3) > \mu(\tau^{-r}P_2) > \mu(\tau^{-(r-1)}P_1)$. Because there are the irreducible monomorphisms $\tau^{-(r-1)}P_3 \to \tau^{-(r-1)}P_2 \to \tau^{-r}P_3$, we have $\mu(\tau^{-(r-1)}P_3) < \mu(\tau^{-r}P_3)$. Note that a predecessor of $\tau^{-(r+1)}P_3$ is either isomorphic to $\tau^{-r}P_2$ or $\tau^{-(r-1)}P_1$, or a predecessor of $\tau^{-r}P_3$. Therefore, we can finish the proof by induction.

Now we show that $\tau^{-i}P_3$ is a GR submodule of $\tau^{-i}P_1$ for each $i \geq 1$. Since there is a sectional path $\tau^{-i}P_3 \to \tau^{-i}P_2 \to \tau^{-i}P_1$, the composition of the irreducible maps is either an epimorphism or a monomorphism. For each $i \geq 1$,

$$\begin{aligned} |\tau^{-i}P_3| - |\tau^{-i}P_1| &= |\tau^{-i}P_3| + |\tau^{-(i-1)}P_1| - |\tau^{-i}P_2| \\ &= (2|\tau^{-i}P_3| + |\tau^{-(i-1)}P_1| - |\tau^{-i}P_2|) - |\tau^{-i}P_3| \\ &= |\tau^{-(i-1)}P_2| - |\tau^{-i}P_3| < 0. \end{aligned}$$

Thus there is a monomorphism from $\tau^{-i}P_3$ to $\tau^{-i}P_1$. Since a GR submodule of $\tau^{-i}P_1$ is one of its predecessors, it is sufficient to show that neither $\tau^{-i}P_2$ nor $\tau^{-(i-1)}P_1$ is a GR submodule of $\tau^{-i}P_1$. But this is obvious because the irreducible map $\tau^{-i}P_2 \to \tau^{-i}P_1$ is an epimorphism and $\operatorname{Hom}(\tau^{-(i-1)}P_1, \tau^{-i}P_1) = 0$.

(3) Since all irreducible maps $N \to \tau^{-i}P_2$ with N indecomposable are monomorphisms, a GR submodule of $\tau^{-i}P_2$ is isomorphic either to $\tau^{-(i-1)}P_1$ or $\tau^{-i}P_3$. Firstly, we show for each $i \geq 1$ that $\tau^{-(i-1)}P_1 \to \tau^{-i}P_2$ being a GR inclusion implies that $\tau^{-(i+1)}P_3 \to \tau^{-(i+1)}P_2$ is a GR inclusion. If this is not the case, then $\tau^{-i}P_1 \to \tau^{-(i+1)}P_2$ is a GR submodule. Then we have

$$\mu(\tau^{-i}P_3) < \mu(\tau^{(-i-1)}P_1) < \mu(\tau^{-i}P_2) < \mu(\tau^{-(i+1)}P_3) < \mu(\tau^{-i}P_1) < \mu(\tau^{-(i+1)}P_2).$$

Since $\tau^{-i}P_3$ is a GR submodule of $\tau^{-i}P_1$, $\mu(\tau^{-(i-1)}P_1)$ starts with $\mu(\tau^{-i}P_3)$. In particular, there exists a submodule X of $\tau^{-(i-1)}P_1$ such that $\mu(X) = \mu(\tau^{-i}P_3)$. Because X is not isomorphic to $\tau^{-i}P_3$, X has to be a predecessor of $\tau^{-i}P_3$ and thus $\mu(X) < \mu(\tau^{-i}P_3)$ by the discussion in (2). This is a contradiction.

Secondly, we show that $\tau^{-i}P_3 \to \tau^{-i}P_2$ is a GR inclusion implies that $\tau^{-i}P_1 \to \tau^{-(i+1)}P_2$ is a GR inclusion. It is sufficient to show $\mu(\tau^{-(i+1)}P_3) < \mu(\tau^{-i}P_1)$. We may assume $i \geq 1$. Then $\tau^{-i}P_3$ is also a GR submodule of $\tau^{-i}P_1$. Since the irreducible map $\tau^{-i}P_2 \to \tau^{-i}P_1$ is an epimorphism, we have $\mu(\tau^{-i}P_1) > \mu(\tau^{-i}P_2)$. Note that $\tau^{-i}P_2$ is a GR submodule of $\tau^{-(i+1)}P_3$. Thus $\mu(\tau^{-(i+1)}P_3) < \mu(\tau^{-i}P_1)$.

Now the statement follows by induction and the facts that P_3 is a GR submodule of P_2 and P_1 is a GR submodule of $\tau^{-1}P_2$. □

The following observations can be easily checked:

- $\mu(\tau^{-i}P_2) > \mu(X)$ if X is a predecessor of $\tau^{-i}P_2$ for every $i \geq 0$.
- $\mu(\tau^{-i}P_1) > \mu(X)$ for all predecessors X of $\tau^{-i}P_1$ if i is even, or i is odd and $X \ncong \tau^{-(i-1)}P_1, \tau^{-i}P_2$.

3.2. The take-off part and the minimal central measure

As before, let P_1 be the indecomposable projective module with $\underline{\dim}\, P_1 = (1, 1, 2)$. If X is a non-injective indecomposable proper factor of P_1, then X has dimension vector $\underline{\dim}\, X = (1, 1, 1)$ and $\mu(X) = \{1, 2, 3\}$. Thus a non-simple indecomposable module $M \ncong P_1$ with $\operatorname{Hom}(P_1, M) \neq 0$ has GR measure $\mu(M) > \mu(P_1) = \{1, 3, 4\}$.

