On extension of isometries between unit spheres of $L^\infty(\Gamma)$-type space and a Banach space $E$

Rui Liu

School of Math. Sci. and LPMC, Nankai University, Tianjin 300071, China

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Abstract

In this paper, we study the isometric extension problem between the unit spheres of $L^\infty(\Gamma)$-type space and a Banach space $E$. We prove that every surjective isometry between the unit spheres $S(L^\infty(\Gamma))$ and $S(E)$ can be extended to be a linear isometry of $L^\infty(\Gamma)$ onto $E$. Moreover, we study this problem between $C(\Omega)$ and a Banach space and also get an affirmative answer.

Keywords: AM-space; Isometric extension; $L^\infty(\Gamma)$-type space; Unit sphere

1. Introduction

Let $E$ and $F$ be two Banach spaces, and let $M$ be a subset of $E$. A mapping $V_0 : M \to F$ is called an isometry if $\|V_0(x_1) - V_0(x_2)\| = \|x_1 - x_2\|$ for all $x_1, x_2 \in M$. The classical Mazur–Ulam theorem [17] describes the relation between isometry and linearity, and states that every surjective isometry $V$ between normed spaces is a linear mapping up to translation. So far, A. Vogt [27], Th.M. Rassias [19–21] and D. Tingly [26] have generalized the well-known theorem in several directions (e.g., see [2,18,22–25]), and one of them is the study of the isometric extension problem.

Traced back to 1972, P. Mankiewicz [16] proved that an isometry from an open connected subset of a normed space $E$ onto an open subset of another normed space $F$ can be extended

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E-mail address: leoru816@yahoo.com.cn.

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to be an affine isometry from \( E \) onto \( F \). In 1987, D. Tingly [26] raised the isometric extension problem: \( \text{“Let } E \text{ and } F \text{ be real Banach spaces with unit spheres } S(E) \text{ and } S(F), \text{ respectively. Suppose } V_0 : S(E) \rightarrow S(F) \text{ is a surjective isometry. Is } V_0 \text{ necessarily the restriction of a linear or affine transformation on } E?\” \) He showed that if \( E \) and \( F \) are finite-dimensional Banach spaces, then \( V_0(x) = -V_0(x) \) for all \( x \in S(E) \). And the answer to this problem is clearly negative in the complex spaces (for example, we take \( E = F = \mathbb{C} \) (complex plane) and define \( V_0(x) = \bar{x} \)), so we always consider the real case. In recent years, G. Ding [3–7] and his students such as G. An [1], R. Liu [11–15] and R. Wang [29–33] keep on working on this topic and have obtained many important results (e.g., see [8] and its references). In [28], J. Wang studied the isometries between the unit spheres of \( \ell^0_n \) and \( \ell^p_n \) \( (0 < \beta_n < 1, n \in \mathbb{N} ) \) type spaces. Moreover, in [34], X. Yang proved that any isometry between the unit spheres of \( L^p(\mu) \) and \( L^p(\nu, H) \) \( (1 < p \neq 2, H \text{ is a Hilbert space}) \) can be extended to be a linear isometry on the whole space.

Recently, G. Ding [3] obtained some conditions under which an isometry between the unit spheres of \( \mathcal{C}(\Omega) \) and a Banach space \( E \) can be linearly extended. These conditions are as follows:

(i) For any \( x_1, x_2 \in S(\mathcal{L}^\infty(\Gamma)) \) and \( \lambda_1, \lambda_2 \in \mathbb{R} \), if \( \| \lambda_1 V_0(x_1) + \lambda_2 V_0(x_2) \| = 1 \), then \( \lambda_1 V_0(x_1) + \lambda_2 V_0(x_2) \in S(\mathcal{L}^\infty(\Gamma)). \)

(ii) For any finitely many mutual disjoint subsets \( \{ \Gamma_k \}_{k=1}^n \), real scalars \( \{ \lambda_k \}_{k=1}^n \) and \( x \in S(\mathcal{L}^\infty(\Gamma)) \) such that \( V_0(x) = \sum_{k=1}^n \lambda_k V_0(\chi_{\Gamma_k}) \), there exist scalars \( \{ \lambda'_k \}_{k=1}^n \) and \( x_0 \in \mathcal{L}^\infty(\Gamma) \) satisfying that \( x = \sum_{k=1}^n \lambda'_k \chi_{\Gamma_k} + x_0 \) and \( \supp(x_0) \subset (\bigcup_{k=1}^n \Gamma_k)^C \).

G. Ding showed that every surjective isometry between \( S(\mathcal{L}^\infty(\Gamma)) \) and \( S(E) \) can be extended to be a linear isometry defined on the whole space if and only if the above (ii) holds. However, the condition (ii) is too formal. Recently, G. Ding conjectured that it could be omitted.

In this paper, we shall study the isometric extension problem for general Banach spaces and give some sufficient conditions under which an isometry between unit spheres can be linearly extended, then prove that: every surjective isometry from the unit sphere of \( \mathcal{L}^\infty(\Gamma) \) (or \( C(\Omega) \)) onto \( S(E) \) can be linearly extended to the whole space. Therefore, the above conjecture is true, and we generalize the corresponding results of [3,5,9].

