On Lih’s Conjecture concerning Spernerity

DAVID G. C. HORROCKS

Let $\mathcal{F}$ be a nonempty collection of subsets of $[n] = \{1, 2, \ldots, n\}$, each having cardinality $t$. Denote by $P_{\mathcal{F}}$ the poset consisting of all subsets of $[n]$ which contain at least one member of $\mathcal{F}$, ordered by set-theoretic inclusion. In 1980, K. W. Lih conjectured that $P_{\mathcal{F}}$ has the Sperner property for all $1 \leq t \leq n$ and every choice of $\mathcal{F}$. This conjecture is known to be true for $t = 1$ but false, in general, for $t \geq 4$. In this paper, we prove Lih’s conjecture in the case $t = 2$.

We make extensive use of fundamental theorems concerning the preservation of Sperner-type properties under direct products of posets.

1. Introduction

The boolean algebra $B_n$ of order $n$ is the poset consisting of all subsets of $[n]$, ordered by inclusion. A classical and fundamental theorem, obtained by Sperner in 1928, asserts that a maximum size antichain $\mathcal{A}$ in $B_n$ has no more than $\binom{n}{\lceil n/2 \rceil}$ members, and that equality occurs if and only if $\mathcal{A}$ consists of all subsets of $[n]$ of size $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$. This property, namely that the maximum size of an antichain equals the size of the largest rank of the poset, is now known as the Sperner property.

In examining how Sperner’s theorem might be generalized, Lih [13] considered the following relativized version of the theorem. Let $Y$ be a fixed subset of $[n]$ of cardinality $k > 0$ and let $P$ be the induced subposet of $B_n$ consisting of all sets having nonempty intersection with $Y$. The relativized problem is to determine the maximum size $f(n, k)$ of an antichain in $P$. Lih [13] proved that

$$f(n, k) = \binom{n}{\lfloor n/2 \rfloor} - \binom{n-k}{\lfloor n/2 \rfloor}$$

and that this bound may be realized by taking all elements of $P$ of size $\lfloor n/2 \rfloor$. In other words, $P$ has the Sperner property.

Before Lih’s conjecture is stated, some explanation of terminology is required. A subset $F$ of the poset $P$ is called a filter if $x \in F$ and $y \geq x$ imply that $y \in F$. The principal filter generated by $x$, and denoted by $\langle x \rangle$, is the set $\{y \in P \mid y \geq x\}$. The filter $F$ is said to be generated by $x_1, x_2, \ldots, x_k$ if $F = \langle x_1 \rangle \cup \langle x_2 \rangle \cup \cdots \cup \langle x_k \rangle$.

Let $C(n, k)$ be the induced subposet of $B_n$ consisting of all sets which intersect a given $k$-set nontrivially. Lih’s result stated above is that $C(n, k)$ has the Sperner property for all $1 \leq k \leq n$. Noting that $C(n, k)$ is a filter in $B_n$ generated by $k$ 1-element subsets, Lih made the following conjecture.

Conjecture 1.1 (Lih [13]). For all $t$, if $F$ is a filter in $B_n$ generated by a nonempty collection of $t$-subsets, then $F$ has the Sperner property.

This conjecture, while true for $t = 1$, is false in general. In [7], Griggs provides an example of a filter in $B_n$ generated by a collection of 4-subsets which does not have the Sperner property. Furthermore, Zha [20] has constructed counterexamples for any $t \geq 4$ and $n \geq 2t - 1$.

The purpose of this paper is to prove Lih’s conjecture in the case $t = 2$. It will be convenient to associate a graph with a collection of 2-sets generating a filter in $B_n$ as follows. Let $G$ be
a graph with vertex set \([n]\) and having at least one edge. Let \(P_G\) denote the collection of all subsets \(H \subseteq [n]\) having the property that the induced subgraph \(G[H]\) contains at least one edge. Then \(P_G\) is a poset, assuming that its members are ordered by the usual set-theoretic inclusion. There is a natural one-to-one correspondence between filters in \(B_n\) generated by 2-sets and posets \(P_G\) as described in the following result.

**Proposition 1.2 (Horrocks [9]).** Let \(S\) be the set of all filters in \(B_n\) which are generated by a nonempty collection of 2-sets. Let \(T\) be the set of all posets \(P_G\) where \(G\) is a graph with vertex set \([n]\) having at least one edge. Then there is a one-to-one correspondence between \(S\) and \(T\).

Thus, the notions of a filter in \(B_n\) generated by a collection of 2-sets and a graph with its corresponding poset are completely interchangeable. We will find the latter viewpoint more convenient and now state our main result, which is therefore equivalent to a proof of Conjecture 1.1 for \(t = 2\), in this form.

**Theorem 1.3.** Let \(n \geq 2\) be a positive integer. For every graph \(G\) on \(n\) vertices having at least one edge, the poset \(P_G\) has the Sperner property.

This paper is devoted to proving Theorem 1.3 and is structured as follows. In Section 2 we review some general terminology and notation from Sperner theory, and in Section 3 we present some required background results. The proof of Theorem 1.3 employs two main techniques which are described in Sections 4 and 5. These techniques enable us to establish the conjecture for several classes of graphs whose union covers almost all cases. For example, in Section 6, using the technique of Section 4.1, we prove Theorem 1.3 for graphs having six disjoint edges. In Section 7, we prove the result for graphs on eight vertices using a theorem from extremal graph theory. Section 8 uses a method, due to Greene and Kleitman, for constructing a symmetric chain partition of \(B_n\), to prove Theorem 1.3 for another class of graphs. The technique of Section 5 is applied in Section 9 where we establish the conjecture for graphs having at most seven nonisolated vertices. We complete the proof in Section 10 by examining the graphs which fall outside of the scope of Sections 6 through 9. The exceptional cases are categorized and dealt with by a case analytic argument where again our two main techniques play a prominent role. While the class of exceptional graphs does contain some infinite families, we will show that each exceptional graph contains, as a subgraph, one of a finite set of special graphs. For completeness, a listing of these special graphs appears in the Appendix.

2. **Terminology**

Let \(P\) be a finite partially ordered set, or poset. A *rank function* for \(P\) is a function \(r : P \rightarrow \{0, 1, 2, \ldots, \}\) such that \(r(y) = r(x) + 1\) whenever \(y\) covers \(x\) in \(P\). If \(P\) admits a rank function then \(P\) is said to be *ranked*. In this case, the *rank* of \(P\) is the maximum value of \(r(x)\) taken over all \(x \in P\). If \(P\) has rank \(n\), then for \(k = 0, 1, 2, \ldots, n\), the set

\[
P_k = \{x \in P \mid r(x) = k\}
\]

is called the *\(k\)th rank* of \(P\). The *\(k\)th Whitney number* of \(P\) is \(N_k = N_k(P) = |P_k|\). Note that \(B_n\) is ranked with \(r(X) = |X|\) and \(N_k(B_n) = \binom{n}{k}\). Moreover, for any graph \(G\) with at least one edge, \(P_G\) is ranked and, for convenience, we take \(r(H) = |H|\) as in \(B_n\) (even though the minimal elements in \(P_G\) are 2-sets).
The sequence $a_0, a_1, \ldots, a_n$ of real numbers is said to be unimodal if there is an integer $k$ such that
$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_{n-1} \geq a_n.$$The same sequence is logarithmically concave if $a_i^2 \geq a_i-1 \cdot a_{i+1}$ for all $1 \leq i \leq n-1$. The finite ranked poset $P$ is called rank unimodal if its Whitney numbers form a unimodal sequence.

A set $C$ of elements of $P$ is called a chain if any two elements of $C$ are comparable. The size of the chain $C = \{x_1 < x_2 < \cdots < x_r\}$ is $r$ and its length is $r-1$. If every maximal chain in $P$ has the same length then $P$ is said to be graded. An antichain is a set of elements of $P$, any two of which are incomparable.

