Zero dimensionality and monotone normality

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Abstract

A proof is given that every separable, compact, monotonically normal space $X$ is the continuous image of a zero dimensional one which shows that $X$ is the continuous image of a linearly ordered one. © 1998 Elsevier Science B.V.

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Nikiel [1] has conjectured that every compact monotonically normal space $X$ is the continuous image of a compact linearly ordered space.

In [2] I prove this conjecture if $X$ is also separable and zero dimensional. Here I prove the conjecture for separable $X$ by proving:

**Theorem 2.** Suppose $X$ is a separable, compact, monotonically normal space. Then there is a separable, compact, zero dimensional, monotonically normal space $\Delta$ and a continuous map $\pi$ from $\Delta$ onto $X$.

The rest of the paper consists of a proof of this theorem. It mimics the proof given in [2] without assuming zero dimensionality. My hope is that generalizing the proof in this way may help someone to prove Nikiel's conjecture.

In Section 1 a breakdown is defined which partitions $X$ into what we might call atoms and I then prove (Theorem 1) that there is a breakdown of $X$ whose atoms each have cardinality at most two. In Section 2, after some preliminary work, constructions of types 1 and 2 are defined. These are used in Section 3 to define a $\Delta$ and $\pi$ with the desired properties.
1. Breakdown with atoms

Assume $X$ is separable, compact, and monotonically normal.

The latter means [3] that points are closed and that $X$ has an “MN operator”. That is, for every closed $A$ in $X$ and open $U$ with $A \subset U$, there is an open $H(A, U)$ with $A \subset H(A, U) \subset U$ such that:

(1) (normality) $A \cap V = \emptyset$ and $B \cap U = \emptyset$ imply $H(A, U) \cap H(B, V) = \emptyset$.
(2) (monotonicity) $A \subset B$ and $U \subset V$ imply $H(A, U) \subset H(B, V)$.

If $A = \{x\}$ we simplify to just $H(x, U)$.

Let $S$ be a countable dense subset of $X$.

If $G$ is a finite open cover of $X$ and $x \in X$, let

$$G(x) = \{G \mid x \in G \text{ and either } G \in G \text{ or } G = X - \overline{G'} \text{ for } G' \in G\}.$$ 

Let $G^*$ be a finite open cover of $X$ such that for all $G \in G^*$ there is $x \in X$ such that $G = H(x, \bigcap G(x))$.

Call $\mathcal{F} = \bigcup\{F_n \mid n \in \omega\}$ a breakdown of $X$ provided:

(1) For $n \in \omega$, $\mathcal{F}_n$ is a finite cover of $X$ by nonempty open sets.
(2) For $n \in \omega$, $\mathcal{F}_{n+1}$ is a refinement of $(\bigcup_{m \leq n} \mathcal{F}_m)^*$.

(3) If $p \neq q$ in $S$, there are $F$ and $F'$ in $\mathcal{F}$ having disjoint closures with $p \in F$ and $q \in F'$. Also $p \in S$ and $\{p\}$ is open imply $\{p\} \in \mathcal{F}$.

Let

$$\mathcal{K} = \left\{ \bigcap_{n \in \omega} F_n \mid F_n \in \mathcal{F}_n \text{ and } \overline{F_{n+1}} \subset F_n \right\}.$$ 

By (2), for all $x \in X$ there is $K \in \mathcal{K}$ with $x \in K$ and the members of $\mathcal{K}$ are disjoint.

By (3), if $K \neq \{x\}$ for some $x \in S$, then $x$ is a limit point of $S - K$.

Let

$$\mathcal{K}_3 = \{K \in \mathcal{K} \mid |K| \geq 3\}.$$ 

If $K \in \mathcal{K}_3$ choose distinct $K_0, K_1, K_2$ from $K$ and disjoint open sets $W_0(K), W_1(K), W_2(K)$ with $K_i \subset W_i(K)$ for all $i < 3$. For $i < 3$ let $V_i(K) = H(K_i, W_i(K))$ and $V_i'(K) = H(K_i, V_i(K))$. Since each $K_i$ is a limit point of $S - K$ we can choose $F(K)$ and $F'(K)$ from $\mathcal{F}$ such that $K \subset F(K) \subset \overline{F(K)} \subset F'(K)$ and, for all $i < 3$, $H(K_i, V_i'(K)) - F'(K) \neq \emptyset$.

Lemma 1. If $K \neq K'$ in $\mathcal{K}_3$, there is at most one $i < 3$ such that $V_i(K') \cap K \neq \emptyset$.

Proof. There are $m \in \omega$ and disjoint $F$ and $F'$ from $\mathcal{F}_m$ with $K \subset F$ and $K' \subset F'$.

There are $n > m$ and $F_n \in \mathcal{F}_n$ with $K \subset F_n \subset \overline{F_n} \subset F$. Thus there are $x \in F$ and $F_{n+1} \in \mathcal{F}_{n+1}$ with $K \subset F_{n+1} \subset H(x, F)$. Since $x \in W_i(K')$ for at most one $i < 3$, $H(K_i', W_i(K')) \cap H(x, F) = \emptyset$ for all $j \neq i$ in 3. □

Lemma 2. There do not exist infinitely many $K \in \mathcal{K}_3$ with the same $F(K)$ and $F'(K)$. (So $\mathcal{K}_3$ is countable.)
Proof. Suppose \( \{K(t) \mid t \in \omega\} \subset \mathcal{K}_3 \) and there are \( F \) and \( F' \) with \( F = F(K(t)) \) and \( F' = F'(K(t)) \) for all \( t \in \omega \).

If \( i < 3 \) and \( j < 3 \) define \( I_{ij} = \{s, t \} \subset \omega \mid s \neq t \) and \( s < t \) implies \( K(t) \cap V_k(K(s)) = \emptyset \) if \( k \neq i \) while \( K(s) \cap V_k(K(t)) = \emptyset \) if \( k \neq j \). By Ramsey's theorem [4], there are an infinite \( T \subset \omega \), an \( i < 3 \), and a \( j < 3 \) such that \( \{s, t\} \in I_{ij} \) for all \( s \neq t \) in \( T \). There is \( k < 3 \) with \( k \neq i \) and \( k \neq j \). Thus, for all \( s \neq t \) in \( T \), \( K(t) \cap V_k(K(s)) = \emptyset \) and \( \{V_k'(K(t) \mid t \in T\} \) are disjoint. For all \( t \in T \) choose \( p_t \in (H(K(t)) \cap V_k'(K(t) - F') \).

Let \( p \) be a limit point of \( \{p_t \mid t \in T\} \); \( p \notin F' \) and \( p \in V_k'(K(t)) \) for at most one \( t \) in \( T \). Therefore, since \( H \) is an MN operator, \( H(p, X - F) \cap H(K(t)) = \emptyset \) for all but perhaps one \( t \) contradicting \( p \) is a limit point of \( \{p_t \mid t \in T\} \).

Theorem 1. There is a breakdown of \( X \) for which \( \mathcal{K}_3 = \emptyset \).

Proof. Otherwise, for each countable ordinal \( \alpha \), by induction we select a breakdown \( \mathcal{F}(\alpha) = \bigcup \{F_n(\alpha) \mid n \in \omega\} \) satisfying (1)-(3) among other things as follows. Then define \( \mathcal{K}(\alpha) \) and \( \mathcal{K}_3(\alpha) \) for this \( \mathcal{F}(\alpha) \) exactly as \( \mathcal{K} \) and \( \mathcal{K}_3 \) were defined for \( \mathcal{F} \). For \( K \in \mathcal{K}_3(\alpha) \) for some \( \alpha < \omega_1 \), choose \( K_i, W_i(K), V_i(K) \) and \( V_i'(K) \) as before for each \( i < 3 \). By our construction which follows, \( K \subset X \) can belong to \( \mathcal{K}_3(\alpha) \) for at most one \( \alpha \); choose \( F(K) \) and \( F'(K) \) with reference to this \( \alpha \).

Let \( \mathcal{F}(0) \) be some arbitrary breakdown of \( X \).

Suppose \( \mathcal{F}(\beta) \) has been chosen and \( \alpha = \beta + 1 \). If \( K \in \mathcal{K}_3(\beta) \), let

\[ \mathcal{G}(K) = \{W_i(K) \mid i < 3\} \cup \left\{X - \bigcup_{i<3} V_i(K)\right\}; \]

\( \mathcal{G}(K) \) is a finite open cover of \( X \). Since by Lemma 2, \( \mathcal{K}_3(\beta) \) is countable, we can index \( \{\mathcal{G}(K) \mid K \in \mathcal{K}_3(\beta)\} \) as \( \{\mathcal{G}_n(\beta) \mid n \in \omega\} \). Choose \( \mathcal{F}_n(\alpha) \) so that it refines both \( \mathcal{F}_n(\beta) \) and \( G_n(\beta) \) and again satisfies (1)-(3).

Suppose \( \alpha \) is a limit ordinal and that \( \mathcal{F}(\beta) \) has been chosen for all \( \beta < \alpha \). Choose the \( \mathcal{F}_n(\alpha) \)s so that (in addition to (1)-(3)):

4. For \( \beta < \alpha \) and \( r \in \omega \) there is \( n \in \omega \) such that \( \mathcal{F}_n(\alpha) \) refines \( \mathcal{F}_r(\beta) \), and

5. For all \( n \in \omega \) there are \( \alpha_n < \alpha \) and \( n_\alpha \in \omega \) such that \( \mathcal{F}_n(\alpha) = \mathcal{F}_{n_\alpha}(\alpha_n) \).

By (5), for limit \( \alpha \) and \( K \in \mathcal{K}_3(\alpha) \), we can and do choose \( F(K) \) and \( F'(K) \) from \( \bigcup_{\beta < \alpha} \mathcal{F}(\beta) \).

One can prove by a straightforward induction argument that this is possible for all limit \( \alpha < \omega_1 \) (see [2]).