Lemma 3.2. *Let M be an indecomposable module, which is neither simple nor injective.*

(1) *The GR measure of M is $\{1, 3, 5, \ldots, 2n + 1\}$ if and only if $\underline{\dim}\, M = (0, n, n + 1)$.*
(2) *The GR measure of M is $\{1, 2, 4, \ldots, 2n\}$ if and only if $\underline{\dim}\, M = (0, n, n)$.*

Proof. We show (1) and (2) follows similarly. By the description of the GR measures of the Kronecker modules, $\underline{\dim}\, M = (0, n, n + 1)$ implies that $\mu(M) = \{1, 3, 5, \ldots, 2n + 1\}$. For the converse implication, we use induction on the length. It is clear that $\mu(M) = \{1, 3\}$ if and only if M is the projective module P_2, i.e., $\underline{\dim}\, M = (0, 1, 2)$. Now assume that $\mu(M) = \{1, 3, 5, \ldots, 2n+1, 2n+3\}$ with $n \geq 1$. Then a GR submodule X of M has GR measure $\{1, 3, 5, \ldots, 2n+1\}$. Thus by induction $\underline{\dim}\, X = (0, n, n + 1)$. Since the GR factor M/X has length 2, its dimension vector is $\underline{\dim}\, M/X = (1, 1, 0)$ or $(0, 1, 1)$. In the first case, M/X is the indecomposable injective module Q_2. However, Q_2 cannot be a GR factor module, since there is an irreducible monomorphism $S_2 \to I_2$ (Proposition 2.2 (2)). Therefore, $\underline{\dim}\, M/X = (0, 1, 1)$ and $\underline{\dim}\, M = (0, n+1, n+2)$. □

Proposition 3.3. *A non-simple indecomposable module M is a take-off module if and only if $\underline{\dim}\, M = (0, n, n + 1)$ for some $n \geq 1$. Thus the take-off measures are of the form $\{1, 3, 5, \ldots, 2n + 1\}$ for $n \geq 0$.*

Proof. Let $\mu_n = \{1, 3, 5, \ldots, 2n+1\}$. Then by Lemma 3.2 it is sufficient to show that μ_{n+1} is a direct successor of μ_n for each $n \geq 0$. Assume for a contradiction that

$$\{1, 3, \ldots, 2n+1\} = \mu_n < \mu < \mu_{n+1} = \{1, 3, \ldots, 2n+1, 2n+3\}.$$

Then $\mu = \{1, 3, \ldots, 2n+1, m_1, m_2, \ldots, m_s\}$ with $m_1 > 2n+3$. In particular, there exists an indecomposable module X with length $|X| = m_1$ containing some Y with $\underline{\dim}\, Y = (0, n, n+1)$ as a GR submodule such that the corresponding GR factor X/Y has length $m_1 - (2n+1) > 2$. Assume that $\underline{\dim}\, X = (a, b, c)$. By the description of the GR measures of the Kronecker modules, we have $a \neq 0$ and thus $\mathrm{Hom}\,(P_1, X) \neq 0$. Note that X is obvious not injective. Therefore either there is a monomorphism $P_1 \to X$, or X contains an indecomposable submodule with dimension vector $(1, 1, 1)$. It follows that $\mu \geq \mu(X) > \mu(P_1) > \mu_r$ for all $r \geq 0$. This contradiction implies that μ_{n+1} is a direct successor of μ_n for each $n \geq 1$. □

Proposition 3.4. *The indecomposable projective module P_1 is a central module and $\mu(P_1)$ is the minimal central measure. In particular, $\mu(P_1)$ does not have a direct predecessor.*

Proof. Since $\mu(P_1) = \{1, 3, 4\} > \mu_r = \{1, 3, 5, \ldots, 2r+1\}$ for all $r \geq 0$, P_1 is a central module. Assume for a contradiction that μ is a central measure such that $\mu < \mu(P_1)$. Then

$$\{1, 3, 5, \ldots, 2r+1\} = \mu_r < \mu < \mu(P_1)$$

since μ_r is a take-off measure for each $r \geq 0$. It follows that μ starts with $\{1, 3, 5, \ldots, 2n+1, 2n+2\}$ for some $n \geq 2$. Therefore, there is an indecomposable module M with length $2n+2$ such that it contains an indecomposable module X with $\underline{\dim} = (0, n, n+1)$ as a GR submodule by Lemma 3.2. If the dimension vector of M is $(1, n, n+1)$, then $\mathrm{Hom}\,(P_1, M) \neq 0$ and $\mu(M) > \mu(P_1)$. However, this is not possible because $\mu(M) \leq \mu < \mu(P_1)$. Thus the only possibility is that $\underline{\dim}\, M = (0, n+1, n+1)$ and $\mu(M) = \{1, 2, 4, \ldots, 2n\}$ by Lemma 3.2. This is again a contradiction. Therefore, $\mu(P_1)$ is the minimal central measure and thus does not admit a direct predecessor. □

3.3. The Gabriel–Roiter submodules of $\tau^{-i}M$ with M a Kronecker module

For each integer $a > 0$ and $\lambda \in \mathbb{P}^1_k$, we denote by $H(a)_\lambda$ a representative of the isomorphism class of the indecomposable modules with dimension vector $(0, a, a)$ and parameter λ, and by H_a and H^a for $a \geq 0$ those of indecomposable modules with dimension vectors $(0, a, a+1)$ and $(0, a+1, a)$, respectively. An indecomposable module X is isomorphic to a proper submodule (resp. factor) of H_a (resp. H^a) if and only if X is isomorphic to H_b (resp. H^b) for some $b < a$. Similarly, an indecomposable module Y is isomorphic to a submodule (resp. factor) of $H(a)_\lambda$ if and only if Y is isomorphic to H_b or $H(b)_\lambda$ (resp. H^b or $H(b)_\lambda$) for some $b < a$. The GR measures are $\mu(H_a) = \{1, 3, 5, 7, \ldots, 2a+1\}$, $\mu(H(a)_\lambda) = \{1, 2, 4, 6, \ldots, 2a\}$ and $\mu(H^a) = \{1, 2, 4, 6, \ldots, 2a, 2a+1\}$.

3.3.1. First properties of Kronecker modules

Before we characterize the GR submodules of $\tau^{-i}M$ with M a Kronecker module, we combinatorially describe these modules in the regular components.

