A modified Navier-Stokes equation for incompressible fluid flow

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Abstract

In the actual flow process, due to the unsteady feature of the flow, the rotation of fluid element and variation in the changes of shear rate and other factors, thus the shear force impacted on the fluid should not just follow the Newton’s law of viscosity. Based on this, the additional viscous forces related with variation in time and the history effects, the additional part with spatial variation and the relevant part due to the rotational motion have been proposed. An analogy method has been used to derive the shear rate related forces. This paper presents the corresponding expressions for the four additional forces. The action characteristic of the four additional part have been analyzed, the former two parts show influence on the unsteady movement of fluid element and the other two parts have impact on the lateral motion of the surrounding fluid. On the basis of the above considerations, a modified Navier-Stokes equation containing the additional force related with velocity has been given. The magnitude of the additional forces has been compared, the impact of which on the flow process has been analyzed.

1. Introduction

The turbulent flow is one of the typical process in the study of fluid mechanics [1]. It has many applications in the aerospace engineering, the development of new energy, oceanology and atmospheric sciences and other fields [2]. Because of its complex flow process and mechanism, turbulence is considered to be one of the classical physics problem very difficult to solve [3]. The distinct patterns and processes in fluid movement are controlled by motion equations of fluid. In the classical equations for motion of fluid, the viscous forces fluid suffered origins from Newton's law of viscosity and there exists a linear relationship between shear stress and shear rate. Generally, there

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are two approaches to modify the governing equations describing fluid motion. First, the fluid velocity can be decomposed into the average velocity and the fluctuating component, the two of which satisfy the respective movement equations respectively [4]. The second is to construct stress-strain constitutive relation close to the reality, such as the establishment of a nonlinear model [5] or determination of the additional viscosity coefficients [6].

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>ρ</td>
<td>density of the fluid</td>
</tr>
<tr>
<td>p</td>
<td>pressure of the fluid</td>
</tr>
<tr>
<td>u</td>
<td>velocity in the x direction</td>
</tr>
<tr>
<td>v</td>
<td>velocity in the y direction</td>
</tr>
<tr>
<td>w</td>
<td>velocity in the z direction</td>
</tr>
<tr>
<td>f_u^A</td>
<td>additional unsteady force with u</td>
</tr>
<tr>
<td>f_u^H</td>
<td>additional history force with u</td>
</tr>
<tr>
<td>f_u^R</td>
<td>additional rotational force with u</td>
</tr>
<tr>
<td>f_u^S</td>
<td>additional gradient force</td>
</tr>
<tr>
<td>ν</td>
<td>kinematic viscosity coefficient</td>
</tr>
<tr>
<td>ω_x</td>
<td>vorticity in the x direction</td>
</tr>
</tbody>
</table>

In the actual flow process, due to the instantaneous unsteady flow process and continuously unsteady characteristics, the variation of the velocity shear regularly accompanied by large or small vortices, fluid elements with strong vorticity and changes in shear strain rate, fluid element will suffer additional forces from the surrounding fluid, not just the shear force which follows the linear law of Newtonian shear viscosity. Based on the above considerations, the four additional forces in the movement process of the fluid are proposed and a modified N-S equation has been given.

2. The additional forces related with velocity

2.1. Additional unsteady force

In the shifting process of fluid element movement, it will bring along the surrounding fluid with non-constant velocity [7], the portion of the fluid acts on the reference fluid reversely and will generate a time-varying part of the force. It is analogous to the additional mass force acting on a particle. Although the force on the particle can be considered as derived based on Lagrange system, it is the same when the particle is fixed. On the other hand, the viscous force in the N-S equation is generally obtained in Euler system, it is equivalent to that in Lagrange frame, therefore the following analogies are reasonable and can be performed. Consider a local fluid element with typically spherical configuration, which exhibits isotropic characteristics (for cylinder it is anisotropy), the time related additional force can be expressed as

\[
f_u^A = \frac{1}{2} \frac{\partial (u - \bar{u})}{\partial t}
\]

where \( \bar{u} \) is the space averaged velocity locally.