For \( \alpha < \omega_1 \), \( \mathcal{K}(\alpha) \) partitions \( X \) into disjoint compact sets and, if \( \beta < \alpha \), every member of \( \mathcal{K}(\alpha) \) is a subset of some member of \( \mathcal{K}(\beta) \). If \( \beta < \alpha \) and \( K \in \mathcal{K}_3(\alpha) \), \( K \) is a proper subset of some member of \( \mathcal{K}_3(\beta) \).

If \( \beta \leq \alpha \), \( K \in \mathcal{K}_3(\beta) \), \( K' \in \mathcal{K}_3(\alpha) \) and \( K \cap K' = \emptyset \), then there are disjoint \( F \) and \( F' \) in \( \mathcal{F}_m(\beta) \) for some \( m \in \omega \) with \( K \subset F \) and \( K' \subset F' \). Thus the same proof given for Lemma 1 yields:
Lemma 1'. If $K \neq K'$ in $\bigcup_{\alpha<\omega_1} K_3(\alpha)$ and $K \cap K' = \emptyset$, there is at most one $i < 3$ with $V_i(K') \cap K \neq \emptyset$.

Hence we also have:

Lemma 2'. There do not exist infinitely many disjoint $K \in \bigcup_{\alpha<\omega_1} K_3(\alpha)$ with the same $F(K)$ and $F'(K)$.

We are assuming $K_3(\alpha) \neq \emptyset$ for any $\alpha < \omega_1$.

For all limit $\alpha$, choose $K(\alpha) \in K_3(\alpha)$. Since $F(K(\alpha))$ and $F'(K(\alpha))$ are in $\bigcup_{\beta<\alpha} F_\beta$, by the pressing down lemma, there are $F$ and $F'$ and an uncountable subset $A$ of the limit ordinals such that $F(K(\alpha)) - F$ and $F'(K(\alpha)) = F'$ for all $\alpha \in A$.

If $\tau \in \omega_1$, let $A_\tau = \{ A' \subset (A - \tau) \} \mid \{ K(\alpha) \mid \alpha \in A' \}$ is a maximal disjoint subset of $\{ K(\alpha) \mid \alpha \in (A - \tau) \}$. Each $A' \in A_\tau$ is finite and we choose $A_\tau \in A_\tau$ of minimal cardinality.

Since the cardinalities of the $A_\tau$s can only increase there is $\sigma < \omega_1$ such that $|A_\sigma| = |A_\tau|$ for all $\tau > \sigma$. Choose an uncountable $T \subset (\omega_1 - \sigma)$ such that $\sigma \in T$ and $\tau' < \tau$ in $T$ implies $\tau > \sigma$ for all $\alpha \in A_\tau$. Choose $\alpha_\tau$ minimal in $A_\tau$. For each $\tau \in T$ there is a unique $\alpha_\tau \in A_\tau$ with $K(\alpha_\tau) \subset K(\alpha)$. Since $\{ K(\alpha_\tau) \mid \tau \in T \}$ is an uncountable strictly decreasing sequence of compact sets in the separable, monotonically normal space in which closed set must be $G_\delta$ [5], we have a contradiction.

2. Constructions of types 1 and 2

Having proved Theorem 1, fix a breakdown $\mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n$ of $X$ with associated $\mathcal{K}$ having $\mathcal{K}_3 = \emptyset$. Our primary concern now is $\mathcal{K}_2 = \{ K \in \mathcal{K} \mid |K| = 2 \}$. For $K = \{ K_0, K_1 \} \in \mathcal{K}_2$ choose disjoint $W_0(K)$ and $W_1(K)$ with $K_0 \in W_0(K)$ and $K_1 \in W_1(K)$ and, for $i < 2$, define $V_i(K) = H(K_i, W_i(K))$. Define

$$\mathcal{F}(K) = \{ F \in \mathcal{F} \mid \overline{F} \subset (V_0(K) \cup V_1(K)) \};$$

there is an $n \in \omega$ such that $K \subset F \in \mathcal{F}_n$ implies $F \in \mathcal{F}(K)$. We now have a stronger form of Lemma 1, namely, if $K \neq K'$ in $\mathcal{K}_2$ and $F \in \mathcal{F}(K) \cap \mathcal{F}(K')$, then $K$ is contained in exactly one of $V_0(K')$ and $V_1(K')$. Use $i \neq i'$, $j \neq j'$, and $k \neq k'$ for members of 2.

If $F \in \mathcal{F}$ and $B \subset \overline{F}$ is compact, let

$$\mathcal{K}_F(B) = \{ K \in \mathcal{K}_2 \mid K \subset B \text{ and } F \in \mathcal{F}(K) \}.$$ 

If $K \neq K'$ in $\mathcal{K}_F(B)$, there are $i$ and $j$ with $K' \subset V_i(K)$, $K \subset V_j(K')$ and, since $H$ is an MN operator, $V_i(K) \cap V_j(K') = \emptyset$; so $(V_i(K) \cap B) \subset V_i(K')$ and $(V_j(K') \cap B) \subset V_j(K)$.

Define $\langle \mathcal{L}, \leq \rangle$ to be a linked family in $\mathcal{K}_F(B)$ if $\mathcal{L} \subset \mathcal{K}_F(B)$ and $\leq$ is a total order on $\mathcal{L}$ such that $K < K' < K''$ in $\langle \mathcal{L}, \leq \rangle$, $K' \subset V_i(K)$, and $K'' \subset V_j(K'')$, imply
If $K < K'$ in $(\mathcal{L}, \leq)$ and $K' \subset V_i(K)$ then $K < K''$ in $\mathcal{L}$ if and only if $K'' \subset V_i(K)$, and $K'' < K$ in $\mathcal{L}$ if and only if $K'' \subset V_i(K)$. So $(\mathcal{L}, \leq)$ induces a total order $\leq$ on $\bigcup \mathcal{L}$ with $K' < K$ in this case. Thus, for a given $\mathcal{L} \subset K_p(B)$ there can be at most two orders which make $(\mathcal{L}, \leq)$ into a linked family, namely the one which for some $K \in \mathcal{L}$ induces $K' < K$ and the one which induces $K < K'$, namely its' inverse. Without confusion, I hope, I use $\leq$ for the order on $\mathcal{L}$ as well as the induced order on $\bigcup \mathcal{L}$.

Suppose the $(\mathcal{L}, \leq)$ is a maximal linked family in $K_p(B)$. Reindex each $L \in \mathcal{L}$ so that $L_0 < L_1$ in $(\bigcup \mathcal{L}, \leq$).

**Lemma 3.** Suppose $K \in (K_p(B) - \mathcal{L})$, $L_0 = \{ L \in \mathcal{L} \mid K \subset V_i(L) \}$, and $L_1 = \{ L \in \mathcal{L} \mid K' \subset V_i(L) \}$. Neither $L_0$ nor $L_1$ is empty and there is $i < 2$ such that $\bigcup \mathcal{L} \subset V_i(K)$.

**Proof.** Observe that $L_0$ and $L_1$ partition $(\mathcal{L}, \leq)$ into disjoint subintervals with $L_0 \subset L_1$. If $L_0 = \emptyset$, $\leq$ can be extended to $\mathcal{L} \cup \{ K \}$ by putting $K$ first and, similarly, if $L_1 = \emptyset$, by putting $K$ last. By definition, $V_i(L) \cap V_i(L') = \emptyset$ if $L < L'$ in $(\mathcal{L}, \leq)$, so there are $i$ and $j$ with $L_0 \subset V_i(K)$ and $L_1 \subset V_j(K)$. If $i \neq j$, $L_0 \neq \emptyset$, and $L_1 \neq \emptyset$, $\leq$ can be extended to $\mathcal{L} \cup \{ K \}$ by putting $K$ between $L_0$ and $L_1$ with the induced order on $(\bigcup \mathcal{L} \cup \{ K \})$ being the one with $K < K_i$. All of these extensions contradict the maximality of $\mathcal{L}$.

**Lemma 4.** Suppose $\emptyset \neq L' \subset C$ and $A = \bigcup \{ V_i(L) \cap B \mid L \in L' \}$. Then either $(\mathcal{L'}, \leq)$ has a maximal element $L$, $(\bigcup \mathcal{L'} - L_1) \subset A$, and $A = V_i(L) \cap B$ or there is a unique $a \in (A - A)$, $a \in \bigcup \mathcal{L'}$, and $\bigcup \mathcal{L'} \subset A$.

**Proof.** If $(\mathcal{L'}, \leq)$ has a maximal element $L$, then if $L' \in \mathcal{L'} - \{ L \}$, $(L' \cup (V_i(L') \cap B)) \subset (V_i(L) \cap B)$, so $A = V_i(L) \cap B$. So assume $(\mathcal{L'}, \leq)$ has no maximal element. Again $L' < L$ in $(\mathcal{L'}, \leq)$ implies $(L' \cup (V_i(L') \cap B)) \subset (V_i(L) \cap B)$, so $\bigcup \mathcal{L'} \subset A$. Since $L_0 \notin V_i(L')$ and $B$ is compact, there must be some $a \in \bigcup \mathcal{L'} - A$. Suppose there were $a' \neq a$ in $(A - A)$. Choose disjoint open $U$ and $U'$ with $a \in U$ and $a' \in U'$.

Without loss of generality, there are $L' < L$ in $\mathcal{L'}$ with $V_i(L') \cap H(a', U') \cap B \neq \emptyset$ and $V_i(L') \cap H(a, U) \cap B \neq \emptyset$. Since $a' \notin B$ and $a' \notin V_i(L)$, $a' \in V_i(L)$; thus $a' \notin W_0(L)$ and, since $V_0(L) = H(L_0, W_0(L))$, $L_0 \in U'$. Since $B \subset (V_0(W_0(L)))$, $(B \cap W_0(L)) = V_0(L) \subset A$ and $a \notin W_0(L)$. But $(V_0(L') \cap B) \subset V_0(L) = H(L_0, W_0(L))$. Since $H(a, U) \cap H(L_0, W_0(L)) \neq \emptyset$ and $H$ is an MN operator we have a contradiction.

**Lemma 4'.** Suppose $\emptyset \neq L' \subset C$, and $Z = \bigcup \{ V_i(L) \cap B \mid L \in L' \}$. Then either $(\mathcal{L'}, \leq)$ has a minimal element $L$, $(\bigcup \mathcal{L'} - L_0) \subset Z$, and $Z = V_i(L) \cap B$, or there is a unique $z \in (Z - Z)$, $z \in \bigcup \mathcal{L'}$, and $\bigcup \mathcal{L'} \subset Z$.

