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Linear Spaces of Real Matrices of Constant Rank

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ABSTRACT

The largest possible dimensions of linear spaces of real $n \times n$ matrices of constant rank $n-1$ (or $n-2$) are determined using topological K-theory and expressed in terms of Hurwitz-Radon numbers.

1. INTRODUCTION

It is well known that in the space $M(m, n)$ of real matrices of order $m \times n$, for a fixed $k \leqslant \min(m, n)$, the set of rank k matrices is a smooth manifold $M(m, n; k)$ of dimension $mn - (m-k)(n-k)$. In this paper, we study *linear* subspaces of $M(m, n)$ contained in $M(m, n; k) \cup \{0\}$, with particular interest in the *largest* possible dimension of such subspaces:

$$l(m, n; k) := \max\{\dim V : V \subseteq M(m, n; k) \cup \{0\} \text{ is a linear subspace of } M(m, n).\} \qquad (1)$$

*Research partially supported by NSERC of Canada.
[†]Research partially supported by NSF Grant DMS-9201204.

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655 Avenue of the Americas, New York, NY 10010 0024-3795/93/$6.00

Note that $l(m, n; k)$ is an increasing function of m (and of n).

Assume $m \geqslant n$. Let $A_1, \dots, A_r$ be a basis of a linear subspace of $M(m, n; n) \cup \{0\}$. For a fixed $k \leqslant n$, let P be a fixed (projection) matrix in $M(n, n; k)$. Then $A_1 P, \dots, A_r P$ form a basis of a linear subspace in $M(m, n; k) \cup \{0\}$. Consequently,

$$l(m, n; k) \geqslant l(m, n; n). \tag{2}$$

The determination of $l(m, n; n)$ is equivalent to the *nonsingular bilinear map problem*: given $m \geqslant n$, to determine the largest possible r for the existence of a nonsingular bilinear map $f : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$f(x, y) = 0 \quad \Rightarrow \quad x = 0 \text{ or } y = 0. \tag{3}$$

For $m = n$, the solution was given by J. F. Adams [1], in his celebrated work on vector fields on spheres. For a given integer $n = 2^{4a+b}(2c + 1)$, $0 \leqslant b \leqslant 3$, define the *Hurwitz-Radon function* by

$$\rho(n) = 8a + 2^b. \tag{4}$$

THEOREM 1 (Adams [1]). $l(n, n; n) = \rho(n)$.

The main results of this paper are the determination of $l(n, n; n - 1)$, $l(n, n - 1; n - 2)$ and $l(n, n; n - 2)$.

THEOREM 2. *For* $n \geqslant 2$,

$$l(n, n; n - 1) = \begin{cases} \max\{ \rho(n), \rho(n \pm 1)\}, & n \neq 3, 7, \\ n, & n = 3, 7. \end{cases}$$

THEOREM 3. *For* $n \geqslant 3$,

$$l(n, n - 1; n - 2) = \begin{cases} \max\{ \rho(n), \rho(n \pm 1), \rho(n - 2)\}, & n \neq 3, 7, \\ 3, & n = 3, \\ 6, & n = 7. \end{cases}$$

THEOREM 4. *For* $n \geqslant 3$,

$$l(n, n; n - 2) = \begin{cases} \max\{ \rho(n), \rho(n \pm 1), \rho(n \pm 2)\}, & n \neq 3, 6, 7, \\ 3, & n = 3, \\ 6, & n = 6, 7. \end{cases}$$

These results are obtained by refining the method in Lam [5] of determining $l(n+1, n; n)$ and $l(n+2, n; n)$ (Lam uses different notation). For completeness, we record these numbers below.

THEOREM 5 (Lam [5]; see also Berger and Friedland [4] and Lam and Yiu [7].

(i) $l(n, n+1; n) = \max\{\rho(n), \rho(n+1)\}$.
(ii) $l(n, n+2, n) = \max\{3, \rho(n), \rho(n+1), \rho(n+2)\}$.

2. HURWITZ-RADON NUMBERS AND NORMED BILINEAR MAPS

We begin by recording some elementary properties of the Hurwitz-Radon function. For every positive integer n, let $v_2(n)$ be the unique integer such that $n = 2^{v_2(n)}(2m+1)$ for some integer m.