Throughout this paper, we consider the spaces all over the real field. Assume \( E \) and \( F \) are real Banach spaces and \( V_0 : S(E) \rightarrow S(F) \) is an isometry, the unit sphere \( S(E) = \{ x : x \in E, \| x \| = 1 \} \). Let \( x \in S(E) \), the star of \( x \) with respect to \( S(E) \), \( \text{Str}(x) \), is defined by

\[
\text{Str}(x) = \{ y : y \in S(E), \| y + x \| = 2 \}.
\]

Let \( M \) be a subset of \( S(E) \), then \( V_0(M) = \{ V_0(x) \in S(F) : x \in M \} \). If \( K \) is a convex subset of \( S(E) \), we say \( K \) is a maximal convex subset of \( S(E) \) if it is not properly contained in any other convex subset of \( S(E) \) (cf. [26]).

2. Some lemmas

**Lemma 1.** Let \( E \) and \( F \) be two Banach spaces, and let \( V_0 : S(E) \rightarrow S(F) \) be an isometry such that \( -V_0(S(E)) \subseteq V_0(S(E)) \). Then for any \( x, y \in S(E) \) we have

(i) \( \| V_0(x) + V_0(y) \| = 2 \) whenever \( \| x + y \| = 2 \),

(ii) \( V_0(\text{Str}(x)) \subseteq \text{Str}(V_0(x)) \cap \text{Str}(V_0(-x)) \).
Proof. (i) Fix \( x, y \in S(E) \) with \( \|x + y\| = 2 \), and let \( x_r = (1 - r)x + ry \) for any \( 0 < r < 1/2 \). Then

\[
1 = 2(1 - r) - (1 - 2r) \leq \|(1 - r)(x + y) + (2r - 1)y\|
\]

\[
= \|(1 - r)x + r y\| \leq \|(1 - r)x\| + \|ry\| = 1,
\]

so \( x_r \in S(E) \). By the hypothesis of \( V_0 \), we have

\[
\|x_r - V_0^{-1}(-V_0(x_r))\| = \|V_0(x_r) + V_0(x_r)\| = 2.
\]

By Hahn–Banach Theorem, there exists an \( f_r \in S(E^*) \) such that

\[
2 = f_r(x_r - V_0^{-1}(-V_0(x_r)))
\]

\[
= (1 - r)f_r(x) + rf_r(y) + f_r(-V_0^{-1}(-V_0(x_r)))
\]

\[
\leq (1 - r) + r + 1 = 2,
\]

which implies \( f_r(y) = f_r(-V_0^{-1}(-V_0(x_r))) = 1 \). Then the claim follows, because

\[
\|V_0(x) + V_0(y)\| = \lim_{r \to 0} \|V_0(y) + V_0(x_r)\|
\]

\[
= \lim_{r \to 0} \|y - V_0^{-1}(-V_0(x_r))\|
\]

\[
\geq \lim_{r \to 0} f_r(y - V_0^{-1}(-V_0(x_r))) = 2.
\]

(ii) For any \( x \in S(E) \), the above (i) shows that

\[
V_0(St(x)) \subseteq \{ V_0(y): y \in E, \; \|y + x\| = 2 \}
\]

\[
\subseteq \{ V_0(y): y \in E, \; \|V_0(y) + V_0(x)\| = 2 \}
\]

\[
\subseteq St(V_0(x)).
\]

And since \( V_0 \) is an isometry,

\[
V_0(St(x)) \subseteq \{ V_0(y): y \in E, \; \|y - (-x)\| = 2 \}
\]

\[
\subseteq \{ V_0(y): y \in E, \; \|V_0(y) - V_0(-x)\| = 2 \}
\]

\[
\subseteq St(-V_0(-x)).
\]

Corollary 1. [9,26] Let \( E \) and \( F \) be two Banach spaces, and let \( V_0 : S(E) \to S(F) \) be a surjective isometry. Then for any \( x, y \in S(E) \) we have

(i) \( \|V_0(x) + V_0(y)\| = 2 \) if and only if \( \|x + y\| = 2 \),

(ii) \( V_0(St(x)) = St(V_0(x)) = St(-V_0(-x)) \).

Lemma 2. Let \( E \) and \( F \) be two Banach spaces, and let \( V_0 : S(E) \to S(F) \) be a surjective isometry. Then for any \( x \in S(E) \) we have

\( St(x) \) is convex in \( S(E) \) if and only if \( V_0(St(x)) \) is convex in \( S(F) \).

Proof. Suppose \( St(x) \) is convex. Then, by Corollary 1(ii), it suffices to prove \( St(V_0(x)) \) is convex. For any \( y_1, y_2 \in St(V_0(x)) \), we have
which implies that \((V_0(x) + y_1)/2 \in \text{St}(V_0(x))\). By using Corollary 1(ii) again, we have \(V_0^{-1}((V_0(x) + y_1)/2), V_0^{-1}(y_2) \in \text{St}(x)\). Then from the convexity of \(\text{St}(x)\), we obtain
\[
\|V_0^{-1}((V_0(x) + y_1)/2) + V_0^{-1}(y_2)\| = 2.
\]
Take \(f \in S(E^*)\) such that
\[
2 \geq \left(f(V_0(x)) + f(y_1)\right)/2 + f(y_2)
\]
and
\[
= f\left((V_0(x) + y_1)/2 + y_2\right)
\]
and
\[
= \|V_0^{-1}((V_0(x) + y_1)/2) + V_0^{-1}(y_2)\| = 2.
\]
It follows that \(f(V_0(x)) = f(y_1) = f(y_2) = 1\). Thus,
\[
2 = f(V_0(x) + (y_1 + y_2)/2) \leq \|V_0(x) + (y_1 + y_2)/2\| \leq 2,
\]
which implies that \((y_1 + y_2)/2 \in \text{St}(V_0(x))\). Then the continuity of \(V_0\) yields that \(\text{St}(V_0(x))\) is convex in \(S(F)\).
Similarly, the converse part follows immediately if we notice that \(V_0^{-1}\) is also a surjective isometry. □

The next lemma improves and generalizes the main theorem in [13]. Here, we denote \(\min(a, b)\) by \(a \land b\) for any \(a, b \in \mathbb{R}\).