**Construction 1.** Given $F \in \mathcal{F}$, a compact $B \subset \overline{F}$, a maximal linked family $(\mathcal{L}, \leq)$ from $K_p(B)$ and some $G \in \mathcal{F}$. Index the members of $\mathcal{L}$ so that $L_0 < L_1$ for all $L \in \mathcal{L}$.

For all $x \in B$, let $A_x = \bigcup \{ V_i(L) \cap B \mid L \in \mathcal{L} \text{ and } x \notin V_i(L) \}$ and $Z_x = \bigcup \{ V_i(L) \cap B \mid L \in \mathcal{L} \text{ and } x \notin V_i(L) \}$. Let $M_x = B - (A_x \cup Z_x)$ and $\mathcal{M} = \{ M_x \mid x \in B \}$; $\mathcal{M}$
partitions $B$ into disjoint compact sets. Let $(M, \leq)$ be the total order on $M$ defined by $M_x \leq M_y$ if $A_x \subset A_y$.

Suppose $M = M_x \in M$. If $\mathcal{L}' = \{L \in \mathcal{L} \mid V_0(L) \subset A_x\} \neq \emptyset$, then, by Lemma 4, either $\mathcal{L}'$ has a maximal element $L$ and $A_x = V_0(L) \cap B$ and $L_1 \in M$ or there is a unique $a_M \in (\overline{A}_x - A)$ with $a_M \in M$. Similarly if $\mathcal{L}'' = \{L \in \mathcal{L} \mid V_1(L) \subset Z_x\} \neq \emptyset$, then, by Lemma 4', either $\mathcal{L}''$ has a minimal element $L$ and $Z_x = V_1(L) \cap B$ and $L_0 \in M$ or there is a unique $z_M \in (\overline{Z}_x - Z)$ with $z_M \in M$.

Let

$$\mathcal{L}' = \{L \in \mathcal{L} \mid M \not\subset \{a_M, z_M\}\}.$$ 

By Lemma 3 and the maximality of $(\mathcal{L}, \leq)$, if $K \in (K_F(B) - \mathcal{L})$, there is an $i < 2$ so that $\bigcup \mathcal{L} \subset V_i(K)$. Reindex $K$ so $\bigcup \mathcal{L} \subset V_0(K)$ (and hence $\bigcup \mathcal{L} \cap V_1(K) = \emptyset$).

Suppose $M \in \mathcal{M}^*$. For all $\alpha < 2^{|X|}$ for which it is possible, choose an open $V_\alpha$ by induction as follows. If possible choose a nonempty maximal linked family $(\mathcal{L}_\alpha, \leq\alpha)$ from $K_F(M) - \bigcup_{\beta < \alpha} V_\beta$ with $K_0 <_\alpha K_1$ for all $K \in \mathcal{L}_\alpha$. If there is a term of $K_F(M) - \bigcup_{\beta < \alpha} V_\beta$ contained in $G$, make sure $\mathcal{L}_\alpha$ has such a term. Define

$$V_\alpha = \bigcup \{V_i(K) \mid K \in \mathcal{L}_\alpha\} \quad \text{and} \quad \mathcal{C}_\alpha = B \cap V_\alpha.$$ 

If $\kappa = \{\alpha \mid V_\alpha \text{ is defined}\}$, let $\mathcal{C}_M = \{C_\alpha \mid \alpha < \kappa\}$ and, if $C = C_\alpha$, let $\mathcal{L}_C = \mathcal{L}_\alpha$ and $\leq_C = \leq_\alpha$.

Suppose $\beta < \alpha < \kappa$ and $L \in \mathcal{L}_\alpha$. By Lemma 3 either $\mathcal{L}_\beta \subset V_0(L)$ or $\mathcal{L}_\beta \subset V_1(L)$. Suppose $\mathcal{L}_\beta \subset V_1(L)$. If there is some $K \in \mathcal{L}_\beta$ such that $L \subset V_1(K)$, then $K \subset V_1(L)$ and $L \subset V_1(K)$ implies $V_0(L) \cap V_0(K) = \emptyset$ contradicting $L \subset V_0(L) \cap V_0(K)$. So, for all $K \in \mathcal{L}_\beta$, $L \subset V_0(K)$ and $K \subset V_1(L)$. But this contradicts the maximality of $\mathcal{L}_\beta$. So $\mathcal{L}_\beta \subset V_0(L)$ for all $L \in \mathcal{L}_\alpha$.

For each $C \in \mathcal{C}_M$, by Lemma 4, either there is a minimal $K \in \mathcal{L}_C$ in which case $K_0 = q_C \in M - C$ and $C = V_1(K) \cap B$, or there is a unique $q_C \in (\overline{C} - C)$. Defining $C^* = C \cup \{q_C\}$, $C^*$ is a compact subset of $M$.

Define $D_M = M - \bigcup \mathcal{C}_M$. Let $C = \bigcup \{C_M \mid M \in \mathcal{M}^*\}$ and $D = \{D_M \mid M \in \mathcal{M}^*\}$. Then we call $M, C,$ and $D$ as just defined, "the type 1 construction for $(F, B, (\mathcal{L}, \leq), G)$".

**Lemma 5.** If $C' \subset C$ and $x \in \overline{\bigcup C'} - \overline{\{C^* \mid C \subset C'\}}$, then $x \in \overline{\{q_C \mid C \subset C'\}}$.

**Proof.** Without loss of generality $x \neq q_C$ and $x \notin \mathcal{L}_C$ for any $C \in C'$.

Choose an open $U$ with $x \in U$ and $\overline{U} \cap \{q_C \mid C \subset C'\} = \emptyset$. By the definition of "breakdown", there is an open $U' \subset U$ with $x \in U'$ such that $L \in K_2$ and $L \notin U'$ implies $L \cap U' = \emptyset$. This is obvious if $\{x\} \in \mathcal{K}$. If $x = K_i \in K \in K_2$ let $U^* = (V_i(K) \cup U) \cap V_i(K)$. Again it is obvious that there is an open $U'' \subset U$ with $K \subset U''$ such that $L \in K_2$ and $L \notin U^*$ implies $L \cap U'' = \emptyset$. By an argument analogous to the proof of Lemma 1, if $L \in K_2$, $L \neq K$, and $L \subset U^*$ either $L \subset V_i(K)$ or $L \subset V_i(K)$. Thus $U' = V_i(K) \cap U \cap U''$ have the desired property.

Choose $C \in C$ such that $C \cap H(x, U') \neq \emptyset$. There is some $L \in \mathcal{L}_C$ with $B \cap V_i(L) \cap H(x, U') \neq \emptyset$. For any such $L$, since $x \in B - V_1(L)$ and $C \cap (V_1(L) \cap B) = (W_1(L) \cap B)$, $x \notin W_1(L)$. Since $H$ is an MN operator $L_1 \in U'$ and, by our choice of $U'$, $L_0 \in U$. 
If $\mathcal{L}_C$ has a first term $L'$ we can assume $L' = L$ since $V_i(L) \subseteq V_i(L')$. In this case $q_C = L_0$ and $C = V_i(L) \cap B$. So $L_0 = q_C \in U$ contradicting our choice of $U$.

If $q_C \in (\overline{C} - C)$ and there is some $L \in \mathcal{L}_C$ such that $B \cap V_i(L) \cap H(x, U') \neq \emptyset$, then, since $q_C \notin U$, there is some $L' < L$ in $\langle \mathcal{L}_C, \leq \rangle$ with $L' \cap U = \emptyset$. Since $(B \cap V_i(L)) \subseteq (B \cap V_i(L'))$, $B \cap V_i(L') \cap H(x, U') \neq \emptyset$. Thus $L'_0 \notin U$ is a contradiction. ∎

Construction 2. Given $F \in \mathcal{F}$ and a compact $B \subseteq \overline{F}$. Choose a maximal linked family $\langle \mathcal{L}, \leq \rangle$ from $K_F(B)$ and index its members so $L_0 < L_1$ for all $L \in \mathcal{L}$ induces the $\leq$ order. Also let $\{G_t \mid t \in \omega\}$ be some indexing of $\mathcal{F}$. Then:

Define $C_0 = \{B\}$, $B^* = B$, and $\langle L_B, \leq_B \rangle = \langle \mathcal{L}, \leq \rangle$.

Having chosen $C_t$ and $C^*_t$ and $\langle \mathcal{L}_C, \leq_C \rangle$ for all $C \in C_t$, let $\mathcal{M}(C)$, $\mathcal{M}^*(C)$, $\mathcal{C}(C)$, and $\mathcal{D}(C)$ be the type 1 construction for $(F, C^*_t, \langle \mathcal{L}_C, \leq_C \rangle, G_t)$. Define

$$D_t = \bigcup \{D(C) \mid C \in C_t\} \text{ and } C_{t+1} = \bigcup \{\mathcal{C}(C) \mid C \in C_t\}.$$ 

Recall that in the type 1 construction we indexed all of $\bigcup K_F(B)$ so that for $K \in (K_F(B) - \mathcal{L})$, $\bigcup \mathcal{L} \subseteq V_0(K)$. Observe that we need no further indexing once it is done for $K_F(B)$ since, by induction, if this order is correct for some $B' \in C_t$, then for each $C \in \mathcal{C}(B')$ we choose $\langle \mathcal{L}_C, \leq_C \rangle$ so $L_0 < C L_1$ for every $L \in \mathcal{L}_C$ and for all $K \in (K_F(C^*_t) - \mathcal{L}_C)$, $V_0(K) \supseteq \bigcup \mathcal{L}_C$ because $V_0(K) \cap \bigcup \mathcal{L}_B$.

Define $D^*_\omega = \{\bigcap_{t \in \omega} C_t \mid C^*_t \in C_t \text{ and } C_{t+1} \in \mathcal{C}(C_t)\}$.