LEMMA 6.

(i) $\rho(n) = \rho(2^{v_2(n)})$.
(ii) $\rho(n) \leqslant n$. *Equality holds if and only if* $n = 1, 2, 4, 8$.
(iii) $\rho(2^k)$, $k = 0, 1, 2, \ldots,$ *is an increasing sequence.*
(iv) $n - \rho(n) \geqslant 2$ *except for* $n = 1, 2, 4, 8$.
(v) $n - 2\rho(n) \geqslant 2$ *except for* $n = 1, 2, 3, 4, 8, 16$.

As is well known, the Hurwitz-Radon number $\rho(n)$ arises as the largest possible number r for the existence of a *normed* bilinear map $f: \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$\|f(x, y)\| = \|x\| \|y\|, \qquad x \in \mathbb{R}^r, \quad y \in \mathbb{R}^n. \tag{5}$$

Let ϵ_i, $1 \leqslant i \leqslant r$ (respectively e_j, $1 \leqslant j \leqslant n$), be an orthonormal basis of $\mathbb{R}^r$ (respectively $\mathbb{R}^n$). A bilinear map $f: \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n$ can be conveniently tabulated by listing the images $f(\epsilon_i, e_j)$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$. Let A_i, $1 \leqslant i \leqslant r$, be the matrix of the induced linear map $f_i: \mathbb{R}^n \to \mathbb{R}^n$ [so that the jth row of A_i gives the image $f(\epsilon_i, e_j)$]. Then it is clear that A_i, $1 \leqslant i \leqslant r$, span an r-dimensional linear subspace of $M(n, n; n) \cup \{0\}$. Explicit constructions of normed bilinear maps of type $f: \mathbb{R}^{\rho(n)} \times \mathbb{R}^n \to \mathbb{R}^n$ have been given by various authors. See, for example, Lam and Liu [7]. It is well known that such normed bilinear maps can be constructed so that for each $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$, $f(\epsilon_i, e_j) = \pm e_{k(i,j)}$ for some integer $k = k(i, j)$. Equivalently, each of the matrices $A_1, \ldots, A_{\rho(n)}$ has entries $0, \pm 1$. In particular, one may even take $A_1 = I$, the identity matrix of order n, and if $\rho(n) \geqslant 2$, each of $A_2, \dots, A_{\rho(n)}$ to be *skew*.

EXAMPLE 7. Table 1 shows a normed bilinear map $\mathbb{R}^9 \times \mathbb{R}^{16} \to \mathbb{R}^{16}$. Note that $\rho(16) = 0$.

We remark that the first 8 rows and the first 8 columns restrict to a normed bilinear map $\mathbb{R}^8 \times \mathbb{R}^8 \to \mathbb{R}^8$ giving the 8-dimensional linear subspace of $M(8,8;8) \cup \{0\}$ consisting of the matrices

$$B_x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ -x_2 & x_1 & x_4 & -x_3 & x_6 & -x_5 & -x_8 & x_7 \\ -x_3 & -x_4 & x_1 & x_2 & x_7 & x_8 & -x_5 & -x_6 \\ -x_4 & x_3 & -x_2 & x_1 & x_8 & -x_7 & x_6 & -x_5 \\ -x_5 & -x_6 & -x_7 & -x_8 & x_1 & x_2 & x_3 & x_4 \\ -x_6 & x_5 & -x_8 & x_7 & -x_2 & x_1 & -x_4 & x_3 \\ -x_7 & x_8 & x_5 & -x_6 & -x_3 & x_4 & x_1 & -x_2 \\ -x_8 & -x_7 & x_6 & x_5 & -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}. \tag{6}$$