A ranked poset $P$ has the Sperner property (or simply, $P$ is Sperner) if the maximum size of an antichain in $P$ equals the size of the largest rank of $P$. For any $k \geq 1$, a $k$-family is a subset of $P$ that contains no chain of size $k+1$. Equivalently, a $k$-family is a union of at most $k$ antichains. We say that $P$ has the strong Sperner property if, for all $k \geq 1$, the maximum size of a $k$-family equals the sum of the $k$ largest Whitney numbers. That $B_n$ has the strong Sperner property was first discovered by Erdős [3] in 1945.

A ranked poset $P$ with ranks $P_0, P_1, \ldots, P_n$ is said to have the LYM property if
$$\sum_{k=0}^{n} \frac{|A \cap P_k|}{|P_k|} \leq 1$$for every antichain $A \subseteq P$. Furthermore, Lubell [14], Yamamoto [19], and Meschkalin [15] independently showed that $B_n$ has the LYM property.

We define an isolated vertex in a graph to be a vertex of degree zero. The notation $Z_m$ refers to the graph consisting of $m$ isolated vertices. The other definitions and notations from graph theory used in this paper are standard and for these we refer the reader to Bondy and Murty [1].

3. Background Theory

In a ranked poset $P$, we say that there is a matching between the adjacent ranks $P_i$ and $P_{i+1}$ if there is a matching (in the graph theoretic sense) of size $\min(|P_i|, |P_{i+1}|)$ in the bipartite subgraph of the Hasse diagram of $P$ induced by the ranks $P_i$ and $P_{i+1}$. The following result, which may be found in Griggs [7], describes the connection between the Sperner property and the existence of matchings between all pairs of consecutive ranks.

**Theorem 3.1.** A graded poset $P$ has the Sperner property if and only if it is rank unimodal and there is a matching between every pair of adjacent ranks of $P$.

Therefore, since the problem of finding a maximum size matching in a bipartite graph will be of central importance, we recall the following classical theorem of P. Hall.

**Theorem 3.2 (Hall, 1935).** Let $G$ be a bipartite graph with bipartition $(X, Y)$. Then $G$ contains a matching which meets every vertex of $X$ if and only if, for all $S \subseteq X$,
$$|N(S)| \geq |S|,$$where $N(S)$ is the set of vertices of $G$ adjacent to some vertex of $S$.

Using Theorem 3.2, Zha [20] obtained the following result.
**Lemma 3.3.** Let $G$ be a graph on $n$ vertices with corresponding poset $P = P_G$. Then $P$ is rank unimodal with largest rank $P_{r+1}$ if $n = 2r + 1$ and $P_r$ or $P_{r+1}$ if $n = 2r$. Moreover, a matching exists between every pair of adjacent ranks, except possibly in the case $(P_r, P_{r+1})$ when $n = 2r$.

### 3.1. The proof for the odd case

The following result, which establishes Theorem 1.3 in the case when $n$ is odd, is due to Zhu [21]. It is now immediate from Lemma 3.3 upon applying Theorem 3.1.

**Corollary 3.4 (Zhu [21]).** Let $G$ be a graph on $n$ vertices with corresponding poset $P_G$. If $n$ is odd then $P_G$ has the Sperner property.

### 3.2. An analysis of the even case

In view of Corollary 3.4, we can confine our attention to graphs with an even number of vertices. An integral part of our strategy will be to use the following result.

**Corollary 3.5.** Let $G$ be a graph on $2n$ vertices with corresponding poset $P_G$. Then $P_G$ has the Sperner property if and only if there is a matching between the ranks $(P_G)_n$ and $(P_G)_{n+1}$.

**Proof.** First, suppose that there is a matching between the ranks $(P_G)_n$ and $(P_G)_{n+1}$. By Lemma 3.3, $P_G$ is rank unimodal and a matching exists between every pair of adjacent ranks. Therefore, $P_G$ has the Sperner property by Theorem 3.1.

Conversely, suppose that $P_G$ has the Sperner property and that $|\{P_G\}_n| \geq |\{P_G\}_{n+1}|$. If a matching does not exist between the ranks $(P_G)_n$ and $(P_G)_{n+1}$ then by Hall’s Theorem, there exists $S \subset (P_G)_{n+1}$ such that $|S| > |N(S)|$. Now $A = S \cup ((P_G)_n \setminus N(S))$ is an antichain and

$$|A| = |S| + |\{P_G\}_n| - |N(S)| > |\{P_G\}_n|.$$  

This contradicts $P_G$ having the Sperner property since, by Lemma 3.3, $(P_G)_n$ is the largest rank of $P_G$. The proof for the case $|\{P_G\}_n| < |\{P_G\}_{n+1}|$ is similar and is omitted. □

### 4. Spanning Subgraphs and Spernerity

Let $G$ be a graph with $2n$ vertices and let $P = P_G$ be its corresponding poset. By Corollary 3.5, to show that $P$ has the Sperner property, it is sufficient to prove the existence of a matching between the ranks $P_n$ and $P_{n+1}$. In general, however, we do not know a priori which of the sets $P_n$ or $P_{n+1}$ is to be saturated by the matching. This presents some difficulty to any method which actually constructs the desired matching.

This problem may, however, be surmounted under certain circumstances if $G$ contains a spanning subgraph $H$ such that $P_H$ is Sperner and has $|\{P_H\}_n| \geq |\{P_H\}_{n+1}|$. Assuming that $H$ satisfies a certain technical condition, we will show that the matching between $(P_H)_n$ and $(P_H)_{n+1}$ may be extended to a matching between $P_n$ and $P_{n+1}$, as the edges in $E(G) \setminus E(H)$ are added to $H$ in order to recover $G$.

It is perhaps surprising that there are examples of graphs $H$ having a small numbers of edges which do meet the necessary criteria. For example, in Section 6, we show that $6K_2$ is one such graph. Moreover, when dealing with the exceptional cases in Section 10, we will rely heavily on finding spanning subgraphs with few edges whose corresponding posets have the Sperner property and more $n$-sets than $(n+1)$-sets.

The purpose of this section, then, is to prove the following theorem.
THEOREM 4.1. Let \( G \) be a graph on \( 2n \) vertices, and let \( H \) be a spanning subgraph of \( G \). If

(1) \( P_H \) is Sperner with \( |(P_H)_n| \geq |(P_H)_{n+1}| \), and

(2) for any two isolated vertices \( x \) and \( y \) of \( H \), \( H \setminus \{x, y\} \) has no more than \( \sum_{i=2}^{n-1} \binom{2i-1}{i} \) independent sets of size \( n-1 \),

then \( P_G \) has the Sperner property and \( |(P_G)_n| \geq |(P_G)_{n+1}| \).

The following technical lemma will be required.

LEMMA 4.2 (ZHA [20]). Let \( \mathcal{F} \) be a collection of subsets of size \( s \) of \( [2s] \), and let \( N(\mathcal{F}) \) be the lower shadow of \( \mathcal{F} \), that is,

\[
N(\mathcal{F}) = \{ X \mid |X| = s - 1 \text{ and } X \subseteq Y \text{ for some } Y \in \mathcal{F} \}.
\]

Let \( G \) be the bipartite graph having \( V(G) = \mathcal{F} \cup N(\mathcal{F}) \) with \( X \) adjacent to \( Y \) in \( G \) if and only if \( X \subseteq Y \). If

\[
|\mathcal{F}| \leq \left( \frac{2s - 1}{s} \right) + \left( \frac{2s - 3}{s - 1} \right) + \cdots + \left( \frac{3}{2} \right),
\]

then there is a matching in \( G \) which saturates \( \mathcal{F} \).

To prove Theorem 4.1, consider adding the edges in \( E(G) \setminus E(H) \) one at a time to \( H \) to recover \( G \). After each addition, we show that the two conditions in the theorem continue to hold. This strategy is made precise in the following lemma.

LEMMA 4.3. Let \( G \) be a graph on \( 2n \) vertices, and let \( L \) be a spanning subgraph of \( G \). If

(1) \( P_L \) is Sperner with \( |(P_L)_n| \geq |(P_L)_{n+1}| \), and

(2) for any two isolated vertices \( x \) and \( y \) of \( L \), \( L \setminus \{x, y\} \) has no more than \( \sum_{i=2}^{n-1} \binom{2i-1}{i} \) independent sets of size \( n-1 \),

then \( \tilde{L} = L + e \) also satisfies the above conditions, for any edge \( e \in E(G) \setminus E(L) \).