Suppose $D = \bigcap_{t \in \omega} C_t \in D^*_\omega$ as above. Then for each $t \in \omega$ there is $L_t \in \mathcal{L}_C_t$ such that $C_{t+1} \subseteq V_i(L_t)$. Thus $\{L_t \mid t \in \omega\}$ is a linked family with the induced $(L_t)_0 < (L_t)_1$ order on its union. By Lemma 4, if $A = \bigcup \{V_0(L_t) \cap B \mid t \in \omega\}$ there is a unique $q_D \in D$ in $(\overline{A} - A)$. If $q_D \in K \subseteq K_F(B)$, then $\bigcup \mathcal{L}_C_t \subseteq V_0(K)$ for all $t$, so, since $q_D \in \bigcap_{t \in \omega} \mathcal{L}_C_t$, $q_D = K_0$. If $q_D \in K \subseteq K_F(B)$ and $K \subseteq D$, let $D(D) = \{D \cap V_0(K), D \cap V_i(L)\}$; if $D = \{q_D\}$, let $D(D) = \emptyset$; otherwise let $D(D) = \{D\}$.

Define $D^*_\omega = \{D(D) \mid D \in D^*_\omega\}$.

Let $C = \bigcup_{t \in \omega} C_t$ and $D = \bigcup_{t \leq \omega} D_t$ denote the “result of Construction 2 for $(B, F)$”.

Comment. Since separable monotonically normal spaces are hereditarily separable, the separability of $X$ implies that many things are countable. In Construction 1, $\mathcal{M}^*$ is countable since $\mathcal{M}^*$ includes precisely those members of $\mathcal{M}$, a collection of disjoint sets, containing a point not in the closure of the union of the others. Also, for each $M \in \mathcal{M}^*$, $\mathcal{C}_M$ is a set of disjoint, relatively open subsets of $M$, so $\mathcal{C}_M$ is countable. Thus $\mathcal{C}$ is countable. Hence when we get to Construction 2, by induction, each $C_t$ is countable and $C$ is countable. In Construction 1, $\mathcal{D}$ consists of precisely one closed subset of each of the compact, disjoint $M \in \mathcal{M}^*$. So in Construction 2, by induction, each $D_t$ and $\bigcup_{t \in \omega} D_t$ consist of countable families of disjoint compact sets. The members of $D^*_\omega$ are disjoint, compact, and disjoint from $\bigcup_{t \in \omega} D_t$. Although $D^*_\omega$ may not be countable, $D(D) = \emptyset$ for $D \in D^*_\omega$ unless $D$ contains a point not in the closure of $\bigcup (D^*_\omega - \{D\})$ and each $D(D)$ which is not empty partitions $D$ into at most two compact subsets. Thus $D$ is a countable family of disjoint compact sets.
Lemma 6. If $D \in \mathcal{D}$, $\mathcal{K}_F(D) = \emptyset$.

Proof. Suppose $D \in \mathcal{D}_t$ for some finite $t$. There is $C \in \mathcal{C}_t$ and $M \in \mathcal{M}^*(C)$ with $D = D_M$. For every $\alpha < 2^{n_1}$ for which it was possible we chose $\mathcal{L}_\alpha \subset (\mathcal{K}_F(M) \cap \bigcup_{\beta < \alpha} V_\beta)$ and $\mathcal{V}_\alpha \subset (X - D)$. Thus there is no term of $\mathcal{K}_F(M)$ contained in $D$; but $D \subset M$ so $\mathcal{K}_F(D) = \emptyset$.

Suppose $D \in \mathcal{D}(E)$ for some $E \in \mathcal{D}_{\alpha}$ and $K \in \mathcal{K}_F(D)$. By our choice of $\mathcal{D}(E)$, $q_E \notin K$. So $K \subset (E - \{q_E\})$ which is open in $B$. There is $t \in \omega$ with $K \subset (G_t \cap B) \subset (E - \{q_E\})$. There is a unique $C \in \mathcal{C}_t$ with $E \subset C$, and $\mathcal{L}_C$ was chosen to have a member contained in $G_t$ if any member of $\mathcal{K}_F(C)$ is in $G_t$. But $\mathcal{K}_F(D) \subset \mathcal{K}_F(E) \subset \mathcal{K}_F(C)$ so $\exists L \in \mathcal{L}_C$ with $L \subset G_t$ and $L \subset (E - \{q_E\})$. However, no $L \in \mathcal{L}_C$ is contained in any $M \in \mathcal{M}(C)$, and $E \subset M$ for some $M \in \mathcal{M}(C)$ so we have a contradiction.  

3. Main theorem

Theorem 2. There is a separable, 0-dimensional, compact, monotonically normal space $\Delta$ and continuous map $\pi$ from $\Delta$ onto $X$.

3.1. Construction of $\pi$ and $\Delta$

Suppose $n \in \omega$. For $F_n \in \mathcal{F}_n$, let

$$F^*_n = \{x \in X \mid \forall m < n \exists F_m \in \mathcal{F}_m \text{ with } F_{m+1} \subset F_m\}.$$  

Observe that $\{F^*_n \mid F \in \mathcal{F}_n\}$ is a finite closed cover of $X$. Let $\Sigma_n = \{\langle D_0, F_0, D_1, F_1, \ldots, D_n, F_n \rangle \mid D_0 = X, \forall m F_m \in \mathcal{F}_m, \text{ and } \forall m < n, F_{m+1} \subset F_m \text{ and } D_{m+1} \in D \text{ of the type 2 construction for } \langle D_m \cap F_m^*, F_m \rangle\}$. 

Let $\Sigma_n = \{\sigma = \langle D_0, F_0, D_1, F_1, \ldots \rangle \mid \forall n \in \omega, \sigma \text{ extends some term of } \sigma_n\}$. Let $\Sigma = \bigcup_{n \leq \omega} \Sigma_n$. If $\sigma \in \Sigma_n$, let $\sigma^* = \bigcap \sigma$. If $x \in \sigma^*$, $x \in \bigcap_{n \in \omega} F_n$ and $\sigma^* = \{x\}$. If $K = \{x\}$, $\sigma^* = \{x\}$. If $x = K$, there is $n \in \omega$ with $F_n \in \mathcal{F}(K)$ and thus, by Lemma 6, $K \notin D_{n+1}$, so $\sigma^* = \{x\}$ even in this case. Let $\Delta_\omega = \{\langle x, \sigma \rangle \mid \sigma \in \Sigma_\omega \text{ and } \sigma^* = \{x\}\}$. 

Suppose $n \in \omega$ and $\sigma = \langle D_0, F_0, D_1, F_1, \ldots, D_n, F_n \rangle \in \Sigma_n$; let $\sigma^* = D_n \cap F_n^*$. Let $C(\sigma)$ and $D(\sigma)$ be the type 2 construction for $\langle \sigma^*, F_n \rangle$. Then say:

1. If $C \in C(\sigma)$, $M \in \mathcal{M}(C)$, and $a_M$ exists, $\langle a_M, \sigma, C, 0 \rangle \in \Delta_\sigma$.

2. If $C \in C(\sigma)$, $M \in \mathcal{M}(C)$ and $z_M$ exists, $\langle z_M, \sigma, C, 1 \rangle \in \Delta_\sigma$.

3. If $D \in \mathcal{D}_\omega(\sigma)$, $\langle q_D, \sigma \rangle \in \Delta_\sigma$.

These three cases constitute $\Delta_\sigma$ and $\Delta_n = \bigcup \{\Delta_\sigma \mid \sigma \in \Sigma_n\}$, while $\Delta = \bigcup \{\Delta_n \mid n \leq \omega\}$.

Define $\pi : \Delta \rightarrow X$ by $\pi(\langle x, \sigma \rangle) = x$ and $\pi(\langle x, \sigma, C, k \rangle) = x$.

To define the topology on $\Delta$ we need further definitions. Assume $n \in \omega$ and $\sigma \in \Sigma_n$.  

If \( C \subseteq C(\sigma) \) and \( q_C \) exists choose \( \gamma = \Gamma(C, \sigma) \in \Sigma_m \) extending \( \sigma \) with \( q_C \in \gamma^* \) so that \( m \leq \omega \) is maximal. Since \( q_C \in D \in \mathcal{D}(\sigma) \), \( \gamma \) properly extends \( \sigma \) and, since \( m \) is maximal, \( q_C \notin \mathcal{D}(\gamma) \).

Suppose \( A \subseteq \sigma^* \). Define \( \Delta(A, \sigma) = \{ \delta \in \Delta \mid \delta \text{ is } \langle x, \tau \rangle \text{ or } \langle x, \tau, C, K \rangle , \tau \text{ extends } \sigma \text{, and } x \in A \} \). If \( C \subseteq C(\sigma) \) and \( q_C \in C \), let \( \Delta^*(C, \sigma) = \Delta(C, \sigma) \cup \{ \langle q_C, \sigma, C, 1 \rangle \} \); note that \( q_C \notin C \) but \( q_C = z_M \) where \( M = \{ q_C \} \) is the first term of \( \langle \mathcal{M}(C), \leq \rangle \) in this case. Otherwise let \( \Delta^*(C, \sigma) = \Delta(C, \sigma) \). Define \( C_0(A, \sigma, \mathcal{A}) = \{ \langle C, \tau \rangle \mid q_C \in A, C \subseteq C(\tau) \text{, and } \Gamma(C, \tau) \text{ extends } \sigma \text{ which properly extends } \tau \} \). If \( A \subseteq C_0(A, \sigma, \mathcal{A}) \), define \( C_0(A, \sigma, \mathcal{A}) = \{ C_0(A, \sigma) - \mathcal{A} \} \). For \( m < n - 1 \), define \( C_{m+1}(A, \sigma, \mathcal{A}) = \{ (B, \rho) \in C_0(C, \tau) \mid \langle C, \tau \rangle \in C_m(A, \sigma, \mathcal{A}) \} \). Define \( C(A, \sigma, \mathcal{A}) = \bigcup_{m \leq n} C_m(A, \sigma, \mathcal{A}) \). Finally \( \Delta(A, \sigma, \mathcal{A}) = \Delta(A, \sigma) \cup \bigcup \{ \Delta^*(C, \tau) \mid \langle C, \tau \rangle \in C(A, \sigma, \mathcal{A}) \} \). Observe that \( \Delta^*(C, \tau) \) \( \langle C, \tau \rangle \subseteq C(A, \sigma, \mathcal{A}) \) are disjoint and their union misses \( \Delta(A, \sigma) \); also \( \sigma \) properly extends \( \tau \) if \( \langle C, \tau \rangle \in C(A, \sigma, \emptyset) \), \( \Delta(A, \sigma, \mathcal{A}) \subseteq \Delta(A, \sigma, \emptyset) \), and \( \Delta(A, \sigma, \emptyset) \cup \Delta(A', \sigma, \emptyset) = \emptyset \) unless \( A \cap A' \neq \emptyset \).

