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Indecomposable decompositions of modules whose direct sums are CS ☆

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1. Introduction

It was shown in [6] that if M is a Σ-CS module, i.e., a module such that every direct sum of copies of M is CS, then M is a direct sum of uniform modules. However, while it is known that every countably Σ-injective module is already Σ-injective, a countably Σ-CS-module need not even have an indecomposable decomposition [4, 12.19]. A natural problem is then to find out if there exists a cardinal $\aleph$ such that each $\aleph$-Σ-CS module M (i.e., each M such that every direct sum of copies of M indexed by a set of cardinality $\aleph$ is CS) has an indecomposable decomposition. This problem was studied in [7], where it was shown that every quasi-continuous $\aleph_1$-Σ-CS module is a direct sum of uniform modules. But this response involves the quasi-continuity of the module as an additional condition and so in the same paper it was asked (cf. [7, Remark 2.9]) whether every $\aleph_1$-Σ-CS-module is already a direct sum of uniforms.

The main result of this paper (Theorem 2.6) provides an affirmative answer to this question. Not surprisingly, bearing in mind the fact that cardinal numbers play an important role in this result, the proof relies on infinitary counting arguments based on set-theoretic results introduced by Tarski in the 1920s that were also used by Osofsky in module theory (cf. [10]). Since the analogous result for the Σ-CS case [6] has a much stronger hypothesis and does not depend on cardinality assertions, it seemed reasonable to expect that it should have a proof not requiring counting arguments. On the way to our main result, we show, in Corollary 2.4, that this is indeed the case by proving the existence of indecomposable

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decompositions for Σ-CS-modules in a much more direct and simpler way, using only module-theoretic methods.

Throughout this paper all rings R will be associative and with identity, and Mod-R will denote the category of right R-modules. By a module we will usually mean a right R-module. We refer to [2,11] for all undefined notions used in the text.

2. Results

Recall that a submodule K of an R-module M is said to be closed (in M) when K has no proper essential extensions in M. If $L \subseteq M$, then a closed submodule K of M that contains L as an essential submodule (we then write $L \subseteq_e K$) is called an essential closure of L in M. The module M is called CS (or an extending module, cf. [4]), if every closed submodule is a direct summand. An (internal) direct sum $\bigoplus_I L_i$ of submodules of a module M is called a local direct summand of M if $\bigoplus_{i\in F} L_i$ is a direct summand of M for every finite subset $F \subseteq J$. If, furthermore, $\bigoplus_{i\in I} L_i$ is a direct summand of M, then we will also say that the local direct summand $\bigoplus_i L_i$ is a summand of M.

Recall also that if M is a module, $\sigma[M]$ is defined as the full subcategory of Mod-R whose objects are all the submodules of M-generated modules [11]. $\sigma[M]$ is a Grothendieck category and hence it has injective hulls. The injective objects of $\sigma[M]$ are just the M-injective modules. M is called quasi-injective when it is injective in $\sigma[M]$ and Σ-quasi-injective when every direct sum of copies of M is quasi-injective. The quasi-injective hull of M is precisely the injective hull of M in $\sigma[M]$. We will denote by $|X|$ the cardinality of a set X.

We begin with a technical lemma which will be very useful later on.

Lemma 2.1. *Let M be a CS-module and $p : M \to N$ be an epimorphism. If there exists a submodule $X \subseteq M$ such that $X \cap \operatorname{Ker} p = 0$ and $p(X) \subseteq_e N$, then $\operatorname{Ker} p$ is a direct summand of M.*

Proof. Let $K = \operatorname{Ker} p$ and L an essential closure of K within M, which is a direct summand of M because M is CS. Since $X \cap K = 0$ by hypothesis and K is essential in L, we also have that $X \cap L = 0$. It is then easily checked that $p(X) \cap p(L) = 0$ and, since $p(X)$ is essential in N, it follows that $p(L) = 0$ and hence $L \subseteq K$. Therefore $K = L$ is a direct summand of M. □

By [4, 2.4, 8.2], if M is a CS-module whose quasi-injective hull is Σ-quasi-injective, then M has an indecomposable decomposition. Moreover, a Σ-quasi-injective module is a direct sum of indecomposable quasi-injective modules, but the converse is not true (see B.L. Osofsky's example for a non-artinian commutative self-injective local ring in [5, 24.34]). However, we have:

Lemma 2.2. *If M is a CS-module whose quasi-injective hull is a direct sum of uniform modules, then M itself is a direct sum of uniform modules.*