PROOF. We first show that \( P = P_L \) is Sperner with \( |P_n| \geq |P_{n+1}| \). For \( k = n, n + 1, \) let

\[
A_k = \{ X \in P_n \mid \tilde{L}[X] \text{ contains only edge } e \}.
\]

In other words, \( A_k \) consists of those \( k \)-sets which are in \( P \) but not \( P_L \). Therefore, we may write

\[
P_k = (P_L)_k \cup A_k.
\]

Let \( H \) be the bipartite graph with \( V(H) = P_n \cup P_{n+1} \) where \( X \) is adjacent to \( Y \) if and only if \( X \subseteq Y \). Let \( H_1 \) and \( H_2 \) be the subgraphs of \( H \) induced by \( (P_L)_n \cup (P_L)_{n+1} \) and \( A_n \cup A_{n+1} \) respectively.

By hypothesis, \( P_L \) is Sperner with \( |(P_L)_n| \geq |(P_L)_{n+1}| \), so there is a matching \( M_1 \) in \( H_1 \) which saturates \( (P_L)_{n+1} \).

We claim that there is also a matching in \( H_2 \) which saturates \( A_{n+1} \). To establish this, it is convenient to distinguish two cases depending on whether \( e \) is incident with an edge of \( L \) or not.

First, suppose that \( e \) is incident with some edge \( e_1 \) of \( L \). For definiteness, we take \( V(G) = [2n], e = (i, j) \) and \( e_1 = (i, k) \) where \( k \neq j \). Let \( \mathcal{F} \subseteq A_{n+1} \) and let

\[
N(\mathcal{F}) = \{ X \in P_{n+1} \mid X \sim Y \text{ for some } Y \in \mathcal{F} \}.
\]
Now every $S \in \mathcal{F}$ has degree $n - 1$ in $H_2$ since $\hat{L}[S]$ contains only one edge and, moreover, $N(\mathcal{F}) \subseteq A_n$. The degree of $T \in N(\mathcal{F})$ in $H$ is $n$. Since $T \in A_n$, $\hat{L}[T]$ contains edge $e$ but not $e_1$ so $\{i, j\} \subseteq T$ but $k \notin T$. Now $T \cup \{k\} \notin A_{n+1}$ so the degree of $T$ in $H_2$ is strictly less than $n$. Thus, enumerating edges between $\mathcal{F}$ and $N(\mathcal{F})$ gives

$$(n - 1)|\mathcal{F}| \leq (n - 1)|N(\mathcal{F})|$$

so $|N(\mathcal{F})| \geq |\mathcal{F}|$. By Hall’s theorem, there is a matching $M_2$ in $H_2$ which saturates $A_{n+1}$.

Secondly, suppose that $e = (i, j)$ is incident with no edge of $L$. Thus, $i$ and $j$ are isolated vertices in $L$ so $X \in A_k$ if and only if $X$ is an independent set of size $k - 2$ in $L \setminus \{x, y\}$. Let $a_m(G)$ denote the number of independent sets in $G$ of size $m$. Thus

$$|A_{n+1}| = a_{n-1}(L \setminus \{i, j\}) \leq \sum_{i=2}^{n-1} \binom{2i - 1}{i}$$

since $i$ and $j$ are isolated vertices in $L$. Let $\hat{A}_k = \{X \setminus \{i, j\} \mid X \in A_k\}$ so $\hat{A}_{n+1}$ is a collection of $(n - 1)$-subsets of $[2n - 2]$. Since

$$|\hat{A}_{n+1}| = |A_{n+1}| \leq \sum_{i=2}^{n-1} \binom{2i - 1}{i}$$

and $N(\hat{A}_{n+1}) \subseteq \hat{A}_n$, it is possible to match every element of $\hat{A}_{n+1}$ to an element of $\hat{A}_n$ by Lemma 4.2. This matching gives rise in the obvious way to a matching $M_2$ in $H_2$ which saturates $A_{n+1}$.

Taking the union of $M_1$ and $M_2$, we obtain a matching in $H$ which saturates $P_n$. Therefore, $P$ has the Sperner property and $|P_n| \geq |P_{n+1}|$.

It remains to show that the second condition is satisfied by $\hat{L} = L + e$. Let $x$ and $y$ be any two isolated vertices in $\hat{L}$. Clearly $E(L) \subseteq E(\hat{L})$ so

$$a_{n-1}(\hat{L} \setminus \{x, y\}) \leq a_{n-1}(L \setminus \{x, y\}) \leq \sum_{i=2}^{n-1} \binom{2i - 1}{i}$$

since $x$ and $y$ are also isolated vertices in $L$.

\[\square\]

**Proof of Theorem 4.1.** The proof of Theorem 4.1 is now immediate. Apply Lemma 4.3 repeatedly, taking $L = H$ initially, and adding the edges in $E(G) \setminus E(H)$ one at a time, in any order.

\[\square\]

## 5. Adding Isolated Vertices

Frequently, when wishing to apply Theorem 4.1, we are faced with the problem of proving that $P_H$ has the Sperner property where $H$ is a graph having an arbitrarily large number of vertices but only a fixed small number of edges. Indeed, this will be the case in Section 6 when we consider a graph on $2n$ vertices having exactly six disjoint edges, and also in Section 10 pertaining to exceptional cases. Fortunately, it is possible to obtain Sperner-type results for $P_H$ without considering large numbers of isolated vertices, as we now discuss.

Let $G$ be a graph and suppose that $\hat{G}$ is obtained from $G$ by adding some isolated vertices. Perhaps not unexpectedly, there is a close relationship between $P_H$ and $P_{\hat{G}}$, namely, that $P_{\hat{G}}$ is isomorphic to the product of $P_G$ and a boolean algebra. Such a decomposition of $P_{\hat{G}}$ will then
allow us to apply a product theorem concerning the preservation of Sperner-type properties under direct products.

The goal of this section is to prove the following theorem which will be used in later sections, namely in Section 9, and in conjunction with Theorem 4.1 in Section 10.

**Theorem 5.1.** Let $G$ be a graph consisting of a graph $H$ plus some isolated vertices. If $|V(H)|$ is odd and $P_H$ is strongly Sperner, then $P_G$ has the Sperner property.

We begin with the aforementioned decomposition theorem.

**Theorem 5.2 (Horrocks [9]).** Let $G$ be a graph and suppose that $\hat{G}$ is obtained from $G$ by adding $m$ isolated vertices. Then

$$P_{\hat{G}} \cong P_G \times B_m.$$ 

Let $P$ and $Q$ be ranked posets. We say that $P$ and $Q$ are compatible if there exists an integer $d$ such that, for all $i$ and $j$, if $N_i(P) < N_j(P)$, then $N_{d-i}(Q) \leq N_{d-j}(Q)$. We will require the following theorem due to Proctor et al. [16].

**Theorem 5.3.** Let $P$ and $Q$ be rank unimodal posets having the strong Sperner property. If $P$ and $Q$ are compatible, then $P \times Q$ is rank unimodal and Sperner.

Our aim is to prove Theorem 5.1 by applying the above theorem with $P = P_H$ and $Q = B_m$.

First, however, we must show that these posets are compatible and, to do this, it will be convenient to make the following definition. Let $P$ be a rank unimodal poset with largest rank $P_m$. We say that $P$ is balanced if

$$|P_{m-a}| \geq |P_{m+b}| \text{ and } |P_{m+a}| \geq |P_{m-b}|$$

for all $0 \leq a < b$.

**Lemma 5.4 (Horrocks [9]).** Let $P$ and $Q$ be rank unimodal posets. If $P$ is balanced and $Q$ is rank symmetric, then $P \times Q$ is compatible.

It is not true, unfortunately, that every graph yields a balanced poset. If $G$, however, has an odd number of vertices then $P_G$ is a balanced poset.