2.2. Additional history force

When a fluid element experiences shifting translation in the viscous fluid, the shear layer in the vicinity of the element will drive a portion of the surrounding fluids, due to the fluid inertia, when the reference fluid experienced acceleration, they can not be immediately accelerated, when the fluid deceleration, they can not slow down immediately. Thus, due to the instability of the shear layer in surrounding fluid, the fluid element is subjected to a
time varying force, which is related with the acceleration process of the fluid. In other words, taking into account that there exists a certain delay and the cumulative effect in the movement development and the variation of the force, the migrated fluid is subjected to a "history" force associated with the viscosity. The "history" force acting on the fluid can be obtained as

\[ f_u^H = 9 \sqrt{\frac{\nu}{\pi \tau}} \int_{t-\tau}^{t} \frac{\partial}{\partial \tau} \left( \frac{\partial u}{\partial y} \right) d\tau + 9 \sqrt{\frac{\nu}{\pi \tau}} \int_{t-\tau}^{t} \frac{\partial}{\partial \tau} \left( \frac{\partial u}{\partial z} \right) d\tau + 9 \sqrt{\frac{\nu}{\pi \tau}} \int_{t-\tau}^{t} \frac{\partial}{\partial \tau} \left( \frac{\partial u}{\partial x} \right) d\tau \]  

(2)

The history effect is similar with the Basset force [8]. The unsteady hydrodynamic force is caused by delayed development of the interface shear layer which is due to the relative velocity variation with time. The magnitude of the force has direct relationship with the movement history of the local fluid. Therefore, the force can be also called as “history” force. The direction of the force is opposite to that for the increase of the shearing force. The third term in Equation (2) can be considered as the additional pressure gradient.

2.3. Additional rotational force

The transverse velocity gradient in the fluid accompanied by different relative velocity in both sides of the fluid element, which may cause the rotational motion of the element. At moderate relative Reynolds numbers, the rotation drives the motion of the surrounding fluid. The fluid around the side with relative larger velocity accelerates and the pressure decreases. While the fluid velocity on the other side decelerates and the pressure increases, thus resulting in the movement of the fluid element, which is similar to Magnus effect [9]. The force causing the lateral movement of the fluid is called additional rotational force \( f_u^R \).

Take one direction for example, for the fluid element with velocity \( u \), the lift force exerted by the surrounding fluid with rotational movement can be calculated by

\[ f_u^R = \frac{3}{4} \left( \nabla \times \nabla \right) \left( u - \bar{u} \right) = \frac{3}{4} \omega_s \left( u - \bar{u} \right) \]  

(3)

The force acts in the direction perpendicular to the rotational axis of the fluid element.

2.4. Additional gradient force

For fluid in the flow field with shear variation of velocity gradient, even if the fluid element is not rotated, it will suffer the lift force perpendicular to the flow direction. The additional force induced by the shear effect of the velocity gradient can be expressed as

\[ f_u^S = 9.72 \sqrt{\frac{\nu}{\pi}} \left( \frac{\partial u}{\partial y} \right) \sqrt{\frac{\partial u}{\partial y}} + 9.72 \sqrt{\frac{\nu}{\pi}} \left( \frac{\partial u}{\partial z} \right) \sqrt{\frac{\partial u}{\partial z}} + 9.72 \sqrt{\frac{\nu}{\pi}} \left( \frac{\partial u}{\partial x} \right) \sqrt{\frac{\partial u}{\partial x}} \]  

(4)

The shear variation in the velocity gradient plays a role of lift on the fluid, which has similarities with the lift caused by the shear variation of the velocity [10]. Also, the third term in Equation (4) is equivalent to the impact of the additional pressure gradient.

The four additional forces can be taken into the right side of the fluid movement equation, thus the modified Navier-Stokes equation is expressed as follow

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{1}{\rho} \frac{\partial P}{\partial x} + f_u^A + f_u^B + f_u^R + f_u^S \]  

(5)

3. Discussion

Generally, the forces due to the time and space variations described in the above section act with different order of magnitude. Under certain conditions, some force with complicated expression can be ignored, the theoretical
analysis and solving of the fluid movement equations can be performed conveniently. On the other hand, the actual flow process usually exhibits strong unsteady and nonuniform features. The several additional forces need to be considered and the comparison with the viscous force is meaningful.