Lemma 3.5. (1) *$H(a)_\lambda$ is a quasi-simple module for each $a \geq 1$ and $\lambda \in \mathbb{P}^1_k$.*
(2) *H_a is a quasi-simple module for each $a \geq 4$ and H^a is a quasi-simple module for each $a \geq 1$.*
(3) *Any two regular modules of above three kinds are in different regular components except the pair (H^1, H_4), where $\tau^2 H_4 = H^1$.*

Proof. We show (1) and (2) follows similarly. Assume that $H(a)_\lambda$ is not a quasi-simple module. Then there is a quasi-simple module X and an integer $r \geq 2$ such that $X[r] = H(a)_\lambda$. Then $X \cong H_b$ or $H(b)_\lambda$ for some $0 < b < a$. Thus $\underline{\dim}\, X = (0, b, b+1)$ or $\underline{\dim}\, X = (0, b, b)$. However, the dimension vector of $\tau^{-1}X$ is

$$(0, b, b') \begin{pmatrix} -1 & -1 & -2 \\ 1 & 0 & 0 \\ 0 & 2 & 3 \end{pmatrix} = (b, 2b', 3b')$$

for $b' = b$ or $b+1$. It follows that $(\underline{\dim}\, H(a)_\lambda)_1 = (\underline{\dim}\, X[r])_1 \geq b$. This is a contradiction. Therefore $H(a)_\lambda$ is quasi-simple.

Now we prove (3). It is clear that $(\underline{\dim}\, \tau H(1)_\lambda)_1 = 1 = (\underline{\dim}\, \tau^{-1} H(1)_\lambda)_1$. It follows from Lemma 2.4 that $(\underline{\dim}\, \tau^i H(1)_\lambda)_1 \neq 0$ for any $i \neq 0$. In particular, $\tau^i H(1)_\lambda$ does not isomorphic to H_b or H^b for any b, or $H(b)_\gamma$ for any $b > 1$ or $b = 1$ and $\gamma \neq \lambda$. If $a \geq 2$, the short exact sequence $0 \to H(1)_\lambda \to H(a)_\lambda \to H(a-1)_\lambda \to 0$ induces an exact sequence $0 \to \tau^i H(1)_\lambda \to \tau^i H(a)_\lambda \to \tau^i H(a-1)_\lambda \to 0$ for each integer i. Thus $(\underline{\dim}\, \tau^i H(a)_\lambda)_1 > (\underline{\dim}\, \tau^i H(1)_\lambda)_1 \geq 1$ for every $i \neq 0$. It follows that $H(a)_\lambda$ is the unique indecomposable module M with the property $(\underline{\dim}\, M)_1 = 0$ in the component containing $H(a)_\lambda$.

Instead of $H(a)_\lambda$, we simply write $H(a)$ in the following proof, since the parameter is not so important. We show that H^a and H^b (similarly H_a and H_b) are not in the same component. Without loss of generality, we may assume that $\tau^i H^b = H^a$ for some $i > 0$. Since $H(b)$ is a submodule H^b, $\tau^i H(b)$ is thus a submodule of $\tau^i H^b = H^a$. Thus $\tau^i H(b)$ is isomorphic to $H(c)$ or H_c for some $c < a$. It follows that this $H(b)$ and $H(c)$ or H_c are in the same component. This is a contradiction.

To finish the proof, it is sufficient to show that H^a and H_b are not in the same component with only one exception. If $H_b = \tau^i H^a$ for some $i > 0$, then as before, we have a monomorphism $\tau^i H(a) \to \tau^i H^a = H_b$. Thus $\tau^i H(a)$ is isomorphic to H_c for some $c < b$. This is a contradiction since $H(a)$ and H_c are not in the same component. Therefore the only possibility is $H^a = \tau^i H_b$ for some $i > 0$. If $b > 4$, H_{b-1} is a regular submodule of H_b with factor $H(1)$. Thus $\tau^i H_{b-1}$ is a submodule of $\tau^i H_b = H^a$ with factor $\tau^i H(1)$. Since any indecomposable factor of H^a is of the form H^c with $c < a$, $\tau^i H(1)$ is isomorphic to some H^c. This is again a contradiction. Thus $H^a = \tau^i H_b$ may happen only in case $b = 4$. An easy calculation shows that $\tau^2 H_4 = H^1$. □

Lemma 3.6. *Let M be an indecomposable regular module with dimension vector $\underline{\dim}\, M = (a, b, c)$. Then the quasi-length of M satisfies $ql(M) \leq a + 1$. Moreover, if $a = 1$ and $ql(M) = 2$, then $\underline{\dim}\, M = (1, 2, 2)$ or $(1, 3, 4) = (1, 2, 2)\Phi^{-1}$.*

Proof. Assume for a contradiction that $M = X[r]$ for some quasi-simple module X and $r \geq a+2$. Then $\sum_{i=0}^{r-1} (\underline{\dim}\, \tau^{-i} X)_1 = a \leq r - 2$. It follows from previous discussions that there are $0 \leq i < j \leq r - 1$ such that $(\underline{\dim}\, \tau^{-i} X)_1 = 0 = (\underline{\dim}\, \tau^{-j} X)_1$ and $(\underline{\dim}\, \tau^{-s} X)_1 = 1$ for all $0 \leq s \neq i, j \leq r - 1$. The only possibility is that $\underline{\dim}\, \tau^{-i} X = (0, 2, 1)$ and $\underline{\dim}\, \tau^{-j} X = \underline{\dim}\, \tau^{-(i+2)} X = (0, 4, 5)$. But in this case $\underline{\dim}\, \tau^{-(i+1)} X = (2, 2, 3)$, which contradicts $(\underline{\dim}\, \tau^{-(i+1)} X)_1 = 1$.

If $a = 1$ and $\mathrm{ql}\,(M) = 2$, then $(\underline{\dim} X)_1 = 0$ or $(\underline{\dim}\, \tau^{-1} X)_1 = 0$. If $\underline{\dim} X = (0, x, y)$, then $\underline{\dim}\, \tau^{-1} X = (x, 2y, 3y)$. It follows that $x = 1, y = 1$ and thus $\underline{\dim}\, M = (1, 3, 4)$. If $\underline{\dim}\, \tau^{-1} X = (0, x, y)$, then $\underline{\dim} X = (3x-2y, 3x-2y, 2x-y)$. Thus $3x - 2y = 1$ and the only possibility is $x = y = 1$. It follows that $\underline{\dim}\, M = (1, 2, 2)$. Note that $(1, 2, 2) = (1, 3, 4)\Phi$. □

3.3.2. The Gabriel–Roiter submodules

Now we start to calculate the GR submodules of $\tau^{-i} M$ for Kronecker modules M and for all $i \geq 0$.