**Lemma 5.5 (Horrocks [9]).** Let $G$ be a graph on $n$ vertices with corresponding poset $P_G$. If $n$ is odd then $P_G$ is balanced.

**Proof of Theorem 5.1.** By Proposition 5.2, $P_G \cong P_H \times B_m$ for some $m$. Since $|V(H)|$ is odd, $P_H$ is balanced by Lemma 5.5. Furthermore, as $P_H$ is rank unimodal by Lemma 3.3, and $B_m$ is both rank unimodal and rank symmetric, we apply Lemma 5.4 to obtain that $P_H$ and $B_m$ are compatible. Finally, since $P_H$ and $B_m$ both have the strong Sperner property, $P_G$ is Sperner by Theorem 5.3.

6. **Six Disjoint Edges**

With this section, we begin to establish Theorem 1.3 for several classes of graphs. Our first step in this direction is the following theorem which proves the conjecture for all graphs having a set of six disjoint edges. It is perhaps surprising, first that the number of disjoint edges required does not depend on $n$, and secondly, that it is small. Nevertheless, this result will help us to identify the exceptional graphs in Section 10.
THEOREM 6.1. Let G be a graph on 2n vertices which contains a set of six edges, no two of which are incident with the same vertex. Then $P_G$ has the Sperner property.

We begin by showing that the spanning subgraph $H = 6K_2 + Z_{2n-12}$ of $G$ satisfies the technical requirements of Theorem 4.1.

LEMMA 6.2. Let H be a graph on 2n vertices having exactly six edges, no two of which are incident with the same vertex. Then

1. $|P_H_n| \geq |P_{n+1}|$, and
2. for any two isolated vertices x and y of H, $H \setminus \{x, y\}$ has no more than $\sum_{i=2}^{n-1} \binom{2i-1}{i}$ independent sets of size $n - 1$.

PROOF. Let $P = P_H$ and let $a_m$ be the number of independent sets in $H$ of size $m$. We have $|P_m| = \binom{2n}{m} - a_m$, and by routine calculation,

$$a_m = \sum_{i=0}^{6} 2^i \binom{6}{i} \binom{2n-12}{m-i}.$$ 

Thus, after expanding the binomial coefficients, the inequality $|P_n| \geq |P_{n+1}|$ is seen to be equivalent to

$$451n^5 - 4276n^4 - 379n^3 + 111664n^2 - 364500n + 332640 \geq 0$$

which may be verified to hold for $n \geq 6$.

To prove the second condition, we need to show that

$$a_{n-1}(H \setminus \{x, y\}) \leq \sum_{i=2}^{n-1} \binom{2i-1}{i},$$

where x and y are any two isolated vertices of $H$. This inequality may be verified directly for $n = 7$. For $n \geq 8$, it may be checked that $729 \left( \binom{2n-14}{n-7} \right) \leq \left( \binom{2n-14}{n-1} \right)$ and now (1) holds since $\sum_{i=0}^{6} 2^i \binom{6}{i} \binom{2n-14}{n-1-i} \leq 729 \binom{2n-14}{n-1}$ by unimodality of the binomial coefficient.

To complete the proof of Theorem 6.1 using Theorem 4.1, it remains to show that $P_H$ has the Sperner property if $H = 6K_2 + Z_{2n-12}$. This follows immediately from the following result and the fact that the LYM property implies the Sperner property.

THEOREM 6.3 (HORROCKS [10]). Let G consist of $nK_2$, $n \geq 1$ together with any number of isolated vertices. Then $P_G$ has the LYM property and logarithmically concave Whitney numbers.

7. GRAPHS WITH EXACTLY EIGHT VERTICES

Let $G$ be a graph having exactly eight vertices and at least one edge. In this section we show that $P_G$ has the Sperner property.

We will require the following lemma which may be found in Bollobas [2].

LEMMA 7.1. Let $G$ be a bipartite graph and suppose that the vertices of $G$ may be partitioned into two classes of sizes $m$ and $n$. If $G$ does not contain $K_{s,t}$, an induced subgraph then the number of edges in $G$ is strictly less than

$$(s - 1)^{1/t} (n - t + 1) m^{1-1/t} + (t - 1) m.$$
We are now ready to state and prove the main result of this section.

**Theorem 7.2.** Let $G$ be a graph having exactly eight vertices and at least one edge. Then $P_G$ has the Sperner property.

**Proof.** Let $P_2, P_3, \ldots, P_8$ be the ranks of $P_G$. By Corollary 3.5, it suffices to prove the existence of a matching between $P_4$ and $P_5$. Let us suppose to the contrary that there is no such matching.

If $|P_4| \leq |P_5|$ then there is $A \subseteq P_4$ with $|A| > |N(A)|$. Letting $B = P_5 \setminus N(A)$, we have $N(B) = P_4 \setminus A$ so $|B| > |N(B)|$.

On the other hand, if $|P_5| < |P_4|$ then the nonexistence of a matching between $P_4$ and $P_5$ implies that there is a $B \subseteq P_5$ with $|B| > |N(B)|$. Setting $A = P_4 \setminus N(B)$, we have $N(A) = P_5 \setminus B$ so $|A| > |N(A)|$.

Thus, in either case, we have $A \subseteq P_4$ and $B \subseteq P_5$ with $|A| > |N(A)|$ and $|B| > |N(B)|$. Moreover, $N(A) \cup B = P_5$ and $N(A) \cap B = \emptyset$ so

$$|N(A)| + |B| = |P_5| \leq \binom{8}{5}.$$  \hspace{1cm} (2)

Since $|A| > |N(A)|$, by Lemma 4.2, we have

$$|A| > \left(\binom{7}{4} + \binom{5}{3} + \binom{3}{2}\right).$$

Thus

$$|A| \geq \left(\binom{7}{4} + \binom{5}{3} + \binom{3}{2} + \binom{1}{1}\right)$$

so, by the Kruskal–Katona theorem,

$$|N(A)| \geq \left(\binom{7}{3} + \binom{5}{2} + \binom{3}{1} + \binom{1}{0}\right) = 49.$$  \hspace{1cm} (3)

Let $G$ be the bipartite subgraph of $(P_4, P_3)$ induced by $B \cup N(B)$. Since any pair of 5-sets intersect in at most one 4-set, $G$ has no induced $K_{2,2}$. Furthermore, as the degree in $G$ of any element of $B$ is at least three, the number of edges in $G$ is at least $3|B|$, so by Lemma 7.1

$$3|B| < (|N(B)| - 1)\sqrt{|B|} + |B|.$$  

Therefore $|N(B)| > 2\sqrt{|B|} + 1$ and since $|B| > |N(B)|$, we have

$$|B| \geq \left[2\sqrt{|B|}\right] + 2.$$

Checking the initial values, we see that

$$|B| \geq 8.$$  \hspace{1cm} (4)

But now summing inequalities (3) and (4) contradicts (2), completing the proof. \hfill $\Box$

8. **Star Graphs with an Extra Edge**

To show that $P = P_G$ has the Sperner property, where $G$ is a graph on $2n$ vertices, it suffices, by Corollary 3.5, to show that there is a matching between the ranks $P_n$ and $P_{n+1}$.
Unfortunately, as we have noted, it is often difficult to construct such a matching since we do not know in advance the relative sizes of \( P_n \) and \( P_{n+1} \).

In this section, however, we present a class of graphs for which it is possible to explicitly describe the required matching. Our construction uses the method of Greene and Kleitman [4] for finding an explicit partition of \( B_n \) into symmetric chains.

For each subset \( X \) of \([n]\), let \( l(X) = a_1a_2\ldots a_n \) be its characteristic sequence defined by \( a_i = 1 \) if \( i \in X \) and \( a_i = 0 \) otherwise. In \( l(X) \), put brackets around each occurrence of a 0 immediately followed by a 1. Continue this procedure for as long as possible, bracketing a 01 pair if they are separated only by previously bracketed elements. For example, for \( X = \{1, 3, 4, 5, 9, 10, 15, 17, 18\} \), an element of \( B_{19} \), we obtain

\[
1(01)110(0(01)1)00(0(01)(01)1)0.
\]

A digit is called paired if it appears in some bracket and unpaired otherwise. Notice that after the bracketing procedure has been completed, the unpaired digits consist of 1s followed by 0s.