Similarly, suppose \( \delta = \langle x, \sigma, B, k \rangle \in \Delta_\sigma \). Define \( C_0(\delta) = \{ \langle C, \tau \rangle \in C_0\{ x \}, \sigma \mid (C, B) \subseteq \Delta(\sigma) \text{ and, if } x = a_M \text{ for some } M \in \mathcal{M}(B), \text{ then } k = 0 \} \). If \( A \subseteq C_0(\delta) \) then define \( C_0(A, \delta, \mathcal{B}(C, b), \Delta(\delta, \mathcal{A}), \delta) \) exactly as above, replacing \( (A, \sigma, \mathcal{A}) \) by \( (\delta, \mathcal{A}) \).

Let \( U(\sigma) \) be the set of all sets of the following types:

1. \( \Delta(\sigma^*, \sigma, A) \) where \( A \) is a finite subset of \( C_0(\sigma^*, \sigma) \).
2. \( \Delta(C, \sigma, A) \cup \{ \langle q_C, \sigma, C, 1 \rangle \} \) where \( C \subseteq C(\sigma) \) and \( A \) is a finite subset of \( C_0(C, \sigma) \).
3. \( \Delta(C - D, \sigma, 0) \cup \{ \langle q_C, \sigma, C, 1 \rangle \} \cup \{ \langle q_D, \sigma \rangle \} \) where \( D \in \mathcal{D}_C(\sigma) \) and \( D \subseteq C \subseteq C(\sigma) \).
4. \( \Delta(\delta, \mathcal{A}) \cup \Delta(\{ N \mid M < N < M' \text{ in } \langle \mathcal{M}(C), \leq \rangle \}, \sigma, \emptyset) \cup \Delta(\beta, A') \) where \( C \subseteq C(\sigma) \), \( M < M' \text{ in } \langle \mathcal{M}(C), \leq \rangle \), \( A \) is a finite subset of \( C_0(\delta) \), \( A' \) is a finite subset of \( C_0(\beta, \delta) \), \( \delta = \langle z_M, \sigma, C, 1 \rangle \) and \( \beta = \langle a_M, \sigma, C, 0 \rangle \).

Just delete any undefined terms from the above such as the \( \{ \langle q_C, \sigma, C, 1 \rangle \} \) in (2) or (3) in case \( q_C \neq z_M \) for any \( M \in \mathcal{M}(C) \), or \( \Delta(\delta, \mathcal{A}) \) or \( \Delta(\beta, A') \) in case there is no \( z_M \) or \( a_M \) in (4).

Let \( U = \bigcup \{ U(\sigma) \mid \sigma \in \Sigma_n, n \in \omega \} \).

To check that \( U \) is the basis for a topology on \( \Delta \) we suppose that \( \delta \in U \cap V \) for some \( U \in U(\gamma) \) and \( V \in U(\rho) \) and check that there is a member of \( U \) to which \( \delta \) belongs contained in \( U \cap V \).

If \( \delta \in \Delta_\sigma \) for some finite \( \sigma \) we can assume \( \gamma = \rho = \sigma \). Then, if \( \delta = \langle x, \sigma, x = q_D \) for some \( D \in \mathcal{D}_\sigma(\sigma) \) and we can assume there are \( C \) and \( C' \in C(\sigma) \) with \( U \) being the type (3) member of \( U(\sigma) \) for \( C \) and \( D \) the one for \( C' \) and \( D \). One of \( C \) and \( C' \) contains the other; say \( C \subseteq C' \). Then \( \delta \in U \subseteq V \). A similar argument using type (4) basic neighborhoods for \( \delta \) where \( \delta = \langle x, \sigma, C, k \rangle \) for some \( C \subseteq C(\sigma) \) and \( k < 2 \) takes care of \( \delta \) for all finite \( \sigma \).

If \( \delta = \langle x, \tau \rangle \in \Delta_\omega \), there is a finite \( \sigma \) extended by \( \tau \) and properly extending both \( \rho \) and \( \gamma \). Then, for some finite \( A \subseteq C_0(\sigma^*, \sigma) \), \( \Delta(\sigma^*, \sigma, A) \subseteq U \), and, for some finite \( A' \subseteq C_0(\sigma^*, \sigma) \), \( \Delta(\sigma^*, \sigma, A') \subseteq U \); so \( \delta \in \Delta(\sigma^*, \sigma, A \cup A') \subseteq U \cap V \).
3.2. Proof that $\pi$ and $\Delta$ have the desired properties for our theorem

Point are closed. To see this suppose $\delta \neq \beta$ in $\delta$ and $\delta$ is $\langle x, \tau \rangle$ or $\langle x, \tau, B, k \rangle$. We get a contradiction by assuming $\delta \in U$ implies $\beta \in U$.

If $\delta \in \Delta_\omega$, there is a finite $\sigma$ extended by $\tau$ such that either $\tau$ is not extended by $\sigma$ or $\sigma$ has greater length than $\tau$. Then $\delta \in \Delta(\sigma^*, \sigma, 0) \subseteq U$. If $\beta \in \Delta(\sigma^*, \sigma, 0)$ there is some unique $\langle C, \gamma \rangle \in C_0(\sigma^*, \sigma, 0)$ such that $\Gamma(C, \gamma)$ extends $\sigma$ and $\beta \in \Delta^*(C, \gamma) \cup \Delta(C, \gamma, 0)$. So if $\mathcal{A} = \{(C, \gamma)\}$, $\delta \in U = \Delta(\sigma^*, \sigma, \mathcal{A})$ while $\beta \notin U$.

But assuming $\tau$ is finite, the intersection of the members of $\mathcal{U}(\tau)$ for $\delta$ is the singleton $\{\delta\}$; so at least one member of $U(\tau)$ avoids $\beta$.

$\Delta$ is zero dimensional. This time assume $U \in U(\sigma)$ for some finite $\sigma \in \Sigma$, $\delta \in \Delta_\tau$ is $\langle x, \tau \rangle$ or $\langle x, \tau, C, k \rangle$ for some $\tau \in \Sigma$, and $\delta \notin U$. We get a contradiction by assuming $\delta \in V \subseteq U$ implies $U \cap U \neq \emptyset$.

Many of the four types of members of $U(\sigma)$ involve an $A$ (or two). If $\delta \in \Delta^*(C, \gamma) \cup \Delta^C(C, \gamma, 0)$ for some $\langle C, \gamma \rangle \in \mathcal{A}$, $\Delta^*(C, \gamma) \cup \Delta(C, \gamma, 0)$ is an open set containing $\delta$ and missing $U$. So we can assume $A = \emptyset$. If $\delta \in \Delta(\sigma^*, \sigma, 0)$ and $U$ is of type (2), (3) or (4), there is an open set to which $\delta$ belongs missing $U$, so we can assume $U = \Delta(\sigma^*, \sigma, 0)$.

If $\delta = \langle x, \tau \rangle \in \Delta_\omega$ there is a finite $\rho$ extended by $\tau$ of length greater than that of $\sigma$; clearly $\delta \in \Delta(\rho^*, \rho, 0)$. Let

$$\mathcal{A} = \{\langle A, \alpha \rangle \in C_0(\rho^*, \rho) \mid \exists \langle A', \alpha' \rangle \in \bigcup \{\langle A, \alpha \rangle \cup C(A, \alpha, 0)\}$$

such that $\Delta(\sigma^*, \sigma) \subseteq \Delta^*(A', \alpha')\}.$

Since $\{\Delta^*(A', \alpha') \mid \langle A', \alpha' \rangle \in \mathcal{C}(\rho^*, \rho, 0)\}$ are disjoint, $A$ has at most one term and $\delta \in \Delta(\rho^*, \rho, A) = V$. Assume there is $\beta \in U \cap V$ since otherwise $V$ will do for our desired open set. Then $\beta \in \Delta^*(C, \gamma) \cap \Delta^*(C', \gamma')$ for some $\langle C, \gamma \rangle \in C(\sigma^*, \sigma, 0)$ and $\langle C', \gamma' \rangle \in C(\rho^*, \rho, A)$.

Because of $\beta$, either $\gamma$ extends $\gamma'$ or $\gamma'$ extends $\gamma$ and if $\gamma = \gamma'$ either $C \subseteq C'$ or $C' \subseteq C$.

If $\gamma'$ properly extends $\gamma$ or $\gamma' = \gamma$ and $C' \not\subseteq C$, there is $D \in D(\gamma)$ with $\rho^* \subset D \subset C$, so $\rho$ extends $\gamma$ and $\rho^* \subset C$, hence $\delta \in \Delta(\rho^*, \rho) \subseteq \Delta^*(C, \gamma) \subset U$ contradicting $\delta \notin U$. Similarly, if $\gamma$ properly extends $\gamma'$ or $\gamma' = \gamma$ and $C \not\subseteq C'$, $\Delta(\sigma^*, \sigma) \subset \Delta^*(C', \gamma')$ contradicting our choice of $A$.