Proof. Let M be CS and Q the quasi-injective hull of M. Suppose that Q is a direct sum of uniforms but M is not. Then, by [7, Lemma 2.6] there exists an essential local direct summand $\bigoplus_{\mathbb{N}} M_n$ of M and an element $x \in M$ such that $xR \cap (\bigoplus_{\mathbb{N}} M_n) \nsubseteq \bigoplus_F M_n$ for any finite subset $F \subseteq \mathbb{N}$. Now, for each $n \in \mathbb{N}$, let Q_n be an essential closure of M_n in Q. Then $\bigoplus_{\mathbb{N}} Q_n$ is a local direct summand of Q and, in fact, a direct summand of Q by [8, Theorem 2.22]. Since

$$\bigoplus_{\mathbb{N}} M_n \subseteq_e M \subseteq_e Q$$

we see that $Q = \bigoplus_{\mathbb{N}} Q_n$. Thus there exists a finite set $F \subseteq \mathbb{N}$ such that $x \in \bigoplus_F Q_n$ and so

$$xR \cap \left(\bigoplus_{\mathbb{N}} M_n\right) \subseteq xR \cap \left(\bigoplus_{\mathbb{N}} Q_n\right) \cap M \subseteq \left(\bigoplus_F Q_n\right) \cap M = \bigoplus_F M_n,$$

which is a contradiction and completes the proof. □

The next result is a module-theoretic version of Oshiro's [9] characterization of right Σ-CS rings as the rings R such that the class of projective right R-modules is closed under essential extensions, cf. also [4, Corollary 11.11]. Recall that, for a module N, $\operatorname{Add} N$ denotes the class of all the modules isomorphic to direct summands of direct sums of copies of N.

Theorem 2.3. *Let M be a right R-module. Then the following conditions are equivalent*:

(i) *M is Σ-CS.*
(ii) *M is CS and $\operatorname{Add} N$ is closed under N-generated essential extensions, for each direct summand N of M.*

Proof. Suppose first that (i) holds. Since the class of Σ-CS-modules is closed under direct summands, we can take $N = M$ and suppose that X belongs to $\operatorname{Add} M$ and Y is an M-generated essential extension of X. We then have an epimorphism $p : M^{(I)} \to Y$ for some set I and, since $X \in \operatorname{Add} M$, we can assume that the canonical inclusion j of X into Y factors through p, i.e., there exists $q : X \to M^{(I)}$ such that $p \circ q = j$. Therefore, $p(q(X))) = \operatorname{Im}(p \circ q)$ is an essential submodule of Y and, since $q(x) \cap \operatorname{Ker} p = 0$, it follows from Lemma 2.1 that $\operatorname{Ker} p$ is a direct summand of $M^{(I)}$. This shows that Y belongs to $\operatorname{Add} M$.

For the converse, suppose now that M is CS and, for each direct summand N of M, $\operatorname{Add} N$ is closed under N-generated essential extensions. If I is a set and Q denotes the quasi-injective hull of $M^{(I)}$ then, because Q is an M-generated essential extension of $M^{(I)}$, we have that Q belongs to $\operatorname{Add} M$. Thus, by Kaplansky's theorem [2, 26.1], Q is a direct sum of c-generated modules, where $c = \max(\aleph_0, |M|)$ and, using [4, 2.4], we see that Q is in fact a Σ-M-injective module (and a Σ-quasi-injective module). Moreover, it follows from Lemma 2.2, that M is a direct sum of uniform modules, say $M = \bigoplus_i M_i$. For each $i \in I$, let $\widetilde{M}_i$ be the quasi-injective hull of M_i. Then, our hypothesis implies that

$\widetilde{M}_i \in \operatorname{Add} M_i$ and, since $\widetilde{M}_i$ has the exchange property, we have that $\widetilde{M}_i \cong M_i$, so that each M_i is a quasi-injective module. Then it follows from [3, Theorem 3.3] that M is a Σ-CS-module. □

As a consequence of the preceding results we obtain a module-theoretic proof of the existence of indecomposable decompositions for Σ-CS-modules.

Corollary 2.4. *Every Σ-CS-module is a direct sum of uniform modules.*

Proof. It follows from the proof of Theorem 2.3 that the quasi-injective hull of every Σ-CS-module is Σ-quasi-injective, and hence it has an indecomposable decomposition. Then the result follows from Lemma 2.2. □

We are now going to improve this result by showing that, as in the injective case, there exists a fixed cardinal such that if the direct sums of copies of a module indexed by this cardinal are CS, then the module has an indecomposable decomposition. First, we give a useful lemma, which is very likely known, but whose proof we include for completeness.