Let \( X \) and \( Y \) be elements of \( B_n \) giving rise to the same bracketing, that is, their paired digits are identical. By the above observation regarding unpaired digits, we have \( X \subseteq Y \) or \( Y \subseteq X \) so \( X \) and \( Y \) are comparable. Therefore, the subset of elements of \( B_n \) having a given bracketing form a chain. It may be shown that the collection of chains \( C \) obtained from considering all forms of bracketing is a chain partition of \( B_n \). We shall call this collection of chains the standard chain partition of \( B_n \). Given a bracketing, the elements in the corresponding chain from bottom to top may be obtained by first setting all the unpaired digits to 0 and then changing the unpaired digits to 1s, one at a time from left to right.

For \( X \in B_n \), we define the successor of \( X \), denoted \( \text{succ}(X) \), to be the element in the same chain of \( C \) as \( X \) which covers \( X \). If \( X \) is the top element in its chain then \( \text{succ}(X) \) is undefined.

Before proving the main result of this section, we make the following definition.

**Definition 8.1.** A graph \( G \) is starry if there is a vertex of \( G \) which is incident with every edge except possibly one.

**Theorem 8.2.** Let \( G \) be a graph on 2n vertices. If \( G \) is starry then \( P = P_G \) has the Sperner property and \( |P_n| \leq |P_{n+1}| \).

**Proof.** Assign the labels 1, 2, \ldots, 2n to the vertices of \( G \) as follows. Give the label 1 to a vertex which is incident with every edge except at most one. To the ends of the edge (if it exists) which is not incident with vertex 1, assign the labels 2 and 3. Label the remaining vertices in any fashion so that \( V(G) = [2n] \).

Let \( X \) be any element of \( P_n \). Thus \( G[X] \) contains at least one edge so \( 1 \in X \) or \( \{2, 3\} \subseteq X \). If \( 1 \in X \) then the first position of \( l(X) \) contains a 1 and this must be an unpaired digit. If \( \{2, 3\} \subseteq X \) then \( l(X) \) begins with \( a_1a_2a_3 \ldots \) and so the 1 in position 3 is unpaired, regardless of the value of \( a_1 \).

In either event, \( l(X) \) contains unpaired digits so in the symmetric chain partition \( C \) of \( B_n \), \( X \) is the middle element of a chain from a \((2n - j)\)-set to a \((2n + j)\)-set for some \( j \geq 1 \). In particular, \( \text{succ}(X) \) exists and is an element of \( P_{n+1} \).

Since every \( n \)-set may be matched to an \((n + 1)\)-set, we have \( |P_n| \leq |P_{n+1}| \) and that \( P \) is Sperner by Corollary 3.5.\( \square \)

9. **Graphs Having at Most Seven Nonisolated Vertices**

The purpose of this section is to prove Theorem 1.3 for graphs having no more than seven nonisolated vertices. Roughly speaking, this serves to eliminate small graphs from consideration.
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in Section 10. The proof involves the use of our second technique, namely Theorem 5.1, and the following lemma.

**Lemma 9.1.** Let $G$ be a graph with corresponding poset $P_G$. If $|V(G)| = 7$ then $P_G$ has the strong Sperner property.

Our main result now follows immediately.

**Theorem 9.2.** Let $G$ be a graph with corresponding poset $P_G$. If $G$ has no more than seven nonisolated vertices then $P_G$ has the Sperner property.

**Proof.** We may write $G = H + Z$, where $|V(H)| = 7$ and $Z$ is a graph with $|V(H)| - 7$ isolated vertices. By Lemma 9.1, $P_H$ has the strong Sperner property and the result follows from Theorem 5.1.

The remainder of this section, is devoted to proving Lemma 9.1. Accordingly, let $G$ be a graph on seven vertices and let $P_2, P_3, \ldots, P_7$ be the ranks of its corresponding poset $P_G$. By virtue of Section 8, we may suppose that $G$ has a pair of disjoint edges. To show that $P_G$ is strongly Sperner, we will prove that it is $k$-Sperner for all $k$.

In order to show that $P_G$ is 2-Sperner, we will require the following theorem.

**Theorem 9.3 (Proctor et al. [16]).** A rank-unimodal poset $P$ is $k$-Sperner if and only if $P \times C_k$ is Sperner where $C_k$ denotes the chain of size $k$.

**Proposition 9.4.** $P_G$ is both Sperner and 2-Sperner.

**Proof.** By Corollary 3.4, $P_G$ has the Sperner property. To show that $P_G$ is 2-Sperner, consider the graph $\hat{G}$ on eight vertices obtained from $G$ by adding an isolated vertex. Thus $\hat{P}_G \cong P_G \times B_1$ by Proposition 5.2. Now from Section 7 we have that $\hat{P}_G$ is Sperner, and since $B_1$ is isomorphic to a 2-element chain, it follows from Theorem 9.3 that $P_G$ is 2-Sperner.

To show that $P_G$ is $k$-Sperner for $k \geq 3$, we will proceed directly from the definition. Recall that a poset is $k$-Sperner if the maximum size of a $k$-family is equal to $S_k$, the sum of the $k$ largest Whitney numbers. Since the union of any $k$ ranks forms a $k$-family, the maximum size of a $k$-family is at least $S_k$. Therefore, to establish the $k$-Sperner property, it suffices to show that the size of any $k$-family does not exceed $S_k$. Given any $k$-family $A$, our approach will be to show that there is another $k$-family, having $|A|$ members, which is contained in some $k$ ranks of the poset. To do this will involve repeated use of the Lemma 9.5.

We first introduce some notation. Let $P_i$ and $P_j$ be distinct ranks of the ranked poset $P$, and suppose that $A \subseteq P_i$ and $B \subseteq P_j$. The notation $A \rightarrow B$ will indicate the existence of disjoint chains $C_k = (x_k, \ldots, y_k), k = 1, \ldots, |A|$ where $x_k \in A, y_k \in B$, and $|C_k| = |i - j|$.  

**Lemma 9.5.** Let $P$ be a ranked poset and let $A \subseteq P$ be a $k$-family. Let $Q_0$ be the rank of lowest (highest) index which intersects $A$ nontrivially. Let $Q_1, Q_2, \ldots, Q_k$ be the ranks of $P$ immediately above (below) $Q_0$. If 

$$A \cap Q_0 \rightarrow Q_1 \rightarrow Q_2 \rightarrow \cdots \rightarrow Q_k$$

then there is a $k$-family $A'$ in $P$ with $|A'| = |A|$ and $A' \cap Q_0 = \emptyset$.  

PROOF. For definiteness, suppose that \( Q_0 \) is the rank of lowest index which intersects \( A \) nontrivially, and let \( A^{(0)}_1, A^{(0)}_2, \ldots, A^{(0)}_x \) be the elements of \( A \cap Q_0 \). By hypothesis, there exist disjoint chains \( C_1, C_2, \ldots, C_x \), where

\[
C_i = (A^{(0)}_i < A^{(1)}_i < \cdots < A^{(k)}_i)
\]

such that \( A^{(j)}_i \in Q_j \) for all \( 0 \leq j \leq k \).

Observe that \( C_i \not\subseteq A \) since \( |C_i| = k + 1 \) and \( A \) is a \( k \)-family. Therefore, for each \( 1 \leq i \leq x \), let \( s_i \) be the smallest integer for which \( A^{(s_i)}_i \not\in A \). (Since \( A^{(0)}_i \in A \), evidently \( 1 \leq s_i \leq k \).)

Form \( A' \) from \( A \) by replacing each \( A^{(s_i)}_i \) with \( A^{(s_i)}_i \). Clearly \( |A'| = |A| \) and \( A' \cap Q_0 = \emptyset \) so it remains to show that \( A' \) is a \( k \)-family.