So $\gamma = \gamma'$ and $C = C'$; also $C \neq \rho^*$ since this implies $\delta \in U$. Therefore $\langle C, \gamma \rangle \in C(\rho^*, \rho, A) \cap C(\sigma^*, \sigma, 0)$. Let us assume that $\gamma$ has maximal length for there to be some $\langle C, \gamma \rangle$ in this intersection. There is $\langle A, \alpha \rangle \in \{\langle \sigma^*, \sigma \rangle \cup C(\sigma^*, \sigma, 0)\}$ and $\langle A', \alpha' \rangle \in \{\langle \rho^*, \rho \rangle \cup C(\rho^*, \rho, A)\}$ such that $q_C \in A \cap A'$ and $\Gamma(q_C, \gamma)$ extends both $\alpha$ and $\alpha'$, either $\alpha$ extends $\alpha'$ or $\alpha'$ extends $\alpha$. Because of $q_C \in A \cap A'$, if $\alpha = \alpha'$ either $A \subset A'$ or $A' \subset A$. So the argument from the preceding paragraph implies $\alpha = \alpha'$ and $A = A'$ and $\langle A, \alpha \rangle \in C(\rho^*, \rho, A) \cap C(\sigma^*, \sigma, 0)$ contradicting $\gamma$ has length greater than $\gamma$.

It remains to check the situation if $\delta = \langle x, \tau, C, k \rangle$ or $\langle x, \tau \rangle$ for some finite $\tau$. Since $\delta \notin U$, $\tau$ does not extend $\sigma$. So if $\sigma$ does not properly extend $\tau$, the same argument
given for \( \delta \in \Delta_\omega \) applies, replacing \( \rho \) by \( \tau \). Otherwise \( \sigma \) properly extends \( \tau \) and there is \( D \subset D(\tau) \) with \( \sigma^* \subset D \) and thus \( U = \Delta(\sigma^*, \sigma, \emptyset) \subset \Delta(D, \tau, \emptyset) \). There is a basic neighborhood for \( \delta \) in \( U(\tau) \), of type (4) if \( \delta = \langle x, \tau, C, k \rangle \) and type (3) if \( \delta = \langle x, \tau \rangle \), \( \emptyset \); this neighborhood misses \( U \).

\( \Delta \) is separable. Recall that \( X \) is hereditarily separable and that \( \pi^{-1}(x) \) is countable for every \( x \in X \). For each \( n \in \omega \) and \( \sigma \subset \Sigma_\omega \), \( \sigma^* \) has a countable dense in the topology of \( X \) subset \( S_\sigma \). Thus \( \pi^{-1}(S_\sigma) \) is countable and its closure contains \( \Delta_\sigma \). Since \( \bigcup_{n \in \omega} \Sigma_n \) is countable, \( S_\Delta = \bigcup \{ \pi^{-1}(S_\sigma) \mid \sigma \in \Sigma_n \text{ and } n \in \omega \} \) is countable. If \( \langle x, \sigma \rangle \in \Delta_\omega \), then, for each \( n \in \omega \), there is a unique \( \sigma_n \in \Sigma_n \) extended by \( \sigma \), and \( x \in \sigma_n^* \). Since every open set to which \( \langle x, \sigma \rangle \) belongs contains \( \Delta(\sigma_n^*, \sigma_n, A) \) for some finite \( \Delta \subset C_0(\sigma_n^*, \sigma_n) \), and \( \Delta(\sigma_n^*, \sigma_n, A) \cap n^{-1}(S_{\sigma_n}) \not= \emptyset \). So \( \Delta \subset \Sigma_\Delta \).

\( \Delta \) is compact. Assume \( \mathcal{V} \subset \mathcal{U} \) is a cover of \( \Delta \) and define \( \mathcal{V}^* = \{ \Omega \subset \Delta \mid \text{some finite subset of } \mathcal{V} \text{ covers } \Omega \} \). We prove \( \Delta \subset \mathcal{V}^* \) by assuming \( \Delta \not\in \mathcal{V}^* \). Thus, since \( \Sigma_0 \) is finite and \( X = \bigcup \{ \Delta(\sigma^*, \sigma) \mid \sigma \in \Sigma_0 \} \), there is \( \sigma_0 \in \Sigma_0 \) such that \( \Delta(\sigma_0^*, \sigma_0) \not\in \mathcal{V}^* \).

If, for all \( n \in \omega \), there is \( \sigma_n \in \Sigma_n \) extending \( \sigma_m \) for all \( m < n \) and \( \Delta(\sigma_n^*, \sigma_n) \not\in \mathcal{V}^* \), then, if \( \sigma \in \Sigma_\omega \) extends \( \sigma_n \) for all \( n \in \omega \), \( \langle \sigma^*, \sigma \rangle \in V \in \mathcal{V} \) and there is \( n \in \omega \) with \( \Delta(\sigma_n^*, \sigma_n, A) \subset V \) contradicting \( \Delta(\sigma_n^*, \sigma_n) \not\in \mathcal{V}^* \). Hence there is \( \sigma \in \Sigma_n \) for some finite \( n \) such that \( \Delta(\sigma_n^*, \sigma_n) \not\in \mathcal{V}^* \) but \( \Delta(\tau^*, \tau) \in \mathcal{V}^* \) for all \( \tau \in \Sigma_{n+1} \) extending \( \sigma \).

Recall that to achieve \( C(\sigma) \) and \( D(\sigma) \) for \( \langle D_0, F_0, \ldots, D_n, F_n \rangle \), we did a type 2 construction for \( \langle \sigma^*, F_n \rangle \) and for each \( t \in \omega \) we defined a family \( C_t(\sigma) \) of subset of \( \sigma^* \) and for each \( C \in C_t(\sigma) \) we did a type 1 construction for \( \langle F_n, C^*, \langle C, \leq, G_t \rangle \) yielding \( M(C), D(C) \).

Since \( C_0(\sigma) = \{ \sigma^* \}, \emptyset \subset C_0(\sigma) \) implies \( \Delta(C_0, \sigma) \not\in \mathcal{V}^* \). If, for all \( t \in \omega \), there is \( C_t \in C_t(\sigma) \) with \( C_{t+1} \subset C_t \) and \( \Delta^*(C_t, \sigma) \not\in \mathcal{V}^* \), there is \( D = \cap_{t \in \omega} C_t \subset D(\sigma) \). Since \( \langle q_D, \sigma \rangle \in V \in \mathcal{V} \), there are basic open set of type (3) for \( \langle q_D, \sigma \rangle \) in \( V \) and a \( t \in \omega \) with \( \Delta(C_t - D, \sigma) \subset V \). Since \( \Delta^*(C_t, \sigma) \not\in \mathcal{V}^* \), \( \Delta(D, \sigma) \not\in \mathcal{V}^* \) and one, call it \( D_{n+1} \), of the at most two members of \( D(\sigma) \) whose union is \( D \), has \( \Delta(D_{n+1}, \sigma) \not\in \mathcal{V}^* \). Since \( F_{n+1} \) is finite there is some \( F_{n+1} \in F_{n+1} \) such that \( \Delta(D_{n+1} \cap F_{n+1}, \sigma) \not\in \mathcal{V}^* \). But if \( \tau = \langle D_0, \ldots, F_n, D_{n+1}, F_{n+1} \rangle \), there is at most \( \langle q_D, \sigma \rangle \) in \( \Delta(D_{n+1} \cap F_{n+1}, \sigma) - \Delta(\tau^*, \tau) \) so \( \Delta(\tau^*, \tau) \not\in \mathcal{V}^* \) which is a contradiction. Hence there is \( C \in C(\sigma) \) with \( \Delta^*(C, \sigma) \not\in \mathcal{V}^* \) such that \( B \in C(\sigma) \) and \( B \subset C \) implies \( \Delta^*(B, \sigma) \in \mathcal{V}^* \).

Since \( \langle M(C), \leq \rangle \) is Dedekind complete with a first and last term and the topology of \( \Delta \) respects the order topology on \( \langle M(C), \leq \rangle \) (see the definition of type (4) members of \( \mathcal{U}(\sigma) \)), there must be some \( M \in M(\sigma) \) such that \( \Delta(M, \sigma) \not\in \mathcal{V}^* \). Since \( F_{n+1} \) is finite \( T = \{ \langle D_0, \ldots, F_n D_M, F \rangle \mid F \in F_{n+1} \} \) is finite, and, since \( \tau \in T \) implies \( \tau \in \Sigma_{n+1} \) and extends \( \sigma \), there is an open \( V \) in \( \Delta \) with \( \bigcup \{ \Delta(\tau^*, \tau) \mid \tau \in T \} \subset V \in \mathcal{V} \).

Recall that \( M = D_M \cup (\bigcup C(M)) \) and each \( B \in C(M) \) has \( \Delta^*(B, \sigma) \in \mathcal{V}^* \). Let \( B = \{ B \in C_M \mid \Delta^*(B, \sigma) \not\in V \} \) Since

\[ \Delta(M, \sigma) \subset V \cup \bigcup \{ \{ \Delta^*(B, \sigma) \mid B \in B \} \cup \{ \langle q_M, \Sigma, C, 0 \rangle \} \cup \{ \langle z_M, \Sigma, C, 1 \rangle \} \). \]

\( B \) is infinite.
Choose \( \tau_{n+1} \in T \) extended by \( \Gamma(q_B, \sigma) \) for infinitely many \( B \in B \) and, for all \( m > n + 1 \) for which it is possible, choose \( \tau_m \subset \Sigma_m \) extending \( \tau_{m-1} \) and extended by \( \Gamma(q_B, \sigma) \) for infinitely many \( B \in B \). If \( \tau \in \Sigma_\omega \) extends \( \tau_m \) for all \( m \), \( \langle \tau^*, \tau \rangle \in \Delta(\tau_{n+1}, \tau_{n+1}) \subset V \). So for some \( m \) and \( \mathcal{A} \), \( \Delta(\tau_m, \tau_{m-1}) \subset V \). If \( B \in B' = \{ B \in B \mid \Gamma(q_B, \sigma) \subset V \} \) extends \( \tau_m \), \( \langle B, \sigma \rangle \subset C'(\tau_m, \tau_{m-1}) \) and, since \( B' \) is infinite, there is \( \langle B, \sigma \rangle \in \langle B' \rangle \) and \( \Delta^*(B, \sigma) \subset V \) contrary to assumption. Thus there is a maximal \( m \) for which \( \tau_m \) can be defined, call it \( \tau \). Then \( B^+ = \{ B \in B \mid \Gamma(q_B, \sigma) = \tau \} \) is infinite and, if \( B \in B^+ \), \( q_B \notin \Delta(\tau) \). Again \( C_0(\tau) = \{ \tau^* \} \). If for all \( t \in \omega \) there is \( C_t \subset C_t(\tau) \subset C_t \) such that \( \{ B \in B^+ \mid q_B \in C_t \} \) is infinite, there is \( D = \bigcap_{t \in \omega} C_t \subset \Delta^*(\tau) \), and since \( (q_D, \tau) \in V \), for some \( t \), \( \Delta(\tau_t - D, \tau, 0) \subset V \). Since \( D \cap q_B \mid B \in B^+ \} = \emptyset \), there is \( B \in B^+ \) with \( q_B \in C_t \). Thus \( \langle B, \sigma \rangle \subset C(\tau) \); so \( \Delta^*(B, \sigma) \subset \Delta(C_t - D, \tau, 0) \subset V \) which is impossible. Hence there is \( C' \subset C(\tau) \) such that \( \{ B \in B^+ \mid q_B \in C' \} \) is infinite but, for all \( B' \in \mathcal{C}(\tau) \) with \( B' \subset C' \), \( \{ B \in B^+ \mid q_B \in C' \} \) is finite.