Lemma 2.5. *Let $M \subseteq_e Q$ and X a closed submodule of Q. Then $X \cap M$ is closed in M.*

Proof. By [4, 1.10], it is enough to show that if $X \cap M \subseteq Z \subseteq_e M$, then $Z/(X \cap M) \subseteq_e M/(X \cap M)$. To prove this, let $m \in M$ such that $m \notin X$; we must show that there exists $r \in R$ such that $rm \in Z$ but $rm \notin X \cap M$. Since $Z \subseteq_e M \subseteq_e Q$, we have that $Z \subseteq_e Q$ and so $X + Z \subseteq_e Q$. Since $X \subseteq X + Z \subseteq_e Q$ and X is closed in Q we have, again by [4, 1.10], that $(X + Z)/X \subseteq_e Q/X$ and so there exists $r \in R$ such that $rm \in X + Z$ but $rm \notin X$. Then we see that $rm \in M \cap (X + Z) =$ (by modularity) $(M \cap X) + Z = Z$. However, $rm \notin X$ and hence $rm \notin X \cap M$, completing the proof. □

We are now ready to give our main result.

Theorem 2.6. *Every $\aleph_1$-Σ-CS-module is a direct sum of uniform modules.*

Proof. Using Lemma 2.2, it is enough to show that the quasi-injective hull Q of M has an indecomposable decomposition. Suppose, on the contrary, that this is not the case. Then, by [8, Theorem 2.22], there exists a local direct summand $\bigoplus_I Q_i$ in Q which is not a direct summand. In particular, $\bigoplus_I Q_i$ is not M-injective. By Baer's criterion (cf. [11, 16.3]), there exists a cyclic submodule $mR \subseteq M$ and a submodule $X \subseteq mR$, together with a homomorphism $t : X \to \bigoplus_I Q_i$ which does not have an extension to mR. Since each Q_i is M-injective, so is each sum $\bigoplus_F Q_i$, with F a finite subset of I, and so we have that $\operatorname{Im} t \nsubseteq \bigoplus_F Q_i$ for every finite $F \subseteq I$. If we denote by $p_i : \bigoplus_I Q_i \to Q_i$ the canonical projection, this implies that there exists a countable infinite set $J \subseteq I$ such that $p_j \circ t \neq 0$ for each $j \in J$.

Let now $\pi : \bigoplus_I Q_i \to \bigoplus_J Q_j$ be the canonical projection and set $g = \pi \circ t$. Let Q' be an M-injective hull of $\bigoplus_J Q_j$ contained in Q. By the M-injectivity of Q', there exists $h : mR \to Q'$ extending g. Let $x = h(m)$ and choose $x_j = mr_j \in X$ such that $p_j \circ t(x_j) \neq 0$. Then we see that $h(mr_j) = xr_j \in \operatorname{Im} g$ and $p_j(xr_j) = p_j \circ t(x_j) \neq 0$.

Now, since J is countable, we have by [6, Lemma 2.1] that there exist subsets $\mathcal{A}, \mathcal{K} \subseteq 2^J$ such that:

(i) $\mathcal{A}$ is a partition of J with $|\mathcal{A}| = \aleph_0$ and $|A| = \aleph_0$ for every $A \in \mathcal{A}$.
(ii) $\mathcal{A} \subseteq \mathcal{K}$, $|\mathcal{K}| = \aleph_1$ and $|K| = \aleph_0$ for each $K \in \mathcal{K}$.
(iii) $K \cap K'$ is a finite set for all $K, K' \in \mathcal{K}$ such that $K \neq K'$.

Consider, for each $K \in \mathcal{K}$, M-injective hulls in Q' of $\bigoplus_K Q_j$ and $\bigoplus_{J-K} Q_j$, respectively, say, Q_K, and Q'_K, so that, as $\bigoplus_J Q_j$ is essential in Q', we have that $Q' = Q_K \oplus Q'_K$. Now, let $e_K \in \operatorname{End}(Q')$ be the idempotent corresponding to Q_K under this decomposition, so that $Q_K = e_K Q'$ and $Q'_K = (1 - e_K)Q'$. Then we have that $e_K|_{Q_j} = 1_{Q_j}$ if $j \in K$ and $e_K|_{Q_j} = 0$ if $j \notin K$. Set $x_K = e_K(x)$ and $Y_K = Q_K \cap M$; observe that $Y_K \subseteq_e Q_K$ as $M \subseteq_e Q$. Since Q_K is an M-injective hull of $\bigoplus_K Q_j$ in Q' and hence in Q, it is a direct summand of Q and so, as $M \subseteq_e Q$, Y_K is a closed submodule of M by Lemma 2.5. Then, since M is CS, we see that Y_K is, in fact, a direct summand of M.

