Traveling wave solutions of the $n$-dimensional coupled Yukawa equations

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Abstract

We discuss traveling wave solutions to the Yukawa equations, a system of nonlinear partial differential equations which has applications to meson–nucleon interactions. The Yukawa equations are converted to a six-dimensional dynamical system, which is then studied for various values of the wave speed and mass parameter. The stability of the solutions is discussed, and the methods of competitive modes is used to describe parameter regimes for which chaotic behaviors may appear. Numerical solutions are employed to better demonstrate the dependence of traveling wave solutions on the physical parameters in the Yukawa model. We find a variety of interesting behaviors in the system, a few of which we demonstrate graphically, which depend upon the relative strength of the mass parameter to the wave speed as well as the initial data.

1. Introduction

The coupled Yukawa equations (a coupled system of PDEs) for $u = u(x, t)$ and $a = a(x, t)$ (where $x \in \mathbb{R}^n$; hence the problem domain is $\mathcal{D} = \mathbb{R}^n \times [0, \infty)$) read

$$iu_t + \Delta u = -au,$$
$$a_{tt} - \Delta a = m^2 a + |u|^2. \quad (1.1)$$

Note $u$ may be complex, so the $|\cdot|$ denotes the complex modulus; $a$, on the other hand, is a real field. Yukawa [1] give the equation as a local correction to Poisson’s equation, with the purpose of explaining the strong interaction binding neutrons and protons. Due to its physical significance and mathematical structure, the equation has inspired several studies (the original paper of Yukawa has been cited over 700 times). We should also note that certain variants of this system are referred to as relativistic Vlasov–Yukawa systems (see [2–5]). For some more recent references, see, for instance, [2] and the references therein. The coupled Yukawa equation (1.1) is also referred to as the Klein–Gordon–Schrödinger system [6,7].

Assuming a traveling wave solution

$$u(x, t) = U(z), \quad a(x, t) = A(z), \quad (1.2)$$

where

$$z = \frac{1}{\alpha} \sum_{i=1}^{n} \alpha_i x_i - ct, \quad \alpha \equiv \sqrt{\sum_{i=1}^{n} \alpha_i^2}, \quad (1.3)$$
wefindthatthecoupledYukawaequation (1.1) becomes
\[ -ciU' + U'' = -UA, \]
\[ -(1 + c^2)A'' = m^2A + |U|^2. \]  
(1.4)

We can convert this to a system of three real ODEs by setting \( U = U_R + iU_I \), so that
\[ U_R'' + cU_I' = -U_R A, \]
\[ U_I'' - cU_R' = -U_I A \]
\[ (1 + c^2)A'' + m^2A = -(U_R^2 + U_I^2). \]  
(1.5)

Wegivendatafor \( z = 0 \), we may writethis as a six-dimensionaldynamical system, whose only equilibrium is \( 0 \in \mathbb{R}^6 \). In particular let \( U_R = y_1, U_I' = y_2, U_I = y_3, U_I' = y_4, A = y_5, A' = y_6 \). Then, (1.5) becomes
\[ y_1' = y_2, \]
\[ y_2' = -cy_4 - y_1y_5, \]
\[ y_3' = y_4, \]
\[ y_4' = cy_2 - y_3y_5, \]
\[ y_5' = y_6, \]
\[ y_6' = \frac{-1}{1 + c^2}(m^2y_5 + y_1^2 + y_3^2). \]  
(1.6)

2. Stability of the dynamical system

We now turn our attention to the stability of the dynamical system (1.6). From (1.6) it is clear that \( 0 \in \mathbb{R}^6 \) is the unique equilibrium. The Jacobian evaluated at this equilibrium reads
\[
J = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -c & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -m^2/(1 + c^2) & 0
\end{bmatrix}
\]  
(2.1)

and this implies that the characteristic polynomial is
\[ \rho(\lambda) = \lambda^6 + \frac{m^2}{1 + c^2}\lambda^4. \]  
(2.2)

The eigenvalues of \( J \) are then \( \lambda_{1,2,3,4} = 0 \) (multiplicity 4), \( \lambda_5 = \frac{-im}{\sqrt{1+c^2}} \) and \( \lambda_6 = -\frac{-im}{\sqrt{1+c^2}} \).