**Lemma 3.** Let \(E\) and \(F\) be two real Banach spaces, and let \(V : E \to F\) be a positive-homogeneous continuous mapping. If for any \(x, y \in E\) we have
\[
\|V(x) - V(y)\| \land \|V(x) + V(-y)\| \leq \|x - y\| \quad \text{whenever} \ x + y \neq 0, \tag{1}
\]
and \(\|V(x) - V(-x)\| = 2\|x\|\), then we have
\[
\|V(x) - V(y)\| \land \|V(x) + V(-y)\| = \|x - y\| \quad \text{for all} \ x, y \in E.
\]

**Proof.** Without loss of generality, we can assume \(\dim E < +\infty\). For any smooth point \(x \in S(E)\), we can find a functional \(u^* \in S(F^*)\) such that
\[
u^*(V(x) - V(-x)) = \|V(x) - V(-x)\| = 2.
\]
It is clear that \(u^*(V(x)) = -u^*(V(-x)) = 1\), and by the hypothesis that \(V\) is positive-homogeneous, we have \(u^*(V(rx)) = r\) for all \(r \in \mathbb{R}\). Note that \(x\) is a smooth point in \(S(E)\), let \(f_x\) be the unique support functional at \(x\), i.e. \(f_x \in S(E^*)\) and \(f_x(x) = 1\). Define \(P : E \to E\) by
\[
P(y) = y - u^*(V(y))x \quad \text{for all} \ y \in E. \tag{2}
\]
We claim that \(f_x \circ P = 0\). In fact, choose any \(y \in E\). If \(x\) is not orthogonal to \(P(y)\), i.e. there is a real \(\alpha\) such that \(\|x + \alpha P(y)\| < 1\) (evidently \(\alpha \neq 0\)), by (2) we have \(\|y - (u^*(V(y)) - 1/\alpha)x\| < |1/\alpha|\). On the other hand,
\[
\begin{align*}
\frac{1}{\alpha} &= \left| u^*(V(y)) - (u^*(V(y)) - 1/\alpha) \right| = \left| u^*(V(y)) - u^* \circ V((u^*(V(y)) - 1/\alpha)x) \right|, \\
\frac{1}{\alpha} &= \left| u^*(V(y)) + (-u^*(V(y)) + 1/\alpha) \right| \\
&= \left| u^*(V(y)) + u^* \circ V((-u^*(V(y)) + 1/\alpha)x) \right|.
\end{align*}
\]

This implies that
\[
\frac{1}{\alpha} \leq \| V(y) - V((u^*(V(y)) - 1/\alpha)x) \| \land \| V(y) + V((-u^*(V(y)) + 1/\alpha)x) \| \\
\leq \| y - (u^*(V(y)) - 1/\alpha)x \| = \| x + \alpha y \| / |\alpha| < \| 1/\alpha \|,
\]

which leads to a contradiction. Therefore \( x \perp P(y) \), i.e. \( \| x + \alpha P(y) \| \geq 1 \) for all \( \alpha \in \mathbb{R} \). Then we can conclude that \( x \) and \( P(y) \) are linearly independent, so the formula \( g(\lambda_1 x + \lambda_2 P(y)) = \lambda_1 \) (for all \( \lambda_1, \lambda_2 \in \mathbb{R} \)) defines a continuous linear functional on the space spanned by \( x \) and \( P(y) \). By the choice of \( x \) it follows that \( g \) is a restriction of \( f_x \) (by Hahn–Banach Theorem), so \( f_x(P(y)) = g(P(y)) = 0 \).

Hence for any \( y \in E \), \( f_x(y) = f_x(u^*(V(y)))x + P(y) = u^*(V(y)) \). Therefore, for any \( r \in \mathbb{R} \) we have
\[
\| rx \| = \| f_x(rx + y) - f_x(y) \| = \left| u^*(V(rx + y) - V(y)) \right| \leq \| V(rx + y) - V(y) \|,
\]
\[
\| rx \| = \left| f_x(rx + y) + f_x(-y) \right| = \left| u^*(V(rx + y) + V(-y)) \right| \leq \| V(rx + y) + V(-y) \|, 
\]

i.e. \( \| rx \| \leq \| V(rx + y) - V(y) \| \land \| V(rx + y) + V(-y) \| \). From Mazur density theorem (see p. 171 in [10]), we know the smooth points are a residual subset of \( E \). Thus, by the continuity of \( V \), we can complete the proof. \( \square \)

**Corollary 2.** Let \( E \) and \( F \) be two Banach spaces. Suppose that \( V_0 : S(E) \to S(F) \) is a Lipschitz mapping with \( K = 1 \): \( \| V_0(x) - V_0(y) \| \leq \| x - y \| \) for any \( x, y \in S(E) \). Assume also that \( V_0 \) is a surjective mapping such that for any \( x, y \in S(E) \) and \( r > 0 \), we have
\[
\| V_0(x) - r V_0(y) \| \land \| V_0(x) + r V_0(-y) \| \leq \| x - r y \| \tag{3}
\]
and \( \| V_0(x) - V_0(-x) \| = 2 \). Then \( V_0 \) can be extended to be a linear isometry of \( E \) onto \( F \).