To this end, suppose that the chain

\[
C = (Y_1 < Y_2 < \cdots < Y_{k+1})
\]

is wholly contained in \( A' \). We have that \( C \) intersects \( S = \{A^{(s_1)}_1, \ldots, A^{(s_x)}_x\} \) nontrivially lest \( C \subseteq A \) so let \( r \) be the largest index such that \( Y_r = A^{(s)}_j \) for some \( r \) and \( j \). Furthermore, as each \( A^{(s_i)}_i \in Q_1 \cup Q_2 \cup \cdots \cup Q_k \) and \( C \) meets \( k + 1 \) ranks, we have \( C \not\subseteq S \) so \( r \leq k \).

A minimal element in \( A' \) is in the rank \( Q_1 \) or higher so, as \( Y_r = A^{(s)}_j \in Q_{s_j} \), we have \( s_j \geq r \). By the definition of \( r \), \( \{Y_{r+1}, Y_{r+2}, \ldots, Y_{k+1}\} \subseteq A \). Thus

\[
C' = (A^{(0)}_j < A^{(1)}_j < \cdots < A^{(s-1)}_j < Y_{r+1} < Y_{r+2} < \cdots < Y_{k+1})
\]

is a chain in \( A \) of size \( s_j + (k - r + 1) \geq k + 1 \). Since \( A \) is a \( k \)-family, this is a contradiction and the proof is complete. \(\square\)

Using Lemma 9.5, we now show that \( P_7 \) may be ignored when looking for a maximum size \( k \)-family in \( P_G \).

**Lemma 9.6.** Let \( G \) be a graph with 7 vertices and let \( P_2, P_3, \ldots, P_7 \) be the ranks of its corresponding poset \( P_G \). Suppose that \( A \subseteq P_G \) is a \( k \)-family for some \( 1 \leq k \leq 5 \). Then there is a \( k \)-family \( A' \) of \( P_G \) with \( |A'| = |A| \) and \( A' \cap P_7 = \emptyset \).

**Proof.** For definiteness, suppose that \( V(G) = [7] \) and that \( (1, 2) \) is an edge of \( G \). We may match \( P_7 \to P_2 \) using the chain

\[
1234567 \to 123456 \to 12345 \to 1234 \to 123 \to 12
\]

and now the result follows from Lemma 9.5. \(\square\)

**Lemma 9.7.** \( P_G \) is 3-Sperner.

**Proof.** Let \( A \) be a 3-family in \( P_G \). Using the standard chain partition, it is possible to match \( P_2 \to P_3 \) so by Lemma 9.5, we may assume \( A \cap P_2 = \emptyset \). Moreover, we take \( A \cap P_7 = \emptyset \) by Lemma 9.6.

Finally we show that it is possible to match \( P_6 \to P_3 \). Suppose that \( V(G) = [7] \) and that \( (1, 2) \) and \( (3, 4) \) are edges. Thus a maximum independent set in \( G \) has cardinality at most 5.
so $P_G$ consists of all 6-subsets of $V(G)$. The following chains demonstrate that $P_6$ may be matched to $P_3$.

\[
\begin{array}{cccc}
123456 & \rightarrow & 12345 & \rightarrow & 1234 & \rightarrow & 123 \\
123457 & \rightarrow & 12347 & \rightarrow & 1237 & \rightarrow & 127 \\
123467 & \rightarrow & 12346 & \rightarrow & 1236 & \rightarrow & 126 \\
123567 & \rightarrow & 12356 & \rightarrow & 1235 & \rightarrow & 125 \\
124567 & \rightarrow & 12456 & \rightarrow & 1245 & \rightarrow & 124 \\
134567 & \rightarrow & 13456 & \rightarrow & 1345 & \rightarrow & 134 \\
234567 & \rightarrow & 23456 & \rightarrow & 2345 & \rightarrow & 234 \\
\end{array}
\]

Therefore, we may take $A \cap P_6 = \emptyset$ by Lemma 9.5. Thus $A \subseteq P_3 \cup P_4 \cup P_5$ so $P$ is 3-Sperner.

\[\square\]

**LEMMA 9.8.** $P_G$ is 4-Sperner.

**PROOF.** Let $A$ be a 4-family. By Lemma 9.6, we may suppose that $A \cap P_7 = \emptyset$ so $A \subseteq P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6$. If either $A \cap P_2 = \emptyset$ or $A \cap P_6 = \emptyset$ then $A$ is contained in the union of four ranks and we are finished. Therefore, suppose that $A \cap P_2$ and $A \cap P_6$ are both nonempty.

If $|A \cap P_2| \leq 4$ then $A \cap P_2$ may be matched to $P_6$ as follows. Using the standard chain partition of $B_7$, we may obtain disjoint chains from $A \cap P_2$ up to $P_3$. The 5-sets at the top of these chains may be matched to $P_6$ by the Kruskal–Katona theorem (see [5], for example) since their number is at most $\binom{5}{2} + \binom{4}{1} = 4$. Therefore, the disjoint chains from $A \cap P_2$ to $P_3$ may be extended to the rank $P_6$ so we may take $A \cap P_2 = \emptyset$ by Lemma 9.5.

Conversely, suppose that $|A \cap P_2| \geq 5$. Let $A \cap P_2 = \{X_1, X_2, \ldots, X_r\}$ and suppose that $i \in X_j$ for all $1 \leq j \leq r$. Now the sets $X_j \setminus \{i\}$, $j = 1, \ldots, r$ may be viewed as singleton subsets in $B_6$. The standard chain partition of $B_6$ yields disjoint chains from these singletons to the 5-subsets. Adding the element $i$ back to every set in this family of chains gives a collection of disjoint chains matching $A \cap P_2 \rightarrow P_6$ in the poset $P_G$. Therefore, we may again take $A \cap P_2 = \emptyset$ by Lemma 9.5.

Otherwise, there is a pair of disjoint sets among $X_1, X_2, \ldots, X_r$ which we make take, without loss of generality, to be $(1, 2)$ and $(3, 4)$. Consider the following chains in $P_G$.

$C_1 = (12 < 123 < 1234 < 12345 < 123456)$

$C_2 = (12 < 124 < 1245 < 12456 < 124567)$

$C_3 = (12 < 125 < 1235 < 12356 < 123567)$

$C_4 = (12 < 126 < 1236 < 12367 < 123467)$

$C_5 = (12 < 127 < 1237 < 12347 < 123457)$

$C_6 = (34 < 134 < 1345 < 13456 < 134567)$

$C_7 = (34 < 234 < 2345 < 23456 < 234567).$

Since $A$ is a 4-family and $|C_i| = 5$, we have $C_i \notin A$. Since the bottom element of $C_i$ is in $A$, at most three of the other four elements of $C_i$ can be in $A$. Therefore

$$|A| \leq \sum_{i=2}^{6} |P_i| - 7 = \sum_{i=2}^{5} |P_i|$$

since $|P_6| = 7$. The proof is now complete. \[\square\]

**LEMMA 9.9.** $P_G$ is $k$-Sperner for all $k \geq 5$. 

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Let \( A \) be a \( k \)-family for some \( k \geq 5 \). If \( k = 5 \) then, by Lemma 9.6, we may assume that \( A \cap P_5 = \emptyset \) so \( A \subseteq \bigcup_{i=2}^{5} P_i \). That is, \( A \) is contained in the union of five ranks and the result follows. If \( k \geq 6 \) then the result is immediate since \( P_G \) has only six ranks. \( \square \)

10. THE EXCEPTIONAL GRAPHS

Let \( G \) be a graph on \( 2n \) vertices. We have shown that if \( G \) contains a \( 6K_2 \), or has exactly eight vertices, or is starry, or has no more than seven nonisolated vertices then \( P_G \) has the Sperner property. To complete the proof of Theorem 1.3, we consider, in this section, those graphs having none of the above properties. The argument is case analytic and uses the table of special graphs listed in the Appendix (Section A.3). For the special graph \( G \), let \( a_i(G) \) be the number of independent sets of size \( i \) in \( G \). We will require the generating function \( \sum_{i \geq 0} a_i(G)x^i \) in the Appendix (Section A.2) where we verify certain technical conditions regarding the Whitney numbers of \( P_G \). Therefore, these generating functions are also provided in the table.