Again the basic open sets for \( \langle a_M', \tau, 0 \rangle \) and \( \langle a_M', \tau, 1 \rangle \) for \( M' \in \langle \mathcal{M}(C'), \leq \rangle \) ensure that there is some \( M' \in \mathcal{M}(C') \) such that \( \{ B \in B^+ \mid q_B \in M' \} \) is infinite. Since \( \{ B \in B^+ \mid q_B \in D_{M'} \} = \emptyset \) and \( M' = D_{M'} \cup (\bigcup \mathcal{C}(M')) \) and \( \{ B \in B^+ \mid q_B \in B' \} \) is finite for \( B' \in \mathcal{C}(M') \), \( B_0 = \{ B' \in \mathcal{C}(M') \mid \{ B \in B^+ \mid q_B \in B' \} \neq \emptyset \} \) is infinite. Observe that \( \langle B, \sigma \rangle \subset C_0(\tau, \tau) \) if \( B \in B^+ \) and \( q_B \in B' \in \mathcal{C}(M') \).

To sum up our situation so far: We have a finite \( \sigma \in \Sigma \), an \( A \subset \mathcal{C}(\sigma) \), \( M \in \mathcal{M}(C) \), and an open \( V \in V^* \) such that \( B = \{ B \in \mathcal{C}(M) \mid \Delta^*(B, \sigma) \subset V \} \) is infinite. We also have a finite \( \tau \in \Sigma \) properly extending \( \sigma \) with \( \Delta(\tau^*, \tau) \subset V \), a \( C' \subset C(\tau) \), and an \( M' \in \mathcal{M}(C') \) such that \( B_0 = \{ B' \in \mathcal{C}(M') \mid \exists B \in \mathcal{B} \) with \( \langle B, \sigma \rangle \subset C_0(\tau, \tau) \} \) is infinite.

Let \( \gamma_0 = \tau \) and, using exactly the same steps used in the construction of \( \tau, C', M', \) and \( B_0 \) from \( \sigma, C, M, A \), for each \( r \in \omega \) we can choose a finite \( \gamma_r \in \Sigma \) properly extending \( \gamma_{r-1} \), a \( C_r \subset C(\gamma_r) \), and an \( M_r \) in \( \mathcal{M}(C_r) \) such that \( B_r = \{ B' \in \mathcal{C}(M_r) \mid \exists B \in B_{r-1} \) with \( \langle B, \sigma \rangle \subset C_0(\tau_r) \} \) is infinite.

There is \( \gamma \in \Sigma_\omega \) extending \( \gamma_r \) for all \( r \in \omega \) and, since \( \langle \gamma^*, \gamma \rangle \in V \), there are \( r \in \omega \) and \( A \subset C_0(\gamma^*, \gamma_r) \) such that \( \Delta(\gamma_r, A) \subset V \). If \( r = 0 \), \( \gamma_0 = \tau \) and \( \{ B, \sigma \} \) \( B \in B \} \subset C_0(\tau^*, \tau) \) and there is \( B \in B \) with \( \langle B, \sigma \rangle \notin A \). So \( \Delta^*(B, \sigma) \subset V \) which is a contradiction. If \( r > 0 \), \( \{ B', \gamma_r \} \mid B' \in B_{r-1} \) \( C_0(\gamma^*, \gamma_r) \), and there is \( B' \in B_{r-1} \) with \( \langle B', \gamma_r \} \notin A \). Since \( \Delta(\gamma^*, \gamma_r, \emptyset) \subset A \), \( \Delta(\gamma^*, \gamma_r, \emptyset, \emptyset) \subset B \). But there is some \( B \in B \) with \( \langle B, \sigma \rangle \subset C(\gamma, \gamma_r, \emptyset) \) which implies \( \langle B, \sigma \rangle \subset V \) with is impossible.

\( \pi \) is continuous. Assume \( x \in U \) which is open in \( X \). We prove that \( \delta \in \pi^{-1}(x) \) implies \( \delta \subset V \) for some open \( V \) in \( \Delta \) with \( \pi(V) \subset U \).

If \( \delta = \langle x, \rho \rangle \in \Delta_\omega \) there is \( \epsilon \in \Sigma_n \) for some finite \( n \) with \( \rho \) extending \( \sigma \) and \( x \in \sigma^* = A \subset U \). If \( \delta \) is \( \langle x, \sigma, C, k \rangle \in \Delta_n \) for some \( k < 2 \), there is \( M \in \mathcal{M}(C) \) so that \( x = a_M \) if \( k = 0 \) and \( x = z_M \) if \( k = 1 \). If \( k = 0 \) choose \( M' < M \) in \( \mathcal{M}(C), \leq \) so that \( A = \{ x \} \cup \mathcal{N} \subset \mathcal{N}(C) \mid M' \leq N < M \} \subset U \) and if \( k = 1 \) choose \( M'' > M \) in \( \mathcal{M}(C), \leq \) so that \( A = \{ x \} \cup \mathcal{N} \subset \mathcal{M}(C) \mid M < N \leq M'' \} \subset U \). If
(x, σ) ∈ Δₙ choose C ∈ C(σ) and D ∈ Dₙ⁺(σ) so that x = qₐ ∈ A = C - D ⊂ U.

In all cases A is a compact subset of σ* ∪ U and trivially π(Δ(A, σ)) ⊂ U. Observe that if A is a finite subset of C₀(A, σ) and B ⊂ U for all (B, γ) ∈ C(A, σ, A), then there is a basic open set V ∈ U(σ) with δ ∈ V and, since each π(Δ(B, γ)) ⊂ U, π(V) ⊂ π(Δ(A, σ, A)) ⊂ U.

For all m ≤ n choose an open Uₙ in X with A ⊂ U₀ ⊂ U₁ ⊂ U₂ ⊂ · · · ⊂ Uₙ ⊂ U. If γ is properly extended by o let B₀γ = {B ∈ C(γ) | (B, γ) ∈ C₀(A, σ)} and A₀γ = {B ∈ B₀γ | B ⊈ U₀}. We claim A₀γ is finite. If σ = (D₀, F₀, . . . , Dₙ, Fₙ) and γ = (D₀, F₀, . . . , Dₙ, Fₙ), then σ* ⊂ Dₙ⁺ ⊂ D(γ). If B ∈ B₀γ, qₐ ∈ σ* ⊂ Dₙ⁺ = Dₙ for some M ∈ M(γ) and B ∈ C(M). Thus {B - U₀ | B ∈ A₀γ} is a set of disjoint, nonempty, compact subsets of X - U₀ whose union, if A₀γ is infinite, has, by Lemma 5, a limit point in γ*, a compact subset of U₀. This is clearly impossible, so A₀γ is finite.

Note that A₀γ = {B₀γ - A₀γ} ⊂ U₀ ∩ γ* and A₀γ = U₀ ∩ γ* ⊂ U₁ ∩ γ*.

Let Ω = {γ₀, γ₁, . . . , γₙ} | m ≤ n, γ₀ properly extends σ, and γₙ properly extends γₙ⁺ - 1 if r > 0}. By induction on m, for each E = (γ₀, . . . , γₙ) ∈ Ω we choose AₘE ⊂ Uₙ ∩ γₙ⁺ (as well as BₘE and AₘE). Having chosen Aₘ₋₁E ⊂ Uₙ₋₁ ∩ γₙ₋₁⁺ - 1 for E' = (γ₀, . . . , γₙ₋₁), we define BₘE = {B ∈ C(γₙ) | (B, γₙ) ∈ C₀(Aₘ₋₁E')}, AₘE = {B ∈ BₘE | B ⊈ Uₙ}, and AₘE = U(α₀γ - A₀γ) ⊂ Uₙ ∩ γₙ⁺. Repeating the argument for the construction of A₀γ, we learn that AₘE is finite for all m and E.

If E = (γ₀, . . . , γₙ) ∈ Ω and r < m, let E(r) = (γ₀, . . . , γₙ_r). If B ∈ AₘE define Bₙ = B and for r < m, define Bₙ to be the unique term of Bₙ₋₁ - Aₙ₋₁ such that qₐBₙ₋₁ ≤ Bₙ. Let A = {B₀ | B ∈ BₘE for some m ≤ n and E ∈ Ω having m terms}. Since Ω is finite, A is finite. Since for all (B, γ) ∈ C(A, σ, A), B ⊂ U (and A ⊂ U), π(Δ(A, σ, A)) ⊂ U as desired.

Δ is monotonically normal. It suffices to define a monotone normality operator G for Δ.

Suppose δ ∈ V which is open in Δ. If δ ∈ Δ₁ there are n ∈ ω, σ ∈ Σₙ and a finite A ⊂ C₀(σ*, σ) with δ ∈ Δ(σ*, σ, A) ⊂ V. Choose n (which determines σ) minimal, and having made this choice choose A with |A| minimal. Then A is also uniquely determined since δ ∈ Δ(σ*, σ, A) ∩ Δ(σ*, σ, A') implies δ ∈ Δ(σ*, σ, A ∩ A'). Define G(δ, V) = Δ(σ*, σ, A).