3. LOWER BOUNDS

Let $k \leqslant n$. It follows from (2) that $l(n,n;k) \geqslant l(n,n;n) \geqslant \rho(n)$. More generally, for every integer m in the range $k \leqslant m \leqslant n$, a $\rho(m)$-dimensional linear subspace in $M(m,m;m) \cup \{0\}$ gives rise to a subspace of $M(m,m;k) \cup \{0\}$, and (by appending to each matrix $n-m$ extra rows and $n-m$ extra columns of zeros) to a subspace of $M(n,n;k) \cup \{0\}$ of the same dimension. From this,

$$l(n,n;k) \geqslant \max\{\rho(m) : k \leqslant m \leqslant n\}. \tag{7}$$

LEMMA 8. *Let $f : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n$ be a normed bilinear map. Suppose there are linear subspaces $U, V \subseteq \mathbb{R}^n$ of dimensions h and k respectively, satisfying $f(x,U) \perp V$ for every $x \in \mathbb{R}^r$. Then $l(n-h, n-k; n-h-k) \geqslant r$.*

Proof. Choose orthonormal bases e_j, $1 \leqslant j \leqslant n$, and e'_j, $1 \leqslant j \leqslant n$, of $\mathbb{R}^n$ such that e_j, $n-h+1 \leqslant j \leqslant n$, and e'_j, $n-k+1 \leqslant j \leqslant n$, are bases of U and V respectively. For each (nonzero) $x \in \mathbb{R}^r$, consider the matrix A_x of the induced linear map $f_x : \mathbb{R}^n \to \mathbb{R}^n$ relative to these bases. The matrices

TABLE 1

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}
e_2	$-e_1$	e_4	$-e_3$	e_6	$-e_5$	$-e_8$	e_7	e_{10}	$-e_9$	$-e_{12}$	e_{11}	$-e_{14}$	e_{13}	e_{16}	$-e_{15}$
e_3	$-e_4$	$-e_1$	e_2	e_7	e_8	$-e_5$	$-e_6$	e_{11}	e_{12}	$-e_9$	$-e_{10}$	$-e_{15}$	$-e_{16}$	e_{13}	e_{14}
e_4	e_3	$-e_2$	$-e_1$	e_8	$-e_7$	e_6	$-e_5$	e_{12}	$-e_{11}$	e_{10}	$-e_9$	$-e_{16}$	e_{15}	$-e_{14}$	e_{13}
e_5	$-e_6$	$-e_7$	$-e_8$	$-e_1$	e_2	e_3	e_4	e_{13}	e_{14}	e_{15}	e_{16}	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$
e_6	e_5	$-e_8$	e_7	$-e_2$	$-e_1$	$-e_4$	e_3	e_{14}	$-e_{13}$	e_{16}	$-e_{15}$	e_{10}	$-e_9$	e_{12}	$-e_{11}$
e_7	e_8	e_5	$-e_6$	$-e_3$	e_4	$-e_1$	$-e_2$	e_{15}	$-e_{16}$	$-e_{13}$	e_{14}	e_{11}	$-e_{12}$	$-e_9$	e_{10}
e_8	$-e_7$	e_6	e_5	$-e_4$	$-e_3$	e_2	$-e_1$	e_{16}	e_{15}	$-e_{14}$	$-e_{13}$	e_{12}	e_{11}	$-e_{10}$	$-e_9$
e_9	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	$-e_{16}$	$-e_1$	e_2	e_3	e_4	e_5	e_6	e_7	e_8

$\{A_x : x \in \mathbb{R}^r\}$ form an r-dimensional linear subspace of $M(n, n; n) \cup \{0\}$. Indeed, if $x \neq 0$, then the rows of A_x are mutually orthogonal, and of the same length $\|x\|$. Note that A_x being a *square* matrix, its columns are also mutually orthogonal, and of the same length $\|x\|$. The submatrix B_x consisting of the first $n - h$ rows of A_x clearly has rank $n - h$. The $h \times k$ submatrix in the lower right hand corner of A_x being identically zero, each of the first $n - k$ columns of B_x is orthogonal to each of the rightmost k columns, which are mutually orthogonal and of the same length $\|x\|$. It follows that the $(n - h) \times (n - k)$ submatrix in the upper left hand corner of A_x has rank $n - h - k$. From this, we obtain a linear subspace of $M(n - h, n - k; n - h - k) \cup \{0\}$ of dimension r, and the proof is complete. ■

PROPOSITION 9. *If* $n \neq 1, 3, 7$, *then* $l(n, n; n - 1) \geqslant \max\{\rho(n), \rho(n \pm 1)\}$.