**Proof.** Define \( V : E \to F \) by \( V(x) = \| x \| V(x/\| x \|) \) for \( x \neq 0 \) (and put \( V(0) = 0 \)). It is easy to check that \( V \) is a positive-homogeneous continuous mapping from \( E \) onto \( F \), and satisfies \( \| V(x) \| = \| x \|, \| V(x) - V(-x) \| = 2 \| x \| \) for all \( x \in E \). Moreover, we shall prove that \( V \) satisfies (1): for any \( x, y \in E \setminus \{0\} \) we have
\[
\begin{align*}
\| V(x) - V(y) \| &= \| x \| V_0 \left( \frac{x}{\| x \|} \right) - \| y \| V_0 \left( \frac{y}{\| y \|} \right) \\
&= \| x \| \cdot V_0 \left( \frac{x}{\| x \|} \right) - \| y \| \cdot V_0 \left( \frac{y}{\| y \|} \right), \\
\| V(x) + V(-y) \| &= \| x \| V_0 \left( \frac{x}{\| x \|} \right) + \| y \| V_0 \left( \frac{-y}{\| y \|} \right) \\
&= \| x \| \cdot V_0 \left( \frac{x}{\| x \|} \right) + \| y \| \cdot V_0 \left( \frac{-y}{\| y \|} \right),
\end{align*}
\]

Then, by (3),
\[
\| V(x) - V(y) \| \land \| V(x) + V(-y) \| \leq \| x \| \cdot \left| \frac{x}{\| x \|} - \frac{y}{\| y \|} \right| = \| x - y \|,
\]
which remains true when $x$ or $y = 0$. From Lemma 3, we obtain
\[
\|V(x) - V(y)\| \leq \|V(x) + V(-y)\| = \|x - y\|. \tag{4}
\]
Since $V|_{S(E)} = V_0$, it follows that $V_0$ is a surjective isometry and that $V$ is bijective. Hence $V^{-1}$ is positive-homogeneous continuous and satisfies that for all $x', y' \in F$,
\[
\|V^{-1}(x') + V^{-1}(-y')\| \leq \|V^{-1}(x') - V^{-1}(y')\| \leq \|x' - y'\|.
\]
Then, by using Lemma 3 for $V^{-1}$, we have
\[
\|V(x) - V(y)\| \geq \|V(x) - V(y)\| \leq \|V^{-1}(V(x)) - V^{-1}(V(y))\|
\]
\[
= \|V^{-1}(V(x)) - V^{-1}(V(y))\| = \|V(x) - V(y)\|,
\]
so $V$ is a surjective isometry. Thus, by Mazur–Ulam theorem, we complete the proof. \qed

3. On extension of isometries between $S(L^\infty(\Gamma))$ and $S(E)$

To prove our main theorem, the following definitions and results are required.

Given a nonempty set $\Gamma$, we consider the space of all bounded real-valued functions on $\Gamma$. We equip this space with the “sup” norm. Any closed subspace containing all $e_\gamma$'s ($\gamma \in \Gamma$) is called the $L^\infty(\Gamma)$-type space. For example, the spaces $\ell^\infty(\Gamma)$, $c(\Gamma)$ and $c_0(\Gamma)$ (in particular, $\ell^\infty$, $c$ and $c_0$ etc. are the $L^\infty(\Gamma)$-type space (or $L^\infty(\Gamma)$, in brief). For each $\gamma \in \Gamma$ we define
\[
A_\gamma = \{x: x \in S(L^\infty(\Gamma)), x(\gamma) = 1\}.
\]
In this section we always assume that $V_0 : S(L^\infty(\Gamma)) \to S(E)$ is a surjective isometry, where $\Gamma$ is an index set and $E$ is a real Banach space.

Lemma 4. For any $\gamma \in \Gamma$ we have

(i) $V_0(A_\gamma) = \text{St}(V_0(e_\gamma))$,
(ii) $V_0(A_\gamma)$ is a maximal closed convex subset of $S(E)$.

Proof. (i) For any $y \in V_0(A_\gamma)$, we have $\|V_0^{-1}(y) + e_\gamma\| = 2$. From Corollary 1(i), we obtain $\|y + V_0(e_\gamma)\| = \|V_0^{-1}(y) + e_\gamma\| = 2$, so $y \in \text{St}(V_0(e_\gamma))$.

Conversely, for any $y \in \text{St}(V_0(e_\gamma))$, we have $\|V_0^{-1}(y) + e_\gamma\| = \|y + V_0(e_\gamma)\| = 2$, i.e. $V_0^{-1}(y) \in \text{St}(e_\gamma) = A_\gamma$, so $y \in V_0(A_\gamma)$. Thus $V_0(A_\gamma) = \text{St}(V_0(e_\gamma))$.

(ii) Since $A_\gamma$ is a closed convex subset of $S(L^\infty(\Gamma))$ and $V_0$ is an isometry, by Lemma 2, $V_0(A_\gamma)$ is a closed convex subset of $S(E)$. If $M$ is a convex subset of $S(E)$ and $V_0(A_\gamma) \subseteq M$, then for every $y \in M$ we have $\|y + V_0(e_\gamma)\| = 2$. This implies $y \in \text{St}(V_0(e_\gamma)) = V_0(A_\gamma)$, which leads to a contradiction. \qed

Corollary 3. For any $\gamma \in \Gamma$ we have

(i) $V_0(-A_\gamma) = \text{St}(V_0(-e_\gamma))$.
(ii) $V_0(-A_\gamma) = -V_0(A_\gamma)$. 