Proposition 3.7. *Up to isomorphism, $\tau^{-i} H_a$ is the unique GR submodule of $\tau^{-i} H_{a+1}$ for each $a \geq 1$ and $i \geq 0$. It follows that all $\tau^{-i} H(1)_\lambda$ are GR factor modules.*

Proof. Since H_b is a GR submodule of H_{b+1} with regular GR factor module $H(1)_\lambda$ (different embeddings give rise to different factors), we have monomorphisms $\tau^{-i} H_b \to \tau^{-i} H_{b+1}$ with factors $\tau^{-i} H(1)_\lambda$. In particular, $\mu(\tau^{-i} H_b) < \mu(\tau^{-i} H_{b+1}) < \cdots < \mu(\tau^{-i} H_{b+r}) = \mu(\tau^{-i} H_a)$ for all $r = a - b > 0$.

Assume that X is a GR submodule of $\tau^{-i} H_{a+1}$. If X is a regular module, then the monomorphism $X \to \tau^{-i} H_{a+1}$ induces a monomorphism $\tau^i X \to H_{a+1}$. It follows that $\tau^i X \cong H_b$ for some $b \leq a$. However, $\mu(\tau^{-i} H_b) < \mu(\tau^{-i} H_{b+1})$ for all b. Thus $X \cong \tau^{-i} H_a$. Similarly, if $X \cong \tau^{-j} P$ is preprojective for some indecomposable projective module P and some $j > i + 1$, then $\tau^{-(j-i)} P \cong H_b$ and thus $j - i \leq 1$, which is impossible. Thus $X \cong \tau^{-j} P$ for some indecomposable module P and some $j \leq i + 1$.

For $a = 1, 2$, all modules $\tau^{-i} H_{a+1}$ are preprojective and the statement is the same with that $\tau^{-i} P_2$ is a GR submodule of $\tau^{-(i+1)} P_3$ and $\tau^{-i} P_3$ is a GR submodule of $\tau^{-i} P_1$, which we have proved (Proposition 3.1). If $a = 3$, then the GR submodule X of $\tau^{-i} H_4$ has to be preprojective since H_4 contains no regular submodules. In this case, $X \cong \tau^{-j} P$ for some projective module P and some $j \leq i + 1$. Note that X is a predecessor of $\tau^{-i} H_3$ in the preprojective component. If i is odd, then $\mu(\tau^{-i} H_3) = \mu(\tau^{-(i+1)} P_1)$ is larger than all $\mu(Y)$ if Y is one of its predecessors. Thus $\tau^{-i} H_3 \to \tau^{-i} H_4$ is a GR inclusion. If i is even, then the only predecessors of $\tau^{-i} H_3$ with GR measures larger than $\mu(\tau^{-i} H_3)$ are $\tau^{-(i+1)} P_2$ and $\tau^{-i} P_1$. In both cases, we get monomorphisms from $\tau^{-1} P_2$ and P_1 to H_4, respectively. This is a contradiction. Therefore, $\tau^{-i} H_3$ is a GR submodule of $\tau^{-i} H_4$.

Finally, assume that $a \geq 4$. It is sufficient to show that a GR submodule X of $\tau^{-i} H_a$ is regular for each $i \geq 0$. If X is preprojective, then as before, X is a predecessor of $\tau^{-i} H_3$. Again if i is odd, then $\mu(X) \leq \mu(\tau^{-i} H_3) < \mu(\tau^{-i} H_a)$, a contradiction. If i is even, we may repeating the arguments as in the case $a = 3$ and get a contradiction.

We finish the proof. □

Remark. Since $\tau^{-i} H(1)_\lambda$ are GR factors, we may obtain, for any natural number r, that a GR inclusion $X \to Y$ such that $|Y/X| \geq r$. However, for tame quivers, the dimension vectors of the GR factors are always bounded by δ, where δ is the minimal positive imaginary root [4].

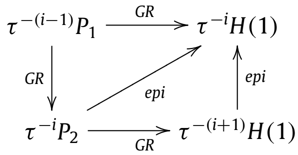
Proposition 3.8. *Fix a $\lambda \in \mathbb{P}_k^1$ and simply denote $H(1)_\lambda$ by $H(1)$. For each $i > 0$, a GR submodule of $\tau^{-i} H(1)$ is isomorphic to*

$$\begin{cases} \tau^{-(i-1)} P_1 & \text{if } i \text{ is odd;} \\ \tau^{-(i-1)} P_2 & \text{if } i \text{ is even.} \end{cases}$$

Proof. Let X be a GR submodule of $\tau^{-i} H(1)$. Then X is preprojective by Lemma 2.3. If $X = \tau^{-j} P$ for some indecomposable projective module P and $j > i$, then we obtain a monomorphism from $\tau^i X$ to $H(1)$. But this is impossible since the unique proper submodule of $H(1)$ is the simple projective module. Thus $X = \tau^{-r} P$ for some indecomposable projective module P and some $r < i$. Clearly, P_1 is a GR submodule of $\tau^{-1} H(1)$ with $H(1)$ as a GR factor. We thus obtain monomorphisms $\tau^{-(i-1)} P_1 \to \tau^{-i} H(1)$ for all $i > 0$. By the same reason, we get monomorphisms $\tau^{-i} P_2 \to \tau^{-i} H(1)$. We can finish the proof by applying the description of the GR measures of the preprojective modules. □

Corollary 3.9. *Fix a $\lambda \in \mathbb{P}_k^1$. Then $\mu(P) < \mu(\tau^i H(1)) < \mu(\tau^j H(1))$ for all $i < j$ and all preprojective modules P.*

Proof. It is sufficient to shown $\mu(\tau^{-i}H(1)) > \mu(\tau^{-(i+1)}H(1))$ for all $i \geq 0$. This is clear for $i = 0$. Now assume $i > 0$ is odd. We use the following diagram to indicate the homomorphisms:



Here *GR* stands for GR inclusions and *epi* for epimorphisms. Since $|\tau^{-i}H(1)| > |\tau^{-i}P_2|$, we have $\mu(\tau^{-i}H(1)) > \mu(\tau^{-i}P_2)$. It follows from $\mu(\tau^{-(i+1)}H(1)) = \mu(\tau^{-i}P_2) \cup \{|\tau^{-(i+1)}H(1)|\}$ that $\mu(\tau^{-i}H(1)) > \mu(\tau^{-(i+1)}H(1))$. The case that $i > 0$ is even follows similarly. Finally, using the description of the GR submodules of $\tau^{-i}H(1)$ and those of preprojective modules, we can easily deduce that $\mu(\tau^{-i}H(1)) > \mu(P)$ for all i and all preprojective modules P. $\square$

Proposition 3.10. *Fix a $\lambda \in \mathbb{P}^1_k$. For each $i \geq 0$ and $a \geq 1$, $\tau^{-i}H(a)$ is the unique, up to isomorphism, GR submodule of $\tau^{-i}H(a+1)$.*

Proof. Since there is a monomorphism $\tau^{-i}H(1) \to \tau^{-i}H(a+1)$, the above corollary implies that the GR submodules of $\tau^{-i}H(a+1)$ are regular modules for all $i \geq 0$. Let X be a GR submodule of $\tau^{-i}H(a+1)$. Then $\tau^{i}X$ is a submodule of $H(a+1)$ and thus isomorphic to $H(b)$ or H_b for some $b < a+1$. Thus $X \cong \tau^{-i}H(b)$ or $X \cong \tau^{-i}H_b$.

Since there is a monomorphism from $\tau^{-i}H(a)$ to $\tau^{-i}H(a+1)$, it is sufficient to show that $\mu(\tau^{-i}H(a)) \geq \mu(X)$. This is obvious for $X \cong \tau^{-i}H(b)$. Assume that $X \cong \tau^{-i}H_b$ for some $b < a+1$. We consider the dimension vectors of the form $(0, c, c+1)$. Note that $(0, c, c+1)\Phi^{-1} = (0, c, c)\Phi^{-1} + (0, 0, 1)\Phi^{-1} = (0, c, c)\Phi^{-1} + (0, 2, 3)$. This implies $|\tau^{-1}H_c| > |\tau^{-1}H(c)|$. Since $(0, 2, 3) = \underline{\dim}\, H_2$, $(0, 2, 3)\Phi^{-i}$ is a positive vector, namely $\underline{\dim}\, \tau^{-i}H_2$ for each $i \geq 0$. It follows that $|\tau^{-i}H_c| > |\tau^{-i}H(c)| \geq |\tau^{-i}H(1)|$ for all $c \geq 1$. Since $\tau^{-i}H_c$ is a GR submodule of $\tau^{-i}H_{c+1}$ and $\mu(\tau^{-i}H_3) < \mu(\tau^{-i}H(1))$, we have $\mu(\tau^{-i}H_c) < \mu(\tau^{-i}H(1)) \leq \mu(\tau^{-i}H(a))$. Thus for $a \geq 1$, $\tau^{-i}H(a)$ is a GR submodule of $\tau^{-i}H(a+1)$. $\square$

The following result is a direct consequence of Proposition 3.10 and Corollary 3.9, which answers partially our conjecture in this special case.

Theorem 3.11. *There are infinitely many GR segments.*

Proof. Let us keep the notations as before and fix a parameter $\lambda \in \mathbb{P}^1_k$. Let $i \geq 1$. Since

$$\tau^{-i}H(1) \subset \tau^{-i}H(2) \subset \tau^{-i}H(3) \subset \cdots \subset \tau^{-i}H(m) \subset \cdots$$

is a sequence of GR inclusions, we have for each $m \geq 1$

$$\mu(\tau^{-i}H(m)) = \mu(\tau^{-i}H(1)) \cup \{|\tau^{-i}H(2)|, \ldots, |\tau^{-i}H(m)|\}.$$

Note that $\mu(\tau^{-i}H(1)) < \mu(\tau^{-(i-1)}H(1))$ by Corollary 3.9 and $|\tau^{-i}H(1)| > |\tau^{-(i-1)}H(1)|$ by Lemma 2.4. Therefore,

$$\mu(\tau^{-(i-1)}H(1)) > \mu(\tau^{-i}H(1)) \cup \{|\tau^{-i}H(2)|, \ldots, |\tau^{-i}H(m)|\} = \mu(\tau^{-i}H(m)).$$

for all $m \geq 1$. In particular, $\tau^{-(i-1)}H(1)$ and $\tau^{-i}H(1)$ are not in the same GR segment since there are infinitely many GR measures lying in between.

The proof is completed. $\square$

Proposition 3.12. *For each $a \geq 1$, the GR submodules of $\tau^{-i}H^a$, $i \geq 0$ are*

$$\begin{cases} \tau^{-i}H(a)_\lambda, \lambda \in \mathbb{P}^1_k, & i = 0, 1; \\ \tau^{-(i-1)}P_1, & i \geq 2, a = 1; \\ \tau^{-i}H(a-1)_\lambda, \lambda \in \mathbb{P}^1_k, & i \geq 2, a \geq 2. \end{cases}$$

Proof. First assume that $a = 1$. Note that $\tau^{-2}H^1$ is H_4 and the GR submodules of $\tau^{-i}H^1$ is already known as $\tau^{-(i-1)}P_1$ for any $i \geq 2$. Every indecomposable module $H(1)_\lambda$ is a GR submodule of H^1 with factor S_2, which is not injective. It follows that there is a monomorphism $\tau^{-1}H(1)_\lambda \to \tau^{-1}(H^1)$. We claim that it is actually a GR inclusion. Note that $\tau^{-1}H(1)_\lambda$ has GR measure $\{1, 3, 4, 6\} > \mu(P)$ for any preprojective module P. Thus a GR submodule of $\tau^{-1}H^1$ has to be regular. Assume that X is a GR submodule of $\tau^{-1}H^1$. We obtain a monomorphism $\tau X \to H^1$, and therefore, X has dimension $(0, 1, 1)$. Thus $\tau^{-1}H(1)_\lambda$ is a GR submodule of $\tau^{-1}H^1$ for each λ.