We know that Q_K is an M-generated module (see [11, 16.3]) and so, if we consider the countable subset of Q_K:

$$\Delta_K := \left\{ p_i(xr_j) \;\middle|\; i \in K,\ j \in J \right\} \cup \{x_K\}$$

there exists a countable set Ω_K and a homomorphism:

$$\pi_K : M^{(\Omega_K)} \to Q_K$$

such that $\Delta_K \subseteq \operatorname{Im} \pi_K$. Consider now the morphism $q_K : Y_K \oplus M^{(\Omega_K)} \to Q_K$ induced by π_K and the inclusion of Y_K in Q_K. Since Y_K is a direct summand of M and Ω_K is countable, $Y_K \oplus M^{(\Omega_K)}$ is a direct summand of $M^{(\aleph_0)}$ and hence a CS-module. Since $q_K(Y_K) = Y_K \subseteq_e Q_K$ and $Y_K \cap \operatorname{Ker} q_K = 0$, $\operatorname{Ker} q_K$ is a direct summand of $Y_K \oplus M^{(\Omega_K)}$ by Lemma 2.1. Hence $\operatorname{Im} q_K$ is isomorphic to a direct summand of $Y_K \oplus M^{(\Omega_K)}$. Call $M_K = \operatorname{Im} q_K \subseteq Q_K = e_K Q'$.

Let now $f : \bigoplus_{\mathcal{K}} M_K \to \sum_{\mathcal{K}} M_K \subseteq \sum_{\mathcal{K}} Q_K \subseteq Q'$ be the epimorphism induced by the inclusions of the M_K in Q'. Let $N = \sum_{\mathcal{A}} M_A = \bigoplus_{\mathcal{A}} M_A \subseteq \sum_{\mathcal{K}} M_K$, where the equality follows from the fact that $\mathcal{A}$ is a partition. Then it is clear that $N \cap \operatorname{Ker} f = 0$. Moreover, since $\bigoplus_J Qj \subseteq_e Q'$ and $\bigoplus_{\mathcal{A}} Q_A$ contains $\bigoplus_J Qj$, we have that $N = \bigoplus_{\mathcal{A}} M_A \subseteq_e \bigoplus_{\mathcal{A}} Q_A \subseteq_e Q'$. Thus we see that $N \subseteq_e \operatorname{Im} f$. Now, for each $K \in \mathcal{K}$, M_K is a direct summand of $M^{(\aleph_0)}$ and $|\mathcal{K}| = \aleph_1$, hence $\bigoplus_{\mathcal{K}} M_K$ is a direct summand of $M^{(\aleph_0)^{(\aleph_1)}} = M^{(\aleph_1)}$. Since M is $\aleph_1$-Σ-CS, so is $\bigoplus_{\mathcal{K}} M_K$, and hence it follows from Lemma 2.1 that $f : \bigoplus_{\mathcal{K}} M_K \to \sum_{\mathcal{K}} M_K$ is a split epimorphism. Let $\varepsilon : \sum_{\mathcal{K}} M_K \to \bigoplus_{\mathcal{K}} M_K$ be such that $f \circ \varepsilon = 1_{\sum_{\mathcal{K}} M_K}$.

Since $\mathcal{A}$ is a partition of J, we know that, for each $i \in J$, there exists $A_i \in \mathcal{A}$ such that $i \in A_i$. If we call $e_i : \bigoplus_J Q_j \to \bigoplus_J Q_j$ the morphism induced by the ith projection then, by construction, $e_i(xr_j) \in M_{A_i}$ for each $j \in J$. In particular, the countable set $\{e_i(xr_j)\}_{i,j\in J}$ is a subset of $\sum_{\mathcal{A}} M_A \subseteq \sum_{\mathcal{K}} M_K$. Let then Z be the submodule of $\sum_{\mathcal{K}} M_K$ generated by this set. Since Z is countably generated, there exists a countable subset $\mathcal{J} \subseteq \mathcal{K}$ such that $\varepsilon(Z) \subseteq \bigoplus_{\mathcal{J}} M_K$. But $\mathcal{K}$ is uncountable and so there exists $K_0 \in \mathcal{K} - \mathcal{J}$. Choose any $i \in K_0$. Since $xr_i \in \bigoplus_J Q_j$ and $e_{K_0}|_{Q_j} = 1_{Q_j}$, for $j \in K_0$, while $e_{K_0}|_{Q_j} = 0$ if $j \notin K_0$, we have that $x_{K_0}r_i = e_{K_0}(xr_i) = \sum_{j\in K_0} e_j(xr_i)$. Now, $\{e_j\}_{j\in J}$ is a set of orthogonal idempotents of $\mathrm{End}(\bigoplus_J Q_j)$ and so $e_i(x_{K_0}r_i) = e_i(\sum_{j\in K_0} e_j(xr_i)) = e_i(xr_i) \neq 0$, because $e_j(xr_j) \neq 0$, by construction, for each $j \in J$.