Clearly, with multiple eigenvalues equal to zero, the long-run behavior of the system may not be stable. For instance, there are two complex conjugate eigenvalues and the other eigenvalues are zero, indicating that generalized Hopf bifurcations are possible. In order to further determine if any interesting behaviors occur in the dynamical system (1.6), we turn to a competitive modes analysis (see, for instance, the Refs. [8–14]). Recall that the following conjecture is posed in [8] (pp. 95).

Conjecture (Yu). The conditions for dynamical systems to be chaotic are given by

1. there exist at least two mode frequencies, labeled by the \( g_i \)’s, in the system and these must be positive;
2. at least two \( g_i \)’s are competitive or nearly competitive, that is, for some \( i \) and \( j \), \( g_i \approx g_j > 0 \) at some \( t \);
3. at least one of the \( g_i \)’s is a function of evolution variables such as \( t \);
4. at least one of the \( h_i \)’s is a function of system variables.

Here we have maintained the language of [8]. To clarify, really the \( g_i \)’s are mode frequencies squared. So, when \( g_i < 0 \), we say that such a frequency does not exist. The functions \( g_i \) themselves will always exist for any dynamical system, although when they are negative the modes (along the lines of the Conjecture of Yu) are not competitive; hence a relevant mode frequency does not exist. To reiterate, existence of a function \( g_i \) is not strong enough. Rather, at the point \( t \) of interest, we require that \( g_i(t) > 0 \) if \( g_i \) is to be used as in the conjecture.
From (1.6), we find that

\[
\begin{align*}
    y_1' &= -g_1 y_1 + h_1 = -y_3 y_1 - cy_3, \\
    y_2' &= -g_2 y_2 + h_2 = -(c^2 + y_3) y_2 + cy_3 y_5 - y_1 y_6, \\
    y_3' &= -g_3 y_3 + h_3 = -m y_3 + cy_2, \\
    y_4' &= -g_4 y_4 + h_4 = -(c^2 + y_3) y_4 - cy_4 y_5 - y_3 y_6, \\
    y_5' &= -g_5 y_5 + h_5 = -m y_5 + cy_6 - y_1^2 - y_3^2, \\
    y_6' &= -g_6 y_6 + h_6 = -\frac{m^2}{1+c^2} y_6 - \frac{2}{1+c^2} (y_1 y_2 + y_3 y_4).
\end{align*}
\]

(2.3)

This tells us that

\[
g_1 = g_3 = y_5, \quad g_2 = g_4 = c^2 + y_5, \quad g_5 = m, \quad g_6 = \frac{m^2}{1+c^2}.
\]

The redundancy \(g_1 = g_3 \) is due to the separation of real and imaginary parts for \( U \); really, we will have four distinct modes. The possible competitiveness conditions are then: (i) \( y_5 = m \), (ii) \( y_5 = m - c^2 \), (iii) \( y_5 = \frac{m^2}{1+c^2} \), (vi) \( m = 1 + c^2 \), and (v) \( y_5 = m^2 - c^2 (1 + c^2) \).

In order to induce competitiveness, it is sufficient to consider the modes to be competitive at \( z = 0 \), so that either (i) \( y_5(0) = m \), (ii) \( y_5(0) = m - c^2 \), (iii) \( y_5(0) = \frac{m^2}{1+c^2} \) or (vi) \( y_5(0) = \frac{m^2-c^2(1+c^2)}{1+c^2} \).

Regarding positivity, \( m > 0 \) and \( c^2 > 0 \) hence \( y_5 > 0 \) implies that \( g_k > 0 \) for all \( k = 1, 2, 3, 4 \) while \( g_5 > 0 \) and \( g_6 > 0 \) are always true.