Now we consider the case $a \geq 2$. Since there is a monomorphism $H(1)_\lambda \to H^a$ for each λ with indecomposable regular factor H^{a-1}, the GR submodules of $\tau^{-i}H^a$ are regular. Let X be a GR submodule of $\tau^{-i}H^a$, then there is a monomorphism $\tau^{i}X \to H^a$. It follows X is either isomorphic to $\tau^{-i}H_b$ or $\tau^{-i}H(b)_\lambda$ for some $b \leq a$. From the description of the GR submodules of these modules, we know that the GR submodules of $\tau^{-i}H^a$ are of the form $\tau^{-i}H(b)_\lambda$ with b as large as possible. We may calculate the dimension vectors as follows:

$$(0, a+1, a)\Phi^{-2} = (a+1, 2a, 3a)\Phi^{-1} = (a-1, 5a-1, 7a-2),$$
$$(0, a, a)\Phi^{-2} = (7a, 2a, 3a)\Phi^{-1} = (a, 5a, 7a),$$
$$(0, a-1, a-1)\Phi^{-2} = (a-1, 2a-2, 3a-3)\Phi^{-1} = (a-1, 5a-5, 7a-7).$$

Comparing the dimension vectors, we conclude that the GR submodules of $\tau^{-i}H^a$ for $a \geq 2$ and $i \geq 2$ are $\tau^{-i}H(a-1)_\lambda$ for all $\lambda \in \mathbb{P}^1_k$, and the GR submodules of $\tau^{-i}H^a$ are $\tau^{-i}H(a)_\lambda$ for $i = 0, 1$. □

In general, for each $i > 0$, $\tau^i H(a)_\lambda$ is not a GR submodule of $\tau^i H(a+1)_\lambda$.

Proposition 3.13. *Any GR factor of $\tau H(a)_\lambda$ is isomorphic to the simple injective module. In particular, $\tau H(a)_\lambda$ is not a GR submodule of $\tau H(a+1)_\lambda$.*

Proof. If $a = 1$, then $\underline{\dim}\,\tau H(1)_\lambda = (1, 1, 1)$ and $\underline{\dim}\,\tau H(2)_\lambda = (2, 2, 2)$. There does not exist an epimorphism $H(1)_\lambda \to H(2)_\lambda / H(1)_\lambda$. Thus $\tau H(1)_\lambda$ is not a GR submodule of $\tau H(2)_\lambda$ by Proposition 2.2(4). Now assume that $a > 1$. As before, it is easily seen that a GR submodule of $\tau^i H(a)_\lambda$ is regular and has GR measure not smaller than $\mu(H(1)_\lambda)$. Let $0 \to X \xrightarrow{f} \tau H(a)_\lambda \to Y \to 0$ be a GR sequence, that is, f is a GR inclusion. If Y is not injective, we get the following exact sequence $0 \to \tau^{-1}X \to H(a)_\lambda \to \tau^{-1}Y \to 0$. Then $\tau^{-1}X$ is isomorphic to $H(b)_\lambda$ or H_b for some $b < a$. Because H_b is cogenerated by $H(1)_\lambda$, τH_b is cogenerated by $\tau H(1)_\lambda$. Thus $\mu(\tau H_b) < \mu(\tau H(1)_\lambda)$ and $\tau^{-1}X$ is not of the form H_b. Assume that $\tau^{-1}X = H(b)_\lambda$ for some $b < a$ and therefore $b = a-1$ since X is a GR submodule. However, an easily calculation shows $\underline{\dim}\,\tau H(a)_\lambda = (a, a, a)$ and

$$\begin{aligned}\underline{\dim}\,\tau H(a-1)_\lambda[2] &= \underline{\dim}\,\tau^{-1}H(a-1)_\lambda + \underline{\dim}\,H(a-1)_\lambda \\ &= (0, a-1, a-1) + (a-1, a-1, a-1) \\ &= (a-1, 2a-2, 2a-2).\end{aligned}$$

Thus, there does not exist an epimorphism from $\tau H(a-1)_\lambda[2]$ to $\tau H(a)_\lambda$. This contradiction implies that in the above GR sequence concerning $\tau H(a)_\lambda$, Y has to be injective. It follows that Y is isomorphic to Q_1 or Q_3, the indecomposable injective modules.

If Y is isomorphic to Q_3, then $\underline{\dim}\,X = (a, a, a) - (2, 2, 1) = (a-2, a-2, a-1)$ and then $\underline{\dim}\,\tau^{-1}X = (0, a, a+1)$. This is impossible since $(\underline{\dim}\,X[2])_1 = a-2 < a = (\underline{\dim}\,\tau H(a)_\lambda)_1$. Thus the GR factor Y of $\tau H(a)$ is isomorphic to the simple injective module Q_1 and the GR submodule X of $\tau H(a)$ has dimension vector $(a-1, a, a)$. In particular, $\tau H(a)_\lambda$ is not a GR submodule of $\tau H(a+1)_\lambda$ because $\underline{\dim}\,\tau H(a)_\lambda = (a, a, a) \neq (a, a+1, a+1)$. □

3.4. The Gabriel–Roiter measures of indecomposable modules of small dimensions

In this section, we try to determine all possible GR measures of indecomposable modules with dimensions not greater than 6.

Lemma 3.14. (1) *An indecomposable module M has GR measure $\{1, 2, 3\}$ if and only if $\underline{\dim}\,M = (1, 1, 1)$ or $(0, 2, 1)$.*
(2) *An indecomposable module M has GR measure $\{1, 2, 3, 4\}$ if and only if $\underline{\dim}\,M = (1, 2, 1)$.*
(3) *An indecomposable module M with dimension vector $(1, 2, 2)$ and $\mathrm{ql}\,(M) = 2$ has GR measure $\{1, 2, 3, 5\}$.*
(4) *An indecomposable module M with dimension vector $(1, 3, 4)$ and $\mathrm{ql}\,(M) = 2$ has GR measure $\{1, 2, 8\}$.*
(5) *An indecomposable module M has GR measure $\{1, 3, 4, 6\}$ if and only if $\underline{\dim}\,M = (1, 2, 3)$, i.e., $M \cong \tau^{-1}H(1)_\lambda$ for some $\lambda \in \mathbb{P}^1_k$.*

Proof. (1) and (2) are obvious.

