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Note

A generalization of the Erdös–Ko–Rado theorem☆

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ABSTRACT

In this note, we investigate some properties of local Kneser graphs defined in [János Körner, Concetta Pilotto, Gábor Simonyi, Local chromatic number and sperner capacity, J. Combin. Theory Ser. B 95 (1) (2005) 101–117]. In this regard, as a generalization of the Erdös–Ko–Rado theorem, we characterize the maximum independent sets of local Kneser graphs. Next, we provide an upper bound for their chromatic number.

1. Introduction

In this section, we elaborate on some basic definitions and facts that will be used later. Throughout the paper the word *graph* is used for a finite simple graph with a prescribed set of vertices. A *homomorphism* $\sigma : G \longrightarrow H$ from a graph G to a graph H is a map $\sigma : V(G) \longrightarrow V(H)$ such that $uv \in E(G)$ implies $\sigma(u)\sigma(v) \in E(H)$. The existence of a homomorphism is indicated by the symbol $G \longrightarrow H$. Two graphs G and H are homomorphically equivalent if $G \longrightarrow H$ and $H \longrightarrow G$ (for more on graph homomorphisms see [6]).

In [3] Bondy and Hell define $\nu(G, K)$, for two graphs G and K, as the maximum number of vertices in an induced subgraph of G that admits a homomorphism to K, and using this they introduce the following generalization of a result from Albertson and Collins [1].

Theorem A (*[3]*). *Let G, H and K be graphs where H is a vertex-transitive graph. If there exists a homomorphism $\sigma : G \longrightarrow H$ then $\frac{|V(G)|}{\nu(G,K)} \le \frac{|V(H)|}{\nu(H,K)}$.*

Hereafter, we denote by $[m]$ the set $\{1, 2, \ldots, m\}$, and denote by $\binom{[m]}{n}$ the collection of all n-subsets of $[m]$. Suppose $m \ge 2n$ are positive integers. The *Kneser graph* $KG(m, n)$ has the vertex set $\binom{[m]}{n}$, in which $A \sim B$ if and only if $A \cap B = \emptyset$. It was conjectured by Kneser [7] in 1955 and proved by Lovász [9] in 1978 that $\chi(KG(m, n)) = m - 2n + 2$.

The *local chromatic number* of a graph was defined in [4] as the minimum number of colors that must appear within distance 1 of a vertex. Here is the formal definition.

Definition 1 (*[4]*). Let G be a graph. Define the local chromatic number of G as follows.

$$\psi(G) \overset{\text{def}}{=} \min_{c} \max_{v \in V(G)} |\{c(u) : u \in V(G), d_G(u, v) \le 1\}|,$$

where the minimum is taken over all proper colorings c of G and $d_G(u, v)$ denotes the distance between u and v in G. ■

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The local chromatic number of graphs has received attention in recent years [2,8,10–12]. Clearly, $\psi(G)$ is always bounded from above by the chromatic number, $\chi(G)$. It is much less obvious that $\psi(G)$ can be strictly less than $\chi(G)$. In fact, it was proved in [4] that there exist graphs with $\psi(G) = 3$ and $\chi(G)$ being arbitrarily large.

One can define $\psi(G)$ via graph homomorphism. In this regard, local complete graphs were defined in [4]. We consider the following definition for local complete graphs.

Definition 2 (*[4,8]*). Let n and r be positive integers where $n \geq r$. Define the local complete graph $U(n, r)$ as follows.

$$V(U(n, r)) = \{(a, A) | \, a \in [n], A \subseteq [n], |A| = r - 1, a \notin A\}$$

and

$$E(U(n, r)) = \{\{(a, A), (b, B)\} | \, a \in B, b \in A\}. \quad \blacksquare$$

The following simple lemma reveals the connection between local complete graphs and local chromatic number. Note that a restatement of the next lemma is proved in the course of proving Lemma 1.1 of [4].