The ratio of the additional unsteady force and the viscous force is calculated by

\[ \frac{f^A_u}{f^T_u} = \frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial y} \right) \frac{\Delta y}{2} \tag{6} \]

where \( \Delta y \) is the characteristic length of the variation in fluid movement, when \( f^A_u / f^T_u \geq 0.1 \), additional unsteady force has the same magnitude with the viscous force. The kinematic viscosity coefficient of water at room temperature \( \nu = 1 \times 10^{-6} \text{m}^2 / \text{s} \), if the characteristic length is taken as 0.2mm, the time rate of the velocity gradient should satisfy \( \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial y} \right) > 2.5 \text{s}^{-1} \frac{\partial u}{\partial y} \). Under such condition, the influence of the additional unsteady force needs to be considered.

The ratio of the additional history force and the viscous force is

\[ \frac{f^H_u}{f^T_u} = \frac{9}{\nu \partial} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) \int_{t_0}^{t} \frac{\partial}{\partial \tau} \left( \frac{\partial u}{\partial y} \right) \sqrt{t - \tau} \, d\tau = \frac{9}{\sqrt{\nu \pi}} \frac{\Delta y}{\sqrt{1 - t_0}} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) \int_{t_0}^{t} \frac{\partial}{\partial \tau} \left( \frac{\partial u}{\partial y} \right) \sqrt{1 - \tau} \, d\tau \tag{7} \]

For \( \frac{f^H_u}{f^T_u} > 1 \), if the characteristic length is taken as 0.2mm, then the time \( t < 0.1 \text{s} \). If the characteristic velocity for fluid flow is taken as \( 2 \text{m/s} \), then within a period \( t = 0.1 \text{s} \), the characteristic distance of fluid movement is \( 0.2 \text{m} \), therefore the flow is generally less affected by the additional history force.

The ratio of the additional rotational force and the viscous force

\[ \frac{f^R_u}{f^T_u} = \frac{3}{4\nu} \left( \nabla \times \mathbf{V} \right) \left( \Delta y \right)^2 \tag{8} \]

When \( f^R_u / f^T_u \geq 0.1 \), the additional history force has the same magnitude with the viscous force. if the characteristic length is 0.2mm with the kinematic viscosity coefficient of water \( \nu = 1 \times 10^{-6} \text{m}^2 \text{s}^{-1} \), the vorticity of the fluid element should satisfy \( \omega > 3.33 \text{rad s}^{-1} \).

For the gradient additional force, the ratio

\[ \frac{f^S_u}{f^T_u} = \frac{9.72}{\nu} \frac{\sqrt{\nu}}{\pi} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial u}{\partial y} \right) \sqrt{\frac{\partial u}{\partial y}} \tag{9} \]

When \( f^S_u / f^T_u = 1 \), the relation between the velocity gradient and the shear rate of the velocity gradient should
be expressed as \[ \frac{\partial u}{\partial y} = 4.7 \times 10^{-3} \text{ms}^{-1} \left( \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) \right). \] The additional gradient force is comparable with the viscous force and it will drive the fluid to move towards the region with larger velocity gradient.

4. Conclusion

Based on the unsteady feature of the flow field, the rotational movement of fluid element, the shear variation in the velocity gradient, the viewpoint that the fluid element will be subject to several additional forces in the flow process. The expression of the four additional forces have been given and the role they played on the flow has been analyzed. The magnitude of the additional forces and viscous forces have been compared through preliminary analysis. The condition under which the impact of the additional forces should be considered is cleared. The relevant additional forces have been added into the movement equations of the fluid, the improved N-S equations have been obtained. The improved equation can provide assistance and inspiration for the study of turbulence. In the future study, quantitative verification with numerical simulations and experiments will be performed.

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