Proof. By the definition of “$St(x)$” and Corollary 1(ii), we have
\[ V_0(-A_\gamma) = V_0(-St(e_\gamma)) = V_0(St(-e_\gamma)) = St(V_0(-e_\gamma)). \]
This implies
\[ V_0(-A_\gamma) = St(V_0(-e_\gamma)) = -St(-V_0(-e_\gamma)) = -V_0(St(e_\gamma)) = -V_0(A_\gamma). \]

Main Theorem. Let $E$ be a real Banach space, and let $V_0 : (L^\infty(\Gamma)) \to S(E)$ be a surjective isometry. Then we have

(i) for every $\gamma \in \Gamma$, there exists an $f_\gamma \in E^*$ such that $\|f_\gamma\| = 1$ and
\[ f_\gamma(V_0(x)) = x(\gamma) \quad \text{for all } x \in S(L^\infty(\Gamma)), \tag{5} \]
(ii) $V_0$ can be extended to be a linear isometry of $L^\infty(\Gamma)$ onto $E$.

Proof. (i) For any fixed $\gamma \in \Gamma$ since $V_0(A_\gamma)$ does not meet the interior of unit ball $B(E)$, by Lemma 4 and Eidelheit separation theorem, there is an $f_\gamma \in S(E^*)$ satisfying $f_\gamma(x) = 1$ whenever $x \in V_0(A_\gamma)$. Now, for any $x \in S(L^\infty(\Gamma))$ set $y_1 = x + (1 - x(\gamma))e_\gamma$ and $y_2 = x - (1 + x(\gamma))e_\gamma$. Clearly we have $y_1 \in A_\gamma$, $y_2 \in -A_\gamma$. Then from Corollary 3(ii) we obtain that $f_\gamma(V_0(y_1)) = 1$ and $f_\gamma(V_0(y_2)) = -1$. Since $|x(\gamma)| \leq 1$, we have
\[
\begin{align*}
 f_\gamma(V_0(x)) &= 1 + f_\gamma(V_0(x)) - f_\gamma(V_0(y_1)) \geq 1 - \|y_1 - x\| \\
 &= 1 - \|(1 - x(\gamma))e_\gamma\| = x(\gamma), \\
 f_\gamma(V_0(x)) &= f_\gamma(V_0(x)) - f_\gamma(V_0(y_2)) - 1 \leq \|y_2 - x\| - 1 \\
 &= \|(1 + x(\gamma))e_\gamma\| - 1 = x(\gamma).
\end{align*}
\]
Thus $f_\gamma(V_0x) = x(\gamma)$ for all $x \in S(L^\infty(\Gamma))$. (ii) By (i), we have
\[
\begin{align*}
 \|V_0^{-1}(x) - rV_0^{-1}(y)\| \wedge \|V_0^{-1}(x) + rV_0^{-1}(-y)\| \\
\leq \|V_0^{-1}(x) - rV_0^{-1}(y)\| \\
= \sup \{\|(V_0^{-1}(x))(\gamma) - r(V_0^{-1}(y))(\gamma)\| : \gamma \in \Gamma\} \\
= \sup \{|f_\gamma(x) - rf_\gamma(y)| : \gamma \in \Gamma\} \\
\leq \|x - ry\| \quad \text{for all } x, y \in S(E) \text{ and } r \geq 0.
\end{align*}
\]
By Corollary 2, there is a linear surjective isometry $V' : E \to L^\infty(\Gamma)$ with $V'|_{S(E)} = V_0^{-1}$. Hence $V'^{-1} : L^\infty(\Gamma) \to E$ is the linear surjective isometry satisfying $V'^{-1}|_{S(L^\infty(\Gamma))} = V_0$. Thus we complete the proof. □

The following corollary improves and generalizes the main results in [3]. Here, the condition (ii) in [3] can be removed.

Corollary 4. Let $E$ be a real Banach space, and let $V_0 : S(L^\infty(\Gamma)) \to S(E)$ be an isometry. Then $V_0$ can be extended to be a linear isometry of $L^\infty(\Gamma)$ into $E$ if and only if the following condition holds:
Let \( E_0 = \bigcup_{r \geq 0} r \cdot V_0(S(\mathcal{L}^\infty(\Gamma))) \), where \( r \cdot V_0(S(\mathcal{L}^\infty(\Gamma))) = \{ rV_0(x) : x \in S(\mathcal{L}^\infty(\Gamma)) \} \) for any \( r \geq 0 \). Then for any \( y_1, y_2 \in E_0 \), there exist \( r_1, r_2 \geq 0 \) and \( x_1, x_2 \in S(\mathcal{L}^\infty(\Gamma)) \) such that \( y_1 = r_1 V_0(x_1), y_2 = r_2 V_0(x_2) \). For any \( \lambda, \lambda_1, \lambda_2 \in \mathbb{R} \) with \( \lambda_1 y_1 + \lambda_2 y_2 \neq \theta \), by the condition (\(*\)) it is easy to prove that \( \lambda_1 y_1 + \lambda_2 y_2 \in r_0 \cdot V_0(S(\mathcal{L}^\infty(\Gamma))) \), where \( r_0 = \| \lambda_1 y_1 + \lambda_2 y_2 \| \). So, \( E_0 \) is a linear subspace of \( E \) with \( V_0(S(\mathcal{L}^\infty(\Gamma))) = S(E_0) \). Since \( V_0 \) is an isometry, it is clear that \( E_0 \) is a Banach space. Thus, by Main Theorem, we complete the proof. \( \square \)