Let now $\alpha : \bigoplus_{\mathcal{K}} M_K \to \bigoplus_{\mathcal{J}} M_K$ and $\beta : \bigoplus_{\mathcal{J}} M_K \to \bigoplus_{\mathcal{K}} M_K$ be the canonical projection, and injection, respectively, and consider the homomorphism:

$$\zeta = f \circ \beta \circ \alpha \circ \varepsilon : \sum_{\mathcal{K}} M_K \to \sum_{\mathcal{K}} M_K.$$

Observe that, since $f \circ \varepsilon = 1$, $\beta \circ \alpha|_{\bigoplus_{\mathcal{J}} M_K}$ is the inclusion, and $\varepsilon(Z) \subseteq \sum_{\mathcal{J}} M_K$, we have that $\zeta|_Z$ is the canonical inclusion of Z in $\sum_{\mathcal{K}} M_K$. On the other hand, $\mathrm{Im}\,\zeta \subseteq \sum_{\mathcal{J}} M_K$ and so $\zeta(x_{K_0}) \subseteq \sum_{\mathcal{J}} M_K$. Thus there exists a finite set $\{K_1, \ldots, K_n\} \subseteq \mathcal{J}$ such that $\zeta(x_{K_0}) \subseteq M_{K_1} + \cdots + M_{K_n}$. Let now $F = K_0 \cap (K_1 \cup \cdots \cup K_n)$. Then F is finite because $K_0 \notin \mathcal{J}$ and so $K_0 \neq K_1, \ldots, K_n$. Since K_0 is an infinite set, there exists $j_0 \in K_0 - F$. Now, as we have seen, $x_{K_0}r_{j_o} = \sum_{j\in K_0} e_j(xr_{j_0}) \in Z$. Therefore, as the restriction of ζ to Z is the inclusion, we have:

$$x_{K_0}r_{j_0} = \zeta(x_{K_0}r_{j_0}) \in M_{K_0} \cap (M_{K_1} + \cdots + M_{K_n}) \subseteq Q_{K_0} \cap (Q_{K_1} + \cdots + Q_{K_n}).$$

Using now [6, Lemma 2.1], we have $Q_{K_0} \cap (Q_{K_1} + \cdots + Q_{K_n}) = \bigoplus_F Q_i$, where F is finite, as we have shown before. Therefore, $e_{j_0}(x_{K_0}r_{j_0}) \in e_{j_0}(\bigoplus_F Q_j) = 0$, since $j_0 \notin F$. But this is a contradiction because we have shown that $e_i(x_{K_0}r_i) \neq 0$ for each $i \in K_0$. This contradiction completes the proof. □

We do not now whether every $\aleph_1$-Σ-CS-module is a Σ-CS-module, although the preceding proof underscores the differences between both concepts and suggests that maybe this is not the case. The following result, however, exhibits another property of Σ-CS-modules which is also enjoyed by $\aleph_1$-Σ-CS-modules.

Corollary 2.7. *The quasi-injective hull of an $\aleph_1$-Σ-CS-module is a Σ-quasi-injective module.*

Proof. Let M be an $\aleph_1$-Σ-CS-module and Q its quasi-injective hull. Let Q' be a quasi-injective hull of $Q^{(\aleph_0)}$. Then Q' is also the quasi-injective hull of the $\aleph_1$-Σ-CS-module $M^{(\aleph_0)}$ and so Q' has an indecomposable decomposition by Theorem 2.6. By [8, Theorem 2.22], every local direct summand of Q' is a direct summand and so $Q^{(\aleph_0)} = Q'$ is a quasi-injective module. □

Finally, we give a necessary and sufficient condition for an $\aleph_1$-Σ-CS-module to be Σ-CS.

Corollary 2.8. *Let M be a right R-module. Then the following conditions are equivalent*:

(i) *M is a Σ-CS-module.*
(ii) *M is an $\aleph_1$-Σ-CS-module such that every uniform direct summand of M has local endomorphism ring.*

Proof. By [1, Proposition 2.3] a uniform Σ-CS-module is quasi-injective and so (i) implies (ii). The converse follows from Theorem 2.6 using [3, Proposition 3.1]. □

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