(3) Note that $M = X[2]$ for a quasi-simple module X with dimension vector $(1, 1, 1)$. Thus $\{1, 2, 3\} < \mu(M) < \mu(Q_3) = \{1, 2, 3, 4, 5\}$, the maximal GR measure. Thus $\mu(M) = \{1, 2, 3, 5\}$.

(4) $M = X[2]$ for a quasi-simple module X with dimension vector $(0, 1, 1)$. It is easily seen that $\mathrm{Hom}\,(H^1, M) = 0 = \mathrm{Hom}\,(\tau H(1)_\lambda, M)$ for all $\lambda \in \mathbb{P}^1_k$. Thus $\{1, 2, 8\} \leq \mu(M) < \{1, 2, 3\}$. In particular, a GR submodule Y of M is regular and thus τY is a submodule of τM with $\underline{\dim}\,\tau M = (1, 2, 2)$. It is not difficult to see that the only possibility is $\underline{\dim}\,Y = (0, 1, 1)$. Thus $\mu(M) = \{1, 2, 8\}$.

(5) If $\underline{\dim}\,M = (1, 2, 3)$, then $M \cong \tau^{-1}H(1)_\lambda$ for some $\lambda \in \mathbb{P}^1_k$. Thus the projective module P_1 is a GR submodule of M. Conversely, if $\mu(M) = \{1, 3, 4, 6\}$, then M contains P_1 as a GR submodule and the corresponding GR factor has dimension vector $(0, 1, 1)$. Thus $\underline{\dim}\,M = (1, 2, 3)$. □

Let M be an indecomposable module with dimension vector $\underline{\dim}\,M = (1, 2, 2)$ or $(1, 3, 4)$. By Lemma 3.6, we know that $\mathrm{ql}\,(M) \leq 2$. The GR measure of M is already determined in case $\mathrm{ql}\,(M) = 2$ by previous lemma. What about the GR measures of those M with $\mathrm{ql}\,(M) = 1$?

Lemma 3.15. *Let M be an indecomposable module with dimension $(1, 2, 2)$ or $(1, 3, 4)$ and $\mathrm{ql}\,(M) = 2$. If N is a quasi-simple module with $\underline{\dim}\,N = \underline{\dim}\,M$, then $\mu(N) < \mu(M)$.*

Proof. Let $\underline{\dim}\,M = (1, 2, 2)$. We have seen that $\mu(M) = \{1, 2, 3, 5\}$. If N is a quasi-simple module with $\underline{\dim}\,N = (1, 2, 2)$, then $\mu(N)$ does not start with $\{1, 2, 3\}$. Otherwise, N contains some indecomposable module X with dimension vector $(1, 1, 1)$ or $(0, 2, 1)$ as a GR submodule. If $\underline{\dim}\,X = (1, 1, 1)$, then there is an epimorphism from $X[2]$ to N, which is impossible since $\underline{\dim}\,X[2] = (1, 2, 2) = \underline{\dim}\,N$. Note that $\underline{\dim}\,X = (0, 2, 1)$ is either not possible, since otherwise, the factor contains the projective simple module. Therefore, $\mu(N) < \{1, 2, 3\} < \mu(M)$.

Now let $\underline{\dim} M = (1, 3, 4)$ and N be a quasi-simple module with dimension vector $\underline{\dim} N = \underline{\dim} M$. Then N does not contain any $H(1)_\lambda$ as a submodule for any λ since otherwise, τN with $\underline{\dim}\, \tau N = (1, 2, 2)$ contains a submodule module with dimension vector $(1, 1, 1)$. This is not possible by above discussion. Therefore $\mu(N) < \{1, 2\} < \mu(M)$. □

We may ask the following question in general.

Question. *Let M and N be indecomposable regular modules such that $\underline{\dim} M = \underline{\dim} N$ and $\mathrm{ql}(M) > \mathrm{ql}(N)$. Does $\mu(M) > \mu(N)$ hold?*

We may calculate precisely the GR measures of the quasi-simple modules N with $\underline{\dim} N = (1, 3, 4)$ and M with $\underline{\dim} M = (1, 2, 2)$. Since $\mu(N) < \{1, 2\}$, $\mu(N)$ starts with $\mu(P_1) = \{1, 3, 4\}$. On the other hand, $\mu(N)$ does not contain 7. Namely, assume that it is not the case and let X be a GR submodule of N. Then the GR factor N/X is simple and thus $\underline{\dim} X = (0, 3, 4)$ or $(1, 2, 4)$. However, the first vector corresponds the preprojective module H_3 and the second vector is not a root. Similarly, a detailed discussion shows that $\mu(N)$ does not contain 5. Therefore, the only possibility is that $\mu(N) = \{1, 3, 4, 6, 8\}$. Thus any GR submodule of N is of dimension vector $(1, 2, 3)$. It follows that any quasi-simple module M with $\underline{\dim} M = (1, 2, 2)$ contains a submodule of dimension vector $(1, 2, 3)\Phi = (0, 1, 1)$. Thus the GR measure of M is either $\{1, 2, 4, 5\}$ or $\{1, 2, 5\}$, and both possibilities occur.

So far, we have known the GR measures of the indecomposable modules M with lengths not greater than 6 except $\underline{\dim} M = (2, 2, 2)$ and $(1, 3, 2)$. If $\underline{\dim} M = (2, 2, 2)$, then a GR factor of M is the simple injective module by Proposition 3.13. Thus a GR submodule of M has dimension vector $(1, 2, 2)$. Note that M always contains an indecomposable module with dimension vector $(1, 1, 1)$. Therefore, $\mu(M) = \{1, 2, 3, 5, 6\}$.

There are several possibilities for $\underline{\dim} M = (1, 3, 2)$. First of all, there is always a monomorphism from any indecomposable module with dimension vector $(0, 1, 1)$. Let X be a GR submodule of M and Y be the corresponding factor. Then $|Y| \neq 2$, since otherwise, X has dimension vector $(1, 2, 1)$. However, there does not exist an epimorphism from $\tau^{-1}X$ to Y because $\underline{\dim}\, \tau^{-1}X = (1, 1, 1)$ and $\underline{\dim} Y = (0, 1, 1)$ (Proposition 2.4(4)). Thus Y is a simple module or $|Y| = 3$. Thus only the following possibilities might occur: $\{1, 2, 3, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 4, 5, 6\}, \{1, 2, 5, 6\}$.