4. On extension of isometries between \( S(C(\Omega)) \) and \( S(E) \)

In 2003, G. Ding \[5\] firstly studied the isometries between \( S(E) \) and \( S(C(\Omega)) \). In \[9\], the authors considered a similar problem and obtained that any isometry from \( S(E) \) onto \( S(C(\Omega)) \) can be extended to be a linear isometry on the whole \( E \), where \( \Omega \) is a “compact metric space.” In this section, we give a new proof of the result which is shorter than the original one. Moreover, we firstly generalize the result to the case when \( \Omega \) is a “compact Hausdorff space” and give an affirmative answer to the corresponding isometric extension problem. Here, for each \( t \in \Omega \), define

\[ A_t = \{ x : x \in S(C(\Omega)), x(t) = 1 \} \] and

\[ F_t = \{ x : x \in S(C(\Omega)), x(t) = 1 \text{ and } |x(t')| < 1 \text{ if } t' \neq t \}. \]

**Lemma 5.** Let \( \Omega \) be a compact metric space, \( C(\Omega) \) the space of all real-valued continuous functions on \( \Omega \), and \( E \) a real Banach space. Let \( V_0 : S(C(\Omega)) \to S(E) \) be a surjective isometry. Then for any \( t \in \Omega \) we have

(i) \( V_0(A_t) = \bigcap_{x \in A_t} St(V_0(x)) \),

(ii) \( -V_0(A_t) = V_0(-A_t) \).

**Proof.** (i) For any \( y \in V_0(A_t) \), we have \( \| V_0^{-1}(y) + x \| = 2 \) for any \( x \in A_t \). By Corollary 1(i), we have \( \| y + V_0(x) \| = \| V_0^{-1}(y) + x \| = 2 \). So \( y \in \bigcap_{x \in A_t} St(V_0(x)) \).

Conversely, for any \( y \in \bigcap_{x \in A_t} St(V_0(x)) \), by Corollary 1(i) we have \( \| V_0^{-1}(y) + x \| = \| y + V_0(x) \| = 2 \) for all \( x \in A_t \). Then for any \( \epsilon > 0 \), define

\[ U_\epsilon = \{ s : s \in \Omega, \| (V_0^{-1}(y))(s) - (V_0^{-1}(y))(t) \| < \epsilon \}, \]

which is an open subset of \( \Omega \) with \( t \in U_\epsilon \). Since \( \Omega \) is a normal space, by the Urysohn’s Lemma there is an \( h_\epsilon \in A_t \) such that \( 0 \leq h_\epsilon \leq 1 \), \( h_\epsilon(t) = 1 \) and \( h_\epsilon(U_\epsilon^c) = 0 \). Hence

\[ 2 = \| V_0^{-1}(y) + h_\epsilon \| = \sup \{ \| (V_0^{-1}(y))(s) + h_\epsilon(s) \| : s \in U_\epsilon \} \]

\[ \leq \sup \{ \| r + (V_0^{-1}(y))(t) \| : 0 \leq r \leq 1 \} + 3\epsilon \leq 2 + 3\epsilon. \]

Let \( \epsilon \to 0 \), we can obtain that \( (V_0^{-1}(y))(t) = 1 \), so \( y \in V_0(A_t) \).
(ii) By (i) and Corollary 1(ii), for any \( t \in \Omega \) we have
\[
V_0(\mathcal{A}_t) = \bigcap_{x \in \mathcal{A}_t} \text{St}(V_0(x)) = \bigcap_{x \in \mathcal{A}_t} \text{St}(V_0(-x))
\]
\[
= - \bigcap_{x \in \mathcal{A}_t} \text{St}(V_0(-x)) = - \bigcap_{y \in -\mathcal{A}_t} \text{St}(V_0(y))
\]
\[
= -V_0(-\mathcal{A}_t).
\]
Thus we complete the proof. \( \square \)

In the following proof, we will use the fact: if \( \Omega \) is a compact metric space, then for any \( t \in \Omega \) we have \( \mathcal{F}_t \neq \emptyset \) and for any \( x \in \mathcal{F}_t, \mathcal{A}_t = \text{St}(x) \) is a convex subset of \( S(C(\Omega)) \).

**Corollary 5.** Let \( \Omega \) be a compact metric space, \( C(\Omega) \) the space of all real-valued continuous functions on \( \Omega \), and \( E \) a real Banach space. Let \( V_0 : S(C(\Omega)) \to S(E) \) be a surjective isometry. Then we have

(i) for every \( t \in \Omega \) there exists an \( f_t \in E^* \) such that \( \| f_t \| = 1 \) and
\[
f_t(V_0(x)) = x(t) \quad \text{for all } x \in S(C(\Omega)),
\]
(ii) \( V_0 \) can be extended to be a linear isometry of \( C(\Omega) \) onto \( E \).