Proof. For $n \geqslant 2$, it follows from (7) that $l(n, n; n - 1) \geqslant \max\{\rho(n), \rho(n - 1)\}$. If $n \neq 3, 7$, then $\rho(n + 1) < n + 1$ by Lemma 6(ii). Consider a normed bilinear map $f : \mathbb{R}^{\rho(n+1)} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Clearly, there are 1-dimensional subspaces of $\mathbb{R}^{n+1}$, say spanned by unit vectors y and z, such that $f(x, y) \perp z$ for every $x \in \mathbb{R}^{\rho(n+1)}$. Indeed, one may choose $y = e_1$ and z to be any unit vector orthogonal to each $f(\epsilon_i, e_1)$, $1 \leqslant i \leqslant \rho(n + 1)$. With n replaced by $n + 1$ and $h = k = 1$ in Lemma 8, we obtain $l(n, n; n - 1) \geqslant \rho(n + 1)$ if $n \neq 1, 3, 7$. ■

PROPOSITION 10. *If* $n = 3, 7$, *then* $l(n, n; n - 1) \geqslant n$.

Proof. Let $W = \{x \in \mathbb{R}^8 : x_1 = 0\}$, and C_x, $x \in W$, be the skew 7×7 matrix obtained by deleting the bottom row and the rightmost column of B_x in (6). Since C_x is skew, rank C_x must be even. If $x \neq 0$, then

$$\operatorname{rank} C_x \geqslant \operatorname{rank} B_x - 2 = 6,$$

and indeed $\operatorname{rank} C_x = 6$. It follows that $\{C_x : w \in W\}$ is a 7-dimensional linear subspace of $M(7, 7; 6) \cup \{0\}$ and $l(7, 7; 6) \geqslant 7$. Similarly, $l(3, 3; 2) \geqslant 3$ by considering the 3×3 submatrix in the upper left hand corner of B_x in (6), with $x_1 = 0$. ■

PROPOSITION 11. *For* $n \neq 3, 7$,

$$l(n, n - 1; n - 2) \geqslant \max\{\rho(n), \rho(n \pm 1), \rho(n - 2)\}.$$

Proof. For $n \geqslant 3$, clearly, $l(n, n-1; n-2) \geqslant \rho(n-2)$. Also, by (2),

$$l(n, n-1; n-2) \geqslant l(n, n-1; n-1).$$

Clearly, $l(n, n-1; n-1) \geqslant \rho(n-1)$. Note that Lemma 8 is valid when one or both of h and k is zero. In particular, starting with a normed bilinear map of the Hurwitz type $\mathbb{R}^{\rho(n)} \times \mathbb{R}^n \to \mathbb{R}^n$, and $h = 0$, $k = 1$, we obtain $l(n, n-1; n-1) \geqslant \rho(n)$. Consequently,

$$l(n, n-1; n-2) \geqslant \max\{\rho(n), \rho(n-1), \rho(n-2)\}.$$

Now consider a normed bilinear map $f: \mathbb{R}^{\rho(n+1)} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. If $n \neq 3, 7$, then $(n+1) - \rho(n+1) \geqslant 2$ by Lemma 6(iv). Let $U = \operatorname{span}(e_1)$ and $V = \operatorname{span}(z_1, z_2)$, where z_1, z_2 are two linearly independent vectors orthogonal to $f(\epsilon_i, e_1)$, $1 \leqslant i \leqslant \rho(n+1)$. An application of Lemma 8 with n replaced by $n+1$ and $h = 1$, $k = 2$ yields $l(n, n-1; n-2) \geqslant \rho(n+1)$. This completes the proof of the proposition. ∎