Lemma A (*[4]*). *Let G be a graph. The graph G admits a proper coloring c with n colors and $\max_{v \in V(G)} |\{c(u)| \, u \in N[v]\}| \leq r$ if and only if there exists a homomorphism from G to $U(n, r)$. In particular $\psi(G) \leq r$ if and only if there exists an n such that G admits a homomorphism to $U(n, r)$.*

In [8] the local complete graphs have been generalized as follows.

Definition 3 (*[8]*). Let n, r and t be positive integers where $n \geq r \geq 2t$. Let $U_t(n, r)$ be a graph whose vertex set contains all ordered pairs (A, B) such that $|A| = t$, $|B| = r - t$, $A, B \subseteq [n]$ and $A \cap B = \varnothing$. Also, two vertices (A, B) and (C, D) of $U_t(n, r)$ are adjacent if $A \subseteq D$ and $C \subseteq B$. $\blacksquare$

Remark. Note that $U_1(n, r) = U(n, r)$, while $U_t(r, r) = KG(r, t)$. Hence the graph $U_t(n, r)$ provides a common generalization of Kneser graphs and local complete graphs $U(n, r)$ in [8]. In this paper, the graph $U_t(n, r)$ is termed *local Kneser graph*.

In the next section, some results concerning the local Kneser graphs are presented. In this regard, as a generalization of the Erdös–Ko–Rado theorem, we characterize the maximum independent sets of local Kneser graphs. Next, we introduce some upper bounds for their chromatic number and local chromatic number.

2. Local Kneser graphs

In this section, we study some properties of the graph $U_t(n, r)$. First, we characterize the maximum independent sets of $U_t(n, r)$. To begin we compute the independence number of $U_t(n, r)$. Now, we introduce some notations which will be used throughout the paper.

Assume that σ is a permutation of $[n]$, $R \subseteq [n]$ and $|R| = r$. It should be noted that σ provides an ordering for $[n]$, i.e., $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$. Define $\min_\sigma R$ as being the minimum member of R according to the ordering σ, i.e., $\min_\sigma R \overset{\text{def}}{=} \sigma(\min\{\sigma^{-1}(r) | r \in R\})$.

Define

$$V_R \overset{\text{def}}{=} \{(A, B) \mid A \cup B = R, |A| = t \text{ and } A \cap B = \varnothing\}$$

and set

$$I_{\sigma,R} \overset{\text{def}}{=} \left\{(A, B) \in V_R | \min_\sigma R \in A\right\}.$$

Also, define

$$S_\sigma \overset{\text{def}}{=} \bigcup_{R \subseteq [n], |R| = r} I_{\sigma,R}.$$

The independence number of $U_t(n, r)$ has been computed in [8] as follows. It is clear that the induced subgraph of $U_t(n, r)$ obtained by the vertices in V_R is isomorphic to the Kneser graph $KG(r, t)$ and it is denoted by $KG_R(r, t)$. That is the reason that the graph $U_t(n, r)$ is called the local Kneser graph. It is straightforward to check that for every $\sigma \in S_n$, $I_{\sigma,R}$ is a maximum independent set of $KG_R(r, t)$. Also, one can easily see that S_σ is an independent set in $U_t(n, r)$ of order $\binom{n}{r}\binom{r-1}{t-1}$. Hence, $\alpha(U_t(n, r)) \geq \binom{n}{r}\binom{r-1}{t-1}$. The reverse inequality $\alpha(U_t(n, r)) \leq \binom{n}{r}\binom{r-1}{t-1}$ follows from the Erdös–Ko–Rado theorem. In fact, once the chosen r-set is fixed, such as R, the induced subgraph $KG_R(r, t)$ has the independence number $\binom{r-1}{t-1}$. Thus, we know $\alpha(U_t(n, r)) = \binom{n}{r}\binom{r-1}{t-1}$.

Also, in view of Bondy and Hell's theorem [3], one can obtain the inequality $\alpha(U_t(n,r)) \le \binom{n}{r}\binom{r-1}{t-1}$. Indeed, $KG(r,t)$ is a subgraph of $U_t(n,r)$. Hence, we have $KG(r,t) \to U_t(n,r)$. If we set $K \stackrel{\text{def}}{=} K_1$ in the Bondy and Hell theorem, then we have

$$\frac{\binom{r}{t}}{\binom{r-1}{t-1}} \le \frac{\binom{n}{r}\binom{r}{t}}{\alpha(U_t(n,r))}.$$

Hence, $\alpha(U_t(n,r)) \le \binom{n}{r}\binom{r-1}{t-1}$. Consequently, $\alpha(U_t(n,r)) = \binom{n}{r}\binom{r-1}{t-1}$ and S_σ is a maximum independent set of $U_t(n,r)$. Now, we are ready to show that for every maximum independent set S in $U_t(n,r)$ there exists a permutation $\sigma \in S_n$ such that $S = S_\sigma$.

Consider a maximum independent set S in $U_t(n,r)$. Note that $|S| = \binom{n}{r}\binom{r-1}{t-1}$. One can easily see that for every $R \subseteq [n]$ ($|R| = r$), $V_R \cap S$ is a maximum independent set in $KG_R(r,t)$. By the Erdös-Ko-Rado theorem [5], if $r > 2t$, then there is an $x(S,R) \in R$ such that $x(S,R) \in \bigcap_{(A,B)\in V_R\cap S} A$.

Lemma 1. *Let S be a maximum independent set in $U_t(n,r)$ where $n \ge r > 2t$. Also, assume that R, R' are two distinct r-subsets of $[n]$. If $x(S,R) = x \in R\cap R'$, then $x(S,R') \notin R\cap R' \setminus \{x\}$.*

Proof. Assume that $x(S,R) = x$ and $x(S,R') = z$. We prove this lemma by induction on $|R \setminus R'|$.
Let $|R\setminus R'| = 1$. Then there are $u \in R$ and $v \in R'$ such that $R = (R' \setminus \{v\}) \cup \{u\}$. If $x(S,R') = z \in R\cap R'\setminus\{x\}$, then there exist $(A,B) \in S$ and $(A',B') \in S$ such that $A, B \subset R$, $x \in A$, $u, z \in B$ and $A', B' \subset R'$, $z \in A'$, $x, v \in B'$, $A' \subset B$, $A \subset B'$. Hence, (A,B) and (A',B') are adjacent, which is a contradiction.

Suppose that $k > 1$ and the lemma holds for $|R\setminus R'| < k$. Now, let $|R\setminus R'| = k$. On the contrary, assume that $z \in R\cap R'\setminus\{x\}$. Choose $y \in R\setminus R'$ and $y' \in R'\setminus R$ and set $R'' = (R'\setminus\{y'\})\cup\{y\}$. Since $|R'\setminus R''| = 1 < k$, we have $x(S,R'') \notin R'\cap R''\setminus\{z\}$, consequently, $x(S,R'') \in \{y,z\}$. On the other hand, $|R\setminus R''| = k-1 < k$, hence, $x(S,R'') \notin R\cap R''\setminus\{x\}$. However, $\{y,z\} \subset R\cap R''\setminus\{x\}$ which is a contradiction. ■

Now, we characterize the maximum independent sets of local Kneser graphs.

Theorem 1. *Let S be a maximum independent set in $U_t(n,r)$. Then there exists a permutation $\sigma \in S_n$ such that $S = S_\sigma$.*

Proof. Suppose that S is a maximum independent set in $U_t(n,r)$. We define a directed graph D_S whose vertex set and edge set are

$$V(D_S) = \{1, 2, \ldots, n\}$$

and

$$E(D_S) \stackrel{\text{def}}{=} \{(i,j) \mid \exists R \subseteq [n], |R| = r, i \ne j, \{i,j\} \subseteq R, i = x(S,R)\},\ \text{respectively.}$$

Assume that $d_1 \ge d_2 \ge \cdots \ge d_n$ is the out degree sequence of D_S where d_i is the out degree of v_i for $i = 1, 2, \ldots, n$. In view of Lemma 1, one can see