Accordingly, let \( \mathcal{E} \) be the set of all graphs on \( 2n \geq 10 \) vertices which are not starry, do not contain a \( 6K_2 \), and which have at least eight nonisolated vertices. For each \( G \in \mathcal{E} \), we will show that one of the following alternatives must hold.

1. \( G = G_i + Z_m \) where \( G_i \) is a special graph and \( m \) is a positive integer. In this case, our strategy is to show that the poset corresponding to \( G_i \) (if \( |V(G_i)| \) is odd) or \( G_i + K_1 \) (if \( |V(G_i)| \) is even) is strongly Sperner, and then deduce the spernerity of \( P_G \) using Theorem 5.1.

2. For some special graph \( G_i \), \( H = G_i + Z_{2n-|V(G_i)|} \) is a spanning subgraph of \( G \). We then show that \( H \) satisfies the hypotheses of Theorem 4.1 and thus conclude that \( P_G \) is Sperner. Let us call the graph \( G_i \) a pillar if \( H = G_i + Z_{2n-|V(G_i)|} \) satisfies the hypotheses of Theorem 4.1. Thus, if \( G \) contains a pillar, then \( P_G \) has the Sperner property.

We now describe the structure of the case analytic argument used to establish Theorem 1.3 for the graphs in \( \mathcal{E} \). Since no graph in \( \mathcal{E} \) contains a \( 6K_2 \), it is possible to characterize the graphs in \( \mathcal{E} \) by the size of a maximum matching using the defect form of Tutte’s one-factor theorem (see [2], for example). We, however, opt for a generic characterization, namely the number of nontrivial components. Since \( G \in \mathcal{E} \) does not contain a \( 6K_2 \), it cannot have more than five nontrivial components. Using this characterization of \( \mathcal{E} \), the case analytic argument is easily seen to be exhaustive.

We now present the analysis using five cases. For brevity, we shall merely state, for example, that a particular special graph \( G_i \) is a pillar. The technical task of verifying such a statement is described in the appendix.

Some explanation regarding notation used in this section is in order. We denote the path on \( n \) vertices by \( P_n \), and \( C(n, m) \) refers to the set of all connected graphs with \( n \) vertices and \( m \) edges. For the nontrivial component \( C_i \) of \( G \), we will denote \( |V(C_i)| \) by \( v_i \).

10.1. \( G \) has exactly one nontrivial component. We will require the following lemma.

**Lemma 10.1.** Let \( G \) be a connected graph on at least eight vertices which is not starry. Then \( G \) contains a subgraph \( T \) on eight vertices which is a tree and not starry.

**Proof.** We note first of all that a starry tree on eight vertices has a vertex of degree at least six. The lemma follows immediately then, if \( G \) has maximum degree less than six. Therefore, let us assume that there is a vertex \( v \in V(G) \) with \( d(v) \geq 6 \).

**Proof.** Let \( A \) be a \( k \)-family for some \( k \geq 5 \). If \( k = 5 \) then, by Lemma 9.6, we may assume that \( A \cap P_5 = \emptyset \) so \( A \subseteq \bigcup_{i=2}^{5} P_i \). That is, \( A \) is contained in the union of five ranks and the result follows. If \( k \geq 6 \) then the result is immediate since \( P_G \) has only six ranks. \( \square \)
Suppose that \( G \setminus v \) has at least two nontrivial components and that \( v_1, v_2 \in N(\{v\}) \) are in different such components of \( G \setminus v \). Then there exist vertices \( x, y \in V(G \setminus v) \) such that \( (v_1, x) \) and \( (v_2, y) \) are disjoint edges. Now \( xyv_1v_2y \) is a path of length 4 in \( G \) which may be extended to a tree \( T \) on eight vertices. Observe that such a tree will not be starry since a starry tree has diameter at most three.

Conversely, suppose that \( G \setminus v \) has exactly one nontrivial component \( C \) containing vertex \( v_1 \in N(\{v\}) \). Since \( G \) is not starry, \( C \) contains at least two edges and it follows that there is a path \( P \) in \( C \) of length 2 which contains \( v_1 \). If \( P \) has the form \( v_1xy \) then \( v_2uv_1xy \), where \( v_2 \) is any vertex in \( N(\{v\}) \setminus \{x, y\} \) is a path of length 4 in \( G \) and may be extended to a tree on eight vertices which is not starry as above. Otherwise \( P \) has the form \( x v_1 y \). Let \( v_2, v_3 \) be any vertices in \( N(\{v\}) \setminus \{x, y\} \). Then the subgraph \( H \) of \( G \) with \( V(H) = \{v, v_1, v_2, v_3, x, y\} \) and \( E(H) = \{(v, v_1), (v, v_2), (v, v_3), (v_1, x), (v_1, y)\} \) is a tree which is not starry. Therefore, \( H \) may be extended to a tree on eight vertices which is not starry. This completes the proof of the lemma.

In this case, we are supposing that \( G \) has exactly one nontrivial component. Thus \( G \) contains a subgraph of the form \( H + K_1 \) where \( H \) is a tree on eight vertices. Since \( G \) is not starry, by Lemma 10.1, we may assume that \( H \) is also not starry. There are 23 trees on eight vertices, two of which are starry. Therefore, there are 21 possibilities for \( H + K_1 \), namely graphs \( G_1, G_2, \ldots, G_{21} \). Graphs \( G_1 \) to \( G_{13} \) are pillars for \( n \geq 5 \) so if \( G \) contains any of these graphs then \( P_G \) has the Sperner property.

Otherwise, \( G \) contains one of the subgraphs \( G_{19}, G_{20}, \) or \( G_{21} \). One of the following three possibilities must hold.

1. \( G = L + Z_{2n-9} \) where \( L \) is \( G_{19}, G_{20}, \) or \( G_{21} \). In each case, \( P_L \) is strongly Sperner so Theorem 5.1 may be applied.

2. \( G \) contains a subgraph \( H \in C(9, 8) \) and \( H \) contains \( G_{19}, G_{20}, \) or \( G_{21} \). There are 10 possibilities for \( H \) (graphs \( G_{22} \) to \( G_{31} \)) which are all pillars for \( n \geq 5 \) so \( P_G \) is Sperner.

3. \( G \) contains a subgraph \( H = L + K_1 \) where \( L \in C(8, 8) \) and \( H \) contains \( G_{19}, G_{20}, \) or \( G_{21} \). There are 15 possibilities for \( H \) (graphs \( G_{32} \) to \( G_{46} \)) which are all pillars for \( n \geq 5 \) so \( P_G \) is Sperner.

10.2. \( G \) has exactly two nontrivial components. Let \( C_1, C_2 \) be the two nontrivial components of \( G \), and without loss of generality suppose that \( v_1 \geq v_2 \).

10.2.1. \( v_2 = 2 \). Then \( v_1 \geq 6 \) so exactly one of the following four possibilities must hold.

1. \( G \) contains \( P_6 + K_2 + K_1 \) as a subgraph. The graph \( G_{47} = P_6 + K_2 + K_1 \) is a pillar for all \( n \geq 5 \) so \( P_G \) is Sperner.

2. \( G \) consists of \( L = H + K_2 + K_1 \) together with \( 2n - 9 \) isolated vertices, where \( H \in C(6, 5) \setminus P_6 \). There are five possibilities for \( L \) (graphs \( G_{48} \) to \( G_{52} \)) and in each case, \( P_L \) is strongly Sperner so Theorem 5.1 may be applied.

3. \( G \) contains a subgraph of the form \( L = H + K_2 + K_1 \) where \( H \in C(6, 6) \) and \( H \) does not contain \( P_6 \). There are seven possibilities for \( L \) (graphs \( G_{53} \) to \( G_{59} \)) which are all pillars for \( n \geq 5 \), except \( G_{59} \).