PROPOSITION 12. *If $n \neq 3, 6, 7$, then*

$$l(n, n; n-2) \geqslant \max\{\rho(n), \rho(n \pm 1), \rho(n \pm 2)\}.$$

Proof. Clearly, $l(n, n; n-2) \geqslant \max\{\rho(n), \rho(n-1), \rho(n-2)\}$ by (7). For $n \geqslant 3$, consider a normed bilinear map of $\mathbb{R}^{\rho(n+2)} \times \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$. If $n \neq 6, 14$, then $(n+2) - 2\rho(n+2) \geqslant 2$ by Lemma 6(v). In these cases, we can choose 2-dimensional subspaces U and V of $\mathbb{R}^{n+2}$ satisfying $f(x, U) \perp V$ for every $x \in \mathbb{R}^{\rho(n+2)}$. Indeed, the same thing can also be done for $n = 14$: for the normed bilinear map $\mathbb{R}^9 \times \mathbb{R}^{16} \to \mathbb{R}^{16}$ in Example 7, we simply choose $U = \operatorname{span}(e_1, e_2)$ and $V = \operatorname{span}(e_{11}, e_{12})$. It follows from Lemma 8, with n replaced by $n+2$ and $h = k = 2$, that $l(n, n; n-2) \geqslant \rho(n+2)$ for $n \geqslant 3$, $n \neq 6$.

Finally, for $n \neq 3, 7$ it follows from Proposition 11 that

$$l(n, n; n-2) \geqslant l(n, n-1; n-2) \geqslant \rho(n+1).$$

This completes the proof of the proposition. ∎

PROPOSITION 13.

(i) $l(3,3;1) \geqslant l(3,2;1) \geqslant 3$.
(ii) $l(7,7;5) \geqslant l(7,6;5) \geqslant 6$.
(iii) $l(6,6;4) \geqslant 6$.

Proof. (i) is trivial.

Consider the normed bilinear map $\mathbb{R}^6 \times \mathbb{R}^8 \to \mathbb{R}^8$ tabulated by the first 6 rows and the first 8 columns of Table 1. Denote by U the 2-dimensional subspace spanned by e_1 and e_2.

(ii): Let V be the 1-dimensional subspace spanned by e_7. Applying Lemma 8 with $n = 8$, $h = 2$, $k = 1$, we obtain $l(7,6;5) \geqslant 6$. Consequently, $l(7,7;5) \geqslant 6$ also.

(iii): Let V be the 2-dimensional subspace spanned by e_7 and e_8 instead. Applying Lemma 8 with $n = 8$, $h = k = 2$, we obtain $l(6,6;4) \geqslant 6$. ■

4. VECTOR BUNDLES

Let V be a linear subspace of $M(m,n;k) \cup \{0\}$, of dimension r. J. Sylvester [12] has shown how V gives rise to a map between vector bundles over the real projective space $\mathbb{R}P^{r-1}$. Denote by ξ_{r-1} the Hopf line bundle over $\mathbb{R}P^{r-1}$, and by ε the trivial line bundle. A basis $A_1, \ldots, A_r$ of V furnishes a bundle map $f : m\xi_{r-1} \to n\varepsilon$ as follows. For each $x = (x_1, \ldots, x_r) \in S^{r-1}$, let $f_x : \mathbb{R}^m \to \mathbb{R}^n$ be the linear map with matrix

$$A(x) = x_1 A_1 + \cdots + x_r A_r \tag{8}$$

relative to the *canonical* bases $e_1, \ldots, e_m$ of $\mathbb{R}^m$ and $\epsilon_1, \ldots, \epsilon_n$ of $\mathbb{R}^n$. Identifying $m\xi$ with $\xi_{r-1} \otimes (m\varepsilon)$, we define $f : m\xi_{r-1} \to n\varepsilon$ by

$$f(\{\pm x\}, x \otimes y) = (\{\pm x\}, f_x(y)), \qquad x \in S^{r-1}. \tag{9}$$