**Proof.** (i) For any fixed \( t \in \Omega \), from Lemma 2 we get \( V_0(\mathcal{A}_t) \) is a convex subset of \( S(E) \). Then by the Eidelheit separation theorem, there is an \( f_t \in S(E^*) \) satisfying \( f_t(x) = 1 \) if \( x \in V_0(\mathcal{A}_t) \). For any \( x \in S(C(\Omega)) \) and \( \epsilon > 0 \), define
\[
U_\epsilon = \{ s \in \Omega : |x(s) - x(t)| < \epsilon \},
\]
which is an open subset of \( \Omega \). Since \( \Omega \) is a normal space, by the Urysohn’s Lemma there is an \( h_\epsilon \in \mathcal{A}_t \) such that \( 0 \leq h_\epsilon \leq 1 \), \( h_\epsilon(t) = 1 \) and \( h_\epsilon(U_\epsilon^c) = 0 \). For any \( s \in \Omega \) define
\[
y_1(s) = x(s) + (1 - x(s))h_\epsilon(s), \quad y_2(s) = x(s) - (1 + x(s))h_\epsilon(s).
\]
Obviously \( y_1 \in \mathcal{A}_t, y_2 \in -\mathcal{A}_t \) and \( \| y_1 - x \| \leq 1 - x(t) + 2\epsilon, \| y_2 - x \| \leq 1 + x(t) + 2\epsilon \), so
\[
f_t(V_0(x)) = 1 + f_t(V_0(x)) - f_t(V_0(y_1)) \geq 1 - \| y_1 - x \| = x(t) - 2\epsilon,
\]
and by Lemma 5(ii),
\[
f_t(V_0(x)) = f_t(V_0(x)) - f_t(V_0(y_2)) - 1 \leq \| y_2 - x \| - 1 = x(t) - 2\epsilon.
\]
Since \( \epsilon \) can be made arbitrarily small, we have \( f_t(V_0(x)) = x(t) \) for all \( x \in S(C(\Omega)) \).

(ii) The proof of this part is the same as that of Main Theorem (ii), because we have
\[
\| V_0^{-1}(x) - rV_0^{-1}(y) \| = \sup \{ |(V_0^{-1}(x))(t) - r(V_0^{-1}(y))(t)| : t \in \Omega \}
\]
\[
= \sup \{ |f_t(x) - r f_t(y)| : t \in \Omega \}
\]
\[
\leq \| x - ry \| \quad \text{for all } x, y \in S(E) \text{ and } r \geq 0. \quad \square
\]

**Remark.** We think that many problems must be overcome before extending the above conclusion (i). Because, in the proof of the conclusion (i), it is crucial that for every \( \gamma \in I^* \) (or \( t \in \Omega \)), there exists a peak point which peaks at \( \gamma \) (or \( t \)). For \( C(\Omega) \) space this statement is true if \( \Omega \) is completely regular and every singleton point of \( \Omega \) is a \( G_\delta \)-set. The properties of peak points make it possible for us to study the convex subsets of the unit spheres.
However, we can generalize the conclusion (ii) to the following more general case. Here, we denote \( \max(a, b) \) by \( a \lor b \) for any \( a, b \in \mathbb{R} \).

**Corollary 6.** Let \( \Omega \) be a compact Hausdorff space, \( C(\Omega) \) the space of all real-valued continuous functions on \( \Omega \), and \( E \) a real Banach space. Let \( V_0 : S(C(\Omega)) \to S(E) \) be a surjective isometry. Then \( V_0 \) can be extended to be a linear isometry of \( C(\Omega) \) onto \( E \).

**Proof.** For every \( x, y \in S(E) \), \( t \in \Omega \) and \( \epsilon > 0 \) define

\[
U_\epsilon = \{ s \in \Omega : \| (V_0^{-1}(x))(s) - (V_0^{-1}(x))(t) \| < \epsilon \text{ and } \| (V_0^{-1}(y))(s) - (V_0^{-1}(y))(t) \| < \epsilon \} ,
\]

which is an open subset of \( \Omega \). Since \( \Omega \) is a normal space, by the Urysohn’s Lemma, take \( h_\epsilon \in S(C(\Omega)) \) such that \( 0 \leq h_\epsilon \leq 1 \), \( h_\epsilon(t) = 1 \), \( h_\epsilon(U_\epsilon^c) = 0 \). For any \( s \in \Omega \) define

\[
x_1(s) = (V_0^{-1}(x))(s) + (1 - (V_0^{-1}(x))(s))h_\epsilon(s),
\]

\[
x_2(s) = (V_0^{-1}(x))(s) - (1 + (V_0^{-1}(x))(s))h_\epsilon(s),
\]

\[
y_1(s) = (V_0^{-1}(y))(s) + (1 - (V_0^{-1}(y))(s))h_\epsilon(s),
\]

\[
y_2(s) = (V_0^{-1}(y))(s) - (1 + (V_0^{-1}(y))(s))h_\epsilon(s).
\]

From the definitions of \( U_\epsilon, h_\epsilon \) and \( x_i, y_i \) \( (i = 1, 2) \) we obtain that \( x_1, y_1 \in \mathcal{A}_t, 2y_1, y_2 \in -\mathcal{A}_t \) and \( \| x_1 - y_2 \| = \| y_1 - x_2 \| = 2 \). So \( \| V_0(x_1) - V_0(y_2) \| = \| V_0(y_1) - V_0(x_2) \| = 2 \). Then by the Hahn–Banach Theorem we can find \( f_t^{(1)}, f_t^{(2)} \in S(E^*) \) such that

\[
f_t^{(1)}(V_0(x_1) - V_0(y_2)) = f_t^{(2)}(V_0(y_1) - V_0(x_2)) = 2.
\]

It is easy to check that \( f_t^{(1)}(V_0(x_1)) = 1 \), \( f_t^{(1)}(V_0(y_2)) = -1 \) and \( f_t^{(2)}(V_0(y_1)) = 1 \), \( f_t^{(2)}(V_0(x_2)) = -1 \). So we get

\[
\| x_1 - V_0^{-1}(x) \| \leq 1 - (V_0^{-1}(x))(t) + 2\epsilon,
\]

\[
\| x_2 - V_0^{-1}(x) \| \leq 1 + (V_0^{-1}(x))(t) + 2\epsilon,
\]

\[
\| y_1 - V_0^{-1}(y) \| \leq 1 - (V_0^{-1}(y))(t) + 2\epsilon,
\]

\[
\| y_2 - V_0^{-1}(y) \| \leq 1 + (V_0^{-1}(y))(t) + 2\epsilon.
\]