Suppose that \( G \) contains \( G_{59} \) as a subgraph. If \( G = G_{59} + Z_{2n-9} \) then \( P_G \) is Sperner by Theorem 5.1 since \( P_{G_{59}} \) is strongly Sperner. Otherwise, \( G \) contains one of the graphs \( G_{60} \) to \( G_{64} \) as a subgraph. These 5 graphs are all pillars for \( n \geq 5 \) so \( P_G \) is Sperner.
(4) $G$ contains a subgraph of the form $L = H + K_2$, where $H \in \mathcal{C}(7, 6)$ and $H$ does not contain $P_6$. Moreover, we may take $H$ to be different from $K_{1,6}$, lest $G$ itself be starry. Therefore, there are seven possibilities for $L$ (graphs $G_{65}$ to $G_{71}$) which are all pillars for $n \geq 5$, except $G_{71}$.

Suppose that, of the graphs $G_{65}$ to $G_{71}$, $G$ contains only $G_{71} = H_0 + K_2$. If $G = G_{71} + Z_{2n-9}$ then $P_G$ is Sperner by Theorem 5.1 since $P_{G_{71}}$ is strongly Sperner. Otherwise, adding an edge only to $H_0$ results in graph $G_{62}$, $G_{63}$, $G_{64}$, or $G_{72}$, while adding an edge and a vertex gives one of the graphs $G_{73}$ to $G_{76}$. These eight graphs are all pillars for $n \geq 5$ so $P_G$ is Sperner.

10.2.2. $v_2 = 3$. Then $C_2$ equals $K_3$ or $P_3$, and $v_1 \geq 5$. Exactly one of the following two possibilities must hold.

1. $G$ consists of $L = H + K + K_1$ together with $2n - 9$ isolated vertices, where $H$ is a tree on five vertices and $K$ equals $K_3$ or $P_3$. There are six possibilities for $L$ (graphs $G_{77}$ to $G_{82}$) and in each case $P_L$ is strongly Sperner so Theorem 5.1 may be applied.

2. $G$ contains a subgraph of the form $L_1 = H + P_3 + K_1$ where $H \in \mathcal{C}(5, 5)$, or $L_2 = K + P_3$ where $K \in \mathcal{C}(6, 5)$.

If $G$ contains $G_{77} = P_3 + P_3 + K_1$ then $P_G$ is Sperner since $G_{77}$ is a pillar for all $n \geq 5$.

Suppose, to the contrary, that $G$ does not contain $G_{77}$. Then there is only a single possibility for $L_1$, namely $G_{83}$, and three possibilities for $L_2$, namely $G_{84}$, $G_{85}$, and $G_{86}$. Graphs $G_{83}$ to $G_{86}$ are all pillars for $n \geq 5$ so $P_G$ is Sperner.

10.2.3. $v_2 = 4$. In this case $v_1 \geq 4$, and it may be seen that either $G = G_{87} + Z_{2n-9}$ or $G = G_{88} + Z_{2n-9}$ or that $G$ contains one of the graphs $G_{89}$ to $G_{93}$. The graphs $G_{87}$ and $G_{88}$ yield strongly Sperner posets and Theorem 5.1 may be applied. On the other hand, the graphs $G_{89}$ to $G_{93}$ are all pillars for $v_1 \geq 5$.

10.2.4. $v_2 \geq 5$. Then $v_1 \geq 5$ so both $C_1$ and $C_2$ contain a tree on five vertices. It may be verified that $G$ contains one of the graphs $G_{93}$, $G_{94}$, or $G_{95}$ which are all pillars for $n \geq 5$.

10.3. $G$ has exactly three nontrivial components. Let $C_1$, $C_2$, $C_3$ be the three nontrivial components of $G$, and without loss of generality suppose that $v_1 \geq v_2 \geq v_3$.

10.3.1. $v_2 = v_3 = 2$. Then $v_1 \geq 4$. Suppose that $C_1$ has three edges. Then $G$ is of the form $H + Z_{2n-9}$ where $H = C_1 + 2K_2 + K_1$ and $C_1$ is a tree on four vertices. Thus, there are two possibilities for $H$ (graphs $G_{96}$ and $G_{97}$) and both result in $P_H$ being strongly Sperner so Theorem 5.1 applies.

If $C_1$ has four edges then $G = H + Z_{2n-9}$ where $H = L_1 + 2K_2 + K_1$ and $L_1 \in \mathcal{C}(4, 4)$ or $H = L_2 + 2K_2$ where $L_2 \in \mathcal{C}(5, 4)$. There are two choices for $L_1$ yielding graphs $G_{98}$ and $G_{99}$ and both are pillars for $n \geq 5$. There are three choices for $L_2$ which give the graphs $G_{100}$, $G_{101}$, and $G_{102}$. Graphs $G_{100}$ and $G_{101}$ are pillars but $G_{102}$ is not. However, $G_{102}$ does yield a strongly Sperner poset so Theorem 5.1 applies.

Finally, if $G$ contains $G_{102}$ but does not equal $G_{102} + Z_{2n-9}$ then $G$ contains one of the graphs $G_{103}$, $G_{104}$, or $G_{105}$. These three graphs may be verified to be pillars for all $n \geq 5$.
10.3.2. \( v_2 = 3, \ v_3 = 2 \). Then \( v_1 \geq 3 \) so exactly one of the following two possibilities must hold.

1. \( G \) consists of \( G_{106} \) together with \( 2n - 9 \) isolated vertices. It may be shown that \( P_{G_{106}} \)
is strongly Sperner so Theorem 5.1 applies.
2. \( G \) contains \( G_{107}, G_{108}, \) or \( G_{109} \) as a subgraph, and all three graphs are pillars for \( n \geq 5 \).

10.3.3. \( v_3 \geq 3 \). Then \( v_1, v_2 \geq 3 \) so \( G \) contains \( G_{110} = 3P_3 \) as a subgraph which is a pillar for \( n \geq 5 \).

10.4. \( G \) has exactly four nontrivial components. \( G \) has exactly four nontrivial components of \( G \), and without loss of generality, support that \( v_1 \geq v_2 \geq v_3 \geq v_4 \).

10.4.1. \( v_2 = v_3 = v_4 = 2 \). If \( v_1 = 2 \) then \( G = 4K_2 + Z_{2n-8} \) so \( P_G \) has the LYM property by Theorem 6.3.
If \( v_1 = 3 \) then \( G = H + Z_{2n-9} \) where \( H = G_{111} \) or \( G_{112} \). In either case, \( P_H \) is strongly Sperner so Theorem 5.1 may be applied.
If \( v_1 \geq 4 \) then \( G \) contains either \( G_{113} \) or \( G_{114} \) as a subgraph, and both of these graphs are pillars for \( n \geq 5 \).

10.4.2. \( v_2 = v_3 = v_4 = 2 \). Then \( v_1 \geq 3 \) so \( G \) contains \( G_{115} \) as a subgraph which is a pillar for \( n \geq 5 \).

10.4.3. \( v_3 \geq 3 \). Then \( v_1, v_2 \geq 3 \) so \( G \) contains \( G_{110} = 3P_3 \) as a subgraph which is a pillar for \( n \geq 5 \).

10.5. \( G \) has exactly five nontrivial components. \( G \) has exactly five nontrivial components of \( G \), and without loss of generality, support that \( v_1 \geq v_2 \geq v_3 \geq v_4 \).

References


**APPENDIX**

For sake of brevity, the appendix is not included here. It is available from the author upon request. The Appendix consists of three sections, the last of which contains a listing of all special graphs. The first two sections are devoted to verifying the details of the case analytic argument of Section 10. Specifically, Section A.1 discusses the problem of showing that $P_G$ is strongly Sperner where $G$ is a special graph, and Section A.2 deals with the task of verifying that $G$ is a pillar. For sake of brevity, we illustrate the details on only one particular special graph since the arguments may be adjusted to all other cases without difficulty. A complete listing of all necessary verifications is available from the author.

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D. G. C. Horrocks

Department of Mathematics and Computer Science,
University of Prince Edward Island,
Charlottetown, PEI,
Canada C1A 4P3

E-mail: dhorrock@upei.ca