Since the restriction of f to each fiber of $m\xi_{r-1}$ is a linear map of rank k, $\operatorname{Im} f$ is a k-plane bundle of $n\varepsilon$. It follows that there is a complementary $(n-k)$-plane bundle η such that

$$\operatorname{Im} f \oplus \eta^{n-k} = n\varepsilon. \tag{10}$$

On the other hand, $\zeta = \operatorname{Ker} f$ is an $(m-k)$-plane bundle satisfying

$$\zeta^{m-k} \oplus \operatorname{Im} f \simeq m\xi_{r-1}. \tag{11}$$

Consequently,

$$m\xi_{r-1} \oplus \eta^{n-k} \simeq \zeta^{m-k} \oplus n\varepsilon, \tag{12}$$

and $m\xi_{r-1} \oplus \eta^{n-k}$ is *stably equivalent to* ζ. By considering the total Stiefel-Whitney classes of the bundles in (10) and (11), Meshulam [9] has established

PROPOSITION 14. $l(n, n; k) \leqslant n$ *for every* $k \leqslant n$.

We shall determine better upper bounds for $l(n, n; k)$, $n - k \leqslant 2$, using topological K-theory. Adams has calculated the KO-theory of $\mathbb{R}P^{r-1}$, which we now summarize. For each integer m, let $\phi(m)$ be the *Adams function* defined by

$$\phi(m) = \operatorname{Card}\{j : 1 \leqslant j \leqslant m, j \equiv 0, 1, 2, 4 \ (\mathrm{mod}\ 8)\}. \tag{13}$$

THEOREM 15 (Adams [1]). $\mathrm{KO}(\mathbb{R}P^{r-1}) = \mathbb{Z} \oplus \widetilde{\mathrm{KO}}(\mathbb{R}P^{r-1})$, *where* $\widetilde{\mathrm{KO}}(\mathbb{R}P^{r-1})$ *is the cyclic group of order* $2^{\phi(r-1)}$ *with generator* $x = \xi_{r-1} - 1$. *The multiplicative structure of* $\mathrm{KO}(\mathbb{R}P^{r-1})$ *is given by* $\xi_{r-1}^2 = 1$, *or equivalently*, $x^2 = -2x$ *in* $\widetilde{\mathrm{KO}}(\mathbb{R}P^{r-1})$.

A basic relationship between the Adams function and the Hurwitz-Radon function defined in (4) is given by

$$\rho(2^{\phi(r-1)}) \geqslant r \qquad \text{for every} \quad r \geqslant 1. \tag{14}$$

Writing the stable equivalence class of η in (10) as $ax \in \widetilde{\mathrm{KO}}(\mathbb{R}P^{r-1})$ and that of ζ in (11) as bx, we have from (12)

$$(m + a - b)x = 0 \in \widetilde{\mathrm{KO}}(\mathbb{R}P^{r-1}). \tag{15}$$

It follows that $v_2(m + a - b) \geqslant \phi(r - 1)$. By Lemma 6 and (14),

$$\rho(m + a - b) = \rho(2^{v_2(m+a-b)}) \geqslant \rho(2^{\phi(r-1)}) \geqslant r. \tag{16}$$

It is well known that every line bundle over $\mathbb{R}P^{r-1}$ is equivalent to ξ_{r-1} or ε. On the other hand, Levine [8] has shown that every 2-plane bundle over $\mathbb{R}P^{r-1}$ necessarily splits into a direct sum of line bundles. Consequently, for $k = 1, 2$, the stable equivalence class of a k-plane bundle over $\mathbb{R}P^{r-1}$, $r \geqslant 2$, is of the form $ax \in \widetilde{\mathrm{KO}}(\mathbb{R}P^{r-1})$ for some integer a satisfying $0 \leqslant a \leqslant k$.

PROPOSITION 16.

(i) $l(n, n; n-1) \leqslant \max\{\rho(n), \rho(n \pm 1)\}$ *for* $n \geqslant 2$.
(ii) $l(n, n-1; n-2) \leqslant \max\{\rho(n), \rho(n \pm 1), \rho(n-2)\}$ *for* $n \geqslant 3$.
(iii) $l(n, n; n-2) \leqslant \max\{\rho(n), \rho(n \pm 1), \rho(n \pm 2)\}$ *for* $n \geqslant 3$.