4. Gabriel–Roiter measures admitting no direct predecessors

Ringel showed in [13] that each GR measure different from I^1, the maximal GR measure, has a direct successor. However, there are GR measures admitting no direct predecessors in general. We have shown in [6] for path algebras of tame quivers that only finitely many GR measures do not admit direct predecessors. As we have mentioned in introduction, we want to know if the number of the GR measures admitting no direct predecessors relates to the representation type (tame or wild).

It should be very difficult to answer this question in general. Now we come back to the quiver we considered above and keep the notations as before. We have seen that $\mu(P_1) = \{1, 3, 4\}$ is the minimal central measure and it does not have a direct predecessor. In this section, we will show the following theorem:

Theorem 4.1. *Let $n \geq 1$ and $\mu^n = \{1, 2, 4, \ldots, 2n, 2n+1\}$. Then μ^n does not have a direct predecessor for any n.*

Lemma 4.2. (1) *For each $n \geq 1$, μ^n is a GR measure.*
(2) *If M is an indecomposable module with $\mu(M) = \mu^n$. Then $\underline{\dim} M = (1, n, n)$ or $(0, n+1, n)$.*

Proof. It is known that each indecomposable module with dimension vector $(0, n+1, n)$ has GR measure μ^n. Thus μ^n is a GR measure. On the other hand, a non-injective indecomposable module M has GR measure $\{1, 2, 4, \ldots, 2n\}$ if and only if $\underline{\dim} M = (0, n, n)$. Thus an indecomposable module with GR measure μ^n has dimension vector $(1, n, n)$ or $(0, n+1, n)$. □

Lemma 4.3. *Let M be an indecomposable module with GR measure $\mu(M) = \mu^n$. Then each indecomposable regular factor of M contains some indecomposable submodule with dimension vector* $(0, 1, 1)$.

Proof. By above lemma, $\underline{\dim} M = (1, n, n)$ or $(0, n+1, n)$. In each case, we have a short exact sequence

$$0 \to X \stackrel{\iota}{\to} M \to M/X \to 0$$

where ι is a GR inclusion and thus $\underline{\dim} X = (0, n, n)$. Note that the factor M/X is a preinjective simple module. Let $M \stackrel{\pi}{\to} Y$ be an epimorphism with Y an indecomposable regular module. Then $\mathrm{Hom}(M/X, Y) = 0$. Since M/X is the cokernel of ι, the composition $X \stackrel{\pi\iota}{\to} Y$ is not zero. Since an indecomposable non-simple factor of X has dimension vector $(0, a, a)$ or $(0, a+1, a)$, the image of $\pi\iota$ contains a submodule with dimension vector $(0, 1, 1)$. □

Lemma 4.4. *Fix an $n \geq 1$. Let M be an indecomposable module such that $\mu^n < \mu(M)$. Then $\mu(M)$ starts with $\mu^m = \{1, 2, \ldots, 2m, 2m+1\}$ for some $1 \leq m \leq n$. In particular, M contains an indecomposable submodule with GR measure μ^m.*

Proof. This follows directly from the definition of the total order on $\mathcal{P}(\mathbb{N})$. □

Lemma 4.5. *If M is an indecomposable module such that $\mu(M)$ is a direct predecessor of μ^n for some n. Then M is regular.*

Proof. It is easily seen that $\mu(P) < \{1, 2\}$ for any indecomposable preprojective module P. Moreover, for each preprojective module, there are infinitely many indecomposable preprojective ones with greater GR measures. Thus M is not preprojective. Let X be an indecomposable regular with dimension vector $(1, 1, 1)$. Then the only indecomposable preinjective module Q such that $\mathrm{Hom}(X, Q) = 0$ is $S_2 \cong \tau Q_1$. Therefore, if Q is neither isomorphic to the simple modules S_1, S_2, nor isomorphic to the injective module Q_2, there is always a monomorphism from X to Q and thus $\mu(Q)$ starts with $\{1, 2, 3\} = \mu^1 > \mu^n$ for every $n > 1$. It follows that M has to be a regular module. □

Proof of Theorem. For the purpose of a contradiction, we assume that M is an indecomposable module such that $\mu(M)$ is a direct predecessor of μ^n for a fixed n. Thus by Lemma 4.5, we may write $M = X[r]$ for some quasi-simple module X and $r \geq 1$. Since $\{1, 2, 4, \dots 2n\} < \mu(M) < \{1, 2, 4, \dots 2n, 2n+1\} = \mu^n$, it follows that $|M| > 2n + 1$ and thus $|X[r+1]| > 2n + 1$. In particular, $\mu(M) = \mu(X[r]) < \mu^n < \mu(X[r+1])$ since $\mu(X[r+1]) > \mu(X[r])$ and $\mu(X[r])$ is a direct predecessor of μ^n. Thus $X[r+1]$ contains a submodule Y with GR measure μ^m for some $1 \leq m \leq n$ (Lemma 4.4). Note that $\underline{\dim}\, Y = (1, m, m)$ or $(0, m+1, m)$ and $\mu(Y) \geq \mu^n$. We claim that $\mathrm{Hom}(Y, \tau^{-r}X) = 0$. If this is not the case, then by Lemma 4.3, the image of a non-zero homomorphism, in particular $\tau^{-r}X$, contains a submodule Z with dimension vector $(0, 1, 1)$. Therefore, there is a proper monomorphism $\tau^r Z \to X$, and thus

$$\mu^n > \mu(M) \geq \mu(X) > \mu(\tau^r Z) \geq \{1, 2, 3\},$$

which is a contradiction. Since there is a short exact sequence

$$0 \to M = X[r] \to X[r+1] \to \tau^{-r}X \to 0$$

and $\mathrm{Hom}(Y, \tau^{-r}X) = 0$, the inclusion $Y \to X[r+1]$ factors through $X[r]$. In particular, there is a monomorphism $Y \to X[r]$. It follows that

$$\mu(X[r]) \geq \mu(Y) \geq \mu^n > \mu(M) = \mu(X[r]).$$

This contradiction implies that μ^n does not have a direct predecessor for any $n \geq 1$. □

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