For all finite σ ∈ Σ and C ∈ D(σ), let (M(C), <) be a well ordering of M(C).

If δ = (x, σ, C, 0) ∈ Δ - Δ₁, there is M ∈ M(C) such that x = a₉. Since (M(C), <) is Dedekind complete there is a minimal M' < M in (M(C), <) such that [Δ(∪{N | M' < N < M}, σ, 0) ∪ Δ(δ, A)] ⊂ V for some A. Choose A minimal for M'. Since δ ∈ Σₙ is a well ordering there is minimal Mₙ for M' < Mₙ < M in (M(C), <). Define G(δ, V) = Δ(∪{N | M' < N < Mₙ}, σ, 0) ∪ Δ(δ, A).

Similarly, if δ = (x, σ, C, 1) ∈ Δ - Δ₁, there is M ∈ M(C) such that x = z₉. Choose M' > M maximal in (M(C), <) for [Δ(∪{N | M < N < M'}, σ, 0) ∪ Δ(δ, A)] ⊂ V. Choose A minimal. There is a minimal Mₙ for M < Mₙ < M' in (M(C), <). Define G(δ, V) = Δ(∪{N | M < N < Mₙ}, σ, 0) ∪ Δ(δ, A).

If δ = (x, σ) ∈ Δ - Δ₁, x = qₐ for some D ∈ Dₙ⁺(σ) and there is C ∈ C(σ) such that Δ(C - D, σ, 0) ⊂ V. Choose C maximal for this to be so. Since the members
of $\mathcal{C}(\sigma)$ containing $D$ are reverse well ordered by inclusion this is possible. Define $G(\delta, V) = \Delta(C - D, \sigma, \emptyset) \cup \{\delta\}$.

In all of the above cases we say $G(\delta, V) \in \mathcal{G}(\sigma)$. If $E$ is a closed subset of $\Delta$ and $E \subset V$, open, define $G(E, V) = \bigcup\{G(\delta, V) \mid \delta \in E\}$. Clearly $G$ is monotone, each $G(E, V)$ is open, and $E \subset G(E, V) \subset V$. Thus to prove $G$ is a monotonic normality operator for $\Delta$ it suffices to assume that $\delta \in V$, open, and $\beta \in W$, open, $\delta \notin W$ and $\beta \notin V$, and prove that $G(\delta, V) \cap G(\beta, W) = \emptyset$. Suppose $G(\delta, V) \cap G(\beta, W) \neq \emptyset$.

Suppose that $G(\delta, V) \in \mathcal{G}(\sigma)$. Then $\delta \in \Delta_\omega$ implies $G(\delta, V) = \Delta(A_\delta, \sigma, A_\delta)$ where $A_\delta = \sigma^*$ and $A_\delta \subset C_0(\sigma^*, \sigma)$. If $\delta = \langle x, \sigma \rangle \in \Delta_\sigma$, $G(\delta, V) = \Delta(A_\delta, \sigma, \emptyset) \cup \{\delta\}$ and $A_\delta \subset \sigma^*$. If $\delta = \langle x, \sigma, C, k \rangle$, $G(\delta, V) = \Delta(A_\delta, \sigma, \emptyset) \cup \Delta(\delta, A_\delta)$ where $A_\delta \subset C_0(\delta)$ and $A_\delta \subset \sigma^*$. Suppose $G(\beta, W) \in \mathcal{G}(\tau)$ and define $A_\beta$ and $A_\beta$ analogously. Let us now prove that if $\sigma = \tau$, $A_\beta \cap A_\delta = \emptyset$.

Assume $\sigma = \tau$. Clearly neither $\beta$ nor $\delta$ is in $\Delta_\omega$. So assume $\delta = \langle x, \sigma, E, k \rangle$ or $\langle x, \sigma, E', h \rangle$ and $\beta = \langle y, \sigma, F, k \rangle$ or $\langle y, \sigma, F', h \rangle$. There is a maximal $t \in \omega$ such that for some $C_t \in C_t(\sigma)$ both $x$ and $y$ are in $C_t$. (Since $\{x, y\} \subset \bigcap_{1 \in \omega} C_t = D_\omega \subset D_\omega^*(\sigma)$ implies $\delta = \beta = \langle yd, \sigma, \rangle$.) So $x \in M_x \in \mathcal{M}(C_1)$ and $y \in M_y \in \mathcal{M}(C_t)$. If $x \in C_x$ for some $C_x \in C_{t+1}(\sigma)$, with $A_\delta \cap C_x = \emptyset$, then $A_\delta \subset C_x$.

Otherwise $\delta = \langle a_{M_x}, \sigma, C_t, k \rangle$ for some $k < 2$. Thus, if $A_\beta \cap A_\delta \neq \emptyset$, $\delta = \langle a_{M_x}, \sigma, C_t, k \rangle$ and $\beta = \langle z_{M_y}, \sigma, C_t, h \rangle$ for some $k < 2$ and $h < 2$. If $M_x = M_y$, $x$ is $a_{M_x}$ and $y$ is $z_{M_y}$ (or vice versa) and then $A_\delta$ is contained in those $Ms$ below $M_x$ in $\langle \mathcal{M}(C_t), \leq \rangle$ and $A_\beta$ in those above. If $M_x \neq M_y$ our choice of $M^*$ ensures the $Ms$ containing $A_\beta$ and $A_\delta$ are disjoint. So $A_\beta \cap A_\delta = \emptyset$ if $\sigma = \tau$.

Now suppose $\sigma \in \Sigma_n$ and $\tau \in \Sigma_r$. Define $P_{\delta 0} = \langle A_\delta, \sigma \rangle$ and for $m < n$, define $P_{(m+1)} = C_m(A_\delta, \sigma, A_\delta)$ if $\delta \in \Delta_\omega$, $C_m(A_\delta, \sigma, \emptyset) \cup C_m(A_\delta, \sigma, \emptyset) \cup C_m(A_\delta, \sigma, \emptyset)$ if $\delta = \langle x, \sigma \rangle$, and $C_m(A_\delta, \sigma, \emptyset) \cup C_m(A_\delta, \sigma, \emptyset)$ if $\delta = \langle x, \sigma, E, k \rangle$. Then

$$G(\delta, V) = \{\delta\} \cup \bigcup_{m \leq m \leq n} \left\{\Delta^*(C, \gamma) \mid (C, \gamma) \in \bigcup_{m \leq m \leq n} P_{\delta m}\right\}.$$

Define $P_{\beta s}$ for $s \leq r$ similarly.

Since $G(\delta, V) \cap G(\beta, W) \neq \emptyset$, there are $m \leq n$ and $s \leq r$ and $\alpha \in \Delta^*(C, \gamma) \cap \Delta^*(C', \gamma')$ for some $(C, \gamma) \in P_{\delta m}$ and $(C', \gamma') \in P_{\beta s}$ whether $\alpha$ is of the form $\langle p, \eta \rangle$ or $\langle p, \eta, j, j \rangle$, $\eta$ extends both $\gamma$ and $\gamma'$. If $\gamma \neq \gamma'$, $p \in C \cap C'$ and there is $D \in \mathcal{D}(\gamma)$ with $\eta^* \subset \gamma^* \subset D$. If $D \subset C$, $\beta \in \Delta(\gamma^*, \gamma) \subset \Delta(C, \gamma) \subset G(\delta, V)$ contrary to assumption. Otherwise $D \subset C = \emptyset$ contradicting $C \cap C' = \emptyset$.

Thus $\gamma = \gamma'$. We cannot have both $m = 0$ and $r = 0$ since then $\langle C, \gamma \rangle = \langle A_\delta, \sigma \rangle$ and $\langle C', \gamma' \rangle = \langle A_\beta, \sigma \rangle$ and we have shown that $A_\delta \cap A_\beta = \emptyset$ and thus $\Delta(A_\delta, \sigma) \cap \Delta(\beta, \tau) = \emptyset$ contrary to assumption. Hence say $r > 0$ so $C' \subset C(\gamma)$. If $m = 0$ and $\gamma = \sigma$, $C = A_\delta$ which is not cut by any $C' \subset C(\sigma)$. If $m > 0$, $C \subset C(\gamma)$ which also has this property. So in any case, either $C' \subset C$ or $C \subset C'$. If $C' \subset C$ but $C \neq C'$, then $\tau$ properly extends $\gamma$ and there is $D \subset C$ such that $D \in \mathcal{D}(\sigma)$ and $\tau^* \subset D$. Thus $\beta \in \Delta(C, \gamma) \subset \mathcal{W}(\delta, V)$ contrary to assumption. The same argument shows $C \not\subset C'$ unless $C = C'$; hence $C = C'$.
Since \( C = C' \in \mathcal{C}(\gamma) \) and \( m = 0 \) implies \( \gamma = \sigma \) and \( C = A_\delta \notin \mathcal{C}(\sigma) \) unless \( C = \sigma^* \) and \( C' = \sigma^{**} \) and \( \gamma - \sigma \) implies \( \delta \in \Delta(\sigma^*, \sigma) \subset G(\beta, W), \ m \neq 0 \).

Thus \( \langle C, \gamma \rangle = \langle C', \gamma' \rangle \in P_{m\delta} \cap P_{r\beta} \) where both \( m \) and \( r \) are greater than zero. Thus there are \( \langle E, \rho \rangle \in P_{(m-1)\delta} \) and \( \langle E', \rho' \rangle \in P_{(r-1)\beta} \) where \( \Gamma(C, \gamma) \) extends both \( \rho \) and \( \rho' \) and \( Q_C \in E \cap E' \). These facts are precisely what is needed to apply the same argument again to show that \( \langle E', \rho' \rangle = \langle E', \rho' \rangle \) where both \( (m-1) \) and \( (r-1) \) are greater than zero. Repeating the argument \( \leq m \) times clearly leads us to a contradiction, thus proving that \( \Delta \) is monotonically normal.

Since \( \Delta \) is compact, separable, zero dimensional, and monotonically normal and \( \pi: \Delta \to X \) is continuous and onto our theorem is proved. \( \square \)

References