Proof. (i): Let $r = l(n, n; n-1)$ for $n \geqslant 2$. Clearly, $r \geqslant 2$ by Propositions 9 and 10. In (16), we take $m = n$. Since η and ζ are line bundles in (10) and (11), the integers a and b in (16) are 0, 1. It follows that one of $\rho(n-1) \geqslant r$, $\rho(n) \geqslant r$, and $\rho(n+1) \geqslant r$ is true. This proves (i).

For (ii), with $m = n$ in (16), η in (10) is a 2-plane bundle and ζ in (11) is a line bundle. It follows that $a = 0, 1, 2$, and $b = 0, 1$. From (16), one of $\rho(n-2) \geqslant r$, $\rho(n-1) \geqslant r$, $\rho(n) \geqslant r$, and $\rho(n+1) \geqslant r$ is true.

The proof of (iii) is the same except that a and b are in the range $0 \leqslant a, b \leqslant 2$. ∎

5. PROOF OF THEOREMS 2, 3, 4

Theorem 2 follows from Propositions 9, 16(i) for $n \neq 3, 7$, and from Propositions 10, 14 for $n = 3, 7$.

Theorem 3 follows from Propositions 11, 16(ii) for $n \neq 3, 7$, and from Propositions 13(i) and 14 for $n = 3$. It remains to consider $l(7, 6; 5)$.

Theorem 4 follows from Propositions 12, 16(iii) for $n \neq 3, 6, 7$, and from Propositions 13(i), (iii) and 14 for $n = 3, 6$. It remains to consider $l(7, 7; 5)$.

Since $l(7, 7; 5) \geqslant l(7, 6; 5) \geqslant 6$ by Proposition 13(ii), we complete the proof of Theorems 3 and 4 by showing that there is *no* 7-dimensional linear subspace of $M(7, 7; 5) \cup \{0\}$. The existence of such a linear subspace would give, by (11), a splitting

$$7\xi_6 \simeq \zeta^2 \oplus \chi^5. \tag{17}$$

Since the Stiefel-Whitney class $w_6(7\xi_6) \neq 0$, the bundle $7\xi_6$ has exactly one section. It follows that $\zeta^2 = \xi_6 \oplus \varepsilon$ or $2\xi_6$, and its stable equivalence class is $bx \in \widetilde{\mathrm{KO}}(\mathbb{R}P^6)$, $b = 1$ or 2. Since η in (10) is also a 2-plane bundle, its stable equivalence class is $ax \in \widetilde{\mathrm{KO}}(\mathbb{R}P^6)$, $a = 0$, 1 or 2. Note that $\widetilde{\mathrm{KO}}(\mathbb{R}P^6)$ is cyclic of order 8. From (15) with $m = 7$, we see that $a = 2$ and $b = 1$. This means that $\zeta^2 \simeq \xi_6 \oplus \varepsilon$ and the stable equivalence class of χ^5 in (17) is $6x \in \widetilde{\mathrm{KO}}(\mathbb{R}P^6)$. Consequently, the *geometric dimension* of $6x$ is at most 5:

$$6\xi_6 \simeq \chi^5 \oplus \varepsilon.$$

This is a contradiction, since the top Stiefel-Whitney class $w_6(6\xi_6) \neq 0$. The proof of Theorems 3 and 4 is now complete.

6. REMARKS

(1) Let $l_{\mathbb{C}}(m, n; k)$ denote the analogue of $l(m, n; k)$ for matrices with complex entries. L. Smith [11] has solved the nonsingular *complex* bilinear map problem, namely, $l_{\mathbb{C}}(m, n; m) = n - m + 1$ for $m \leqslant n$. More generally, Westwick [13, 14] has shown that $l_{\mathbb{C}}(m, n, k) = n - k + 1$ whenever $n - k + 1$ does not divide $(m - 1)!/(k - 1)!$, and completely determined $l_{\mathbb{C}}(m, n; m - 1)$ for $m \leqslant n$.

(2) The nonsingular *real* bilinear map problem has been extensively studied in the works of J. Adem [2, 3], K. Y. Lam [5, 6] and J. Milgram [10].

The authors wish to express their thanks to the referee for valuable suggestions leading to improvement of this paper.

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Received 29 April 1992; final manuscript accepted 10 December 1992