3. Numerical simulations

In order to obtain numerical solutions for the six-dimensional system arising from the Yakawa equations, we apply the RKF-45 method implemented in Maple 13. We set the system up as an initial value problem and integrate out from \( z = 0 \) in order to obtain the solutions. We are then able to obtain the physical quantities \( |u(x, t)| = |U(z)| \) and \( |a(x, t)| = |A(z)| \). We shall select to distinct parameter regimes:

Case 1. \( y_1(0) = y_2(0) = y_5(0) = 1, y_3(0) = y_4(0) = y_6(0) = 0, m = \sqrt{5}, c = 2 \).

Case 2. \( y_1(0) = y_2(0) = 10^{-4}, y_5(0) = 10^{-2}, y_3(0) = y_4(0) = y_6(0) = 0, m = \frac{1}{10}, c = 2 \).

Note that both combinations of parameters satisfy competitiveness condition (v). In Fig. 1, we plot the phase portrait in \( \mathbb{R}^3 \) for \( (y_1(z), y_3(z), y_5(z)) \) for Case 1(a) and Case 2(b). Observe that, while both cases lead to oscillatory \( y_1(z) \) and \( y_3(z) \), the dynamics are much more regular for the Case 2 parameter regime. This hints at the fact that chaotic behaviors may emerge for some values of the parameters.

In Fig. 2, we plot the value \( |U(z)| \) for each of the two cases. We can do the same for \( |A(z)| \), which has far more dull behavior in each case. In Fig. 3, we show the Case 1 plot for \( |A(z)| \); the Case 2 figure is similar, but scaled by several orders of magnitude in value so that it is too small to show side by side with the Case 1 plot. Hence, it is the dynamics of the complex
valued solution for $U(z)$ which are of the most interest. Note that the pattern of oscillation is tied to the choice of $m$ and $c$, as expected.

In Fig. 4, we plot the mode frequencies for each set of parameter values considered. By construction, we have that $g_1 = g_5$ at $z = 0$ for Case 1 parameter values. Meanwhile, for Case 2, we note that the modes do not become competitive. In Case 1, the trajectories in phase space become more erratic, whereas there is more order present in Case 2, when the modes remain non-competitive. Hence, when two modes become competitive at $z = 0$, chaotic oscillation may result, whereas when the modes are separated slightly, more regular oscillations are noted in the phase portraits.

4. Conclusions

Under the assumption of traveling wave solutions, we have converted the Yukawa equations into a six-dimensional nonlinear dynamical system. After discussing the stability of such a system, we have obtained numerical solutions in order to highlight some of the qualitative features of this system. While the system quickly becomes unstable for large initial data, we find that for small initial data a variety of solutions are possible, depending on the strength of the mass parameter $m$ to the strength of the wave speed $c$. The numerical solutions describing the time evolution show that the system shows oscillatory behavior for negative $z$ after which the system may leave the orbit and fly off toward $\infty$. While specific values of the parameters can be selected to delay this process, for all specific values of the parameters taken it appears that this...
breakdown of stable behavior always occurs for the small initial data regime. As this is a new dynamical system, there may be other behaviors which we have not noticed in the numerical simulations considered. Hence, this system, and related systems could be an area of interest for those studying chaotic behavior in nonlinear dynamical systems related to physical models. Additionally, we have seen one use of competitive modes analysis, namely, determining when a dynamical system will change qualitative behavior from regular to chaotic oscillations.

References


Fig. 4. Plot of the mode frequencies $g_3(z)$, $g_2(z)$, $g_5(z)$ and $g_6(z)$ for Case 1(a) and Case 2(b) parameter values. Notice that, by construction of the parameter sets taken, the two mode frequencies $g_1(z)$ and $g_6(z)$ become equal at $z = 0$ in Case 1, and nearly competitive in Case 2.