Then we have

\[
(V_0^{-1}(x))(t) \leq 1 - \| x_1 - V_0^{-1}(x) \| + 2\epsilon
\]

\[
\leq 1 + f_t^{(1)}(x) - f_t^{(1)}(V_0(x_1)) + 2\epsilon = f_t^{(1)}(x) + 2\epsilon,
\]

\[
(V_0^{-1}(y))(t) \geq \| y_2 - V_0^{-1}(y) \| - 1 - 2\epsilon
\]

\[
\geq f_t^{(1)}(y) - f_t^{(1)}(V_0(y_2)) - 1 - 2\epsilon = f_t^{(1)}(y) - 2\epsilon,
\]

and

\[
(V_0^{-1}(y))(t) \leq 1 - \| y_1 - V_0^{-1}(y) \| + 2\epsilon
\]

\[
\leq 1 + f_t^{(2)}(x) - f_t^{(2)}(V_0(y_1)) + 2\epsilon = f_t^{(2)}(y) + 2\epsilon,
\]

\[
(V_0^{-1}(x))(t) \geq \| x_2 - V_0^{-1}(x) \| - 1 - 2\epsilon
\]

\[
\geq f_t^{(2)}(x) - f_t^{(2)}(V_0(x_2)) - 1 - 2\epsilon = f_t^{(2)}(x) - 2\epsilon.
\]
From the definition of norm, we have
\[ \| V_0^{-1}(x) - r V_0^{-1}(y) \| = \sup \{ |(V_0^{-1}(x))(t) - r(V_0^{-1}(y))(t)| : t \in \Omega \} \]
\[ = \sup \{ (V_0^{-1}(x))(t) - r(V_0^{-1}(y))(t) \vee (r(V_0^{-1}(y))(t) - (V_0^{-1}(x))(t)) : t \in \Omega \} \]
\[ \leq \sup \{ |f_t^{(1)}(x) - r f_t^{(1)}(y)| \vee (r f_t^{(2)}(y) - f_t^{(2)}(x)) : t \in \Omega \} + (2 + 2r)\epsilon \]
\[ = \sup \{ f_t^{(1)}(x) - r y \vee f_t^{(2)}(r y - x) : t \in \Omega \} + (2 + 2r)\epsilon \]
\[ \leq \| x - r y \| + (2 + 2r)\epsilon \]
holds for any \( r \geq 0 \). Let \( \epsilon \to 0 \), we can obtain that \( \| V_0^{-1}(x) - r V_0^{-1}(y) \| \leq \| x - r y \| \). Thus it is easy to complete the proof if we proceed as the proof of Main Theorem (ii). \( \square \)

**Corollary 7.** Let \( \Omega \) be a compact Hausdorff space, \( C(\Omega) \) the space of all real-valued continuous functions on \( \Omega \), and \( E \) a real Banach space. Let \( V_0 : S(C(\Omega)) \to S(E) \) be an isometry. Then \( V_0 \) can be extended to a linear isometry of \( C(\Omega) \) into \( E \) if and only if the following condition holds:

\( ** \) For any \( x_1, x_2 \in S(C(\Omega)) \) and \( \lambda_1, \lambda_2 \in \mathbb{R} \), if \( \| \lambda_1 V_0(x_1) + \lambda_2 V_0(x_2) \| = 1 \), then \( \lambda_1 V_0(x_1) + \lambda_2 V_0(x_2) \in V_0(S(C(\Omega))) \).

**Example.** In fact, the natural conditions (\( * \)) and (\( ** \)) cannot be omitted, because a counterexample can be given. Let \( T_c : S(\ell_2^\infty) \to S(\ell_3^\infty) \) be defined by
\[ T_c(x) = (\sin \xi_1, \xi_1, \xi_2) \quad \text{for all} \quad x = (\xi_1, \xi_2) \in S(\ell_2^\infty). \]
It is easy to know that \( T_c \) is an isometry with \( T_c(-x) = -T_c(x) \). If there exists a linear isometry \( \tilde{T}_c : \ell_2^\infty \to \ell_3^\infty \) satisfying \( \tilde{T}_c|_{S(\ell_2^\infty)} = T_c \), then we have
\[ (\sin 1, 1, 1) = \tilde{T}_c((1, 1)) = \tilde{T}_c\left( \left( \frac{1}{2}, 1 \right) + \frac{1}{2}(1, 0) \right) \]
\[ = \frac{1}{2} \tilde{T}_c\left( \left( \frac{1}{2}, 1 \right) \right) + \frac{1}{2} \tilde{T}_c((1, 0)) = \left( \sin \frac{1}{2} + \frac{1}{2} \sin 1, 1, 1 \right), \]
which leads to a contradiction.

Please recall the well-known representation: a Banach lattice \( E \) is isometrically isomorphic to a space \( C(\Omega) \) of continuous function on a compact Hausdorff space \( \Omega \) if and only if \( E \) is an \( AM \)-space with unit. Moreover, the unit of \( E \) corresponds to the one-function in \( C(\Omega) \), and \( \Omega \) may be chosen as \( \Omega = \operatorname{ext} E_+ \) equipped with the \( w^* \)-topology. Thus, we propose the following problem:

**Problem.** Let \( F \) be an \( AM \)-space, and \( E \) a real Banach space. Let \( V_0 : S(F) \to S(E) \) be a surjective isometry. Is \( V_0 \) necessarily the restriction of a linear or affine transformation on \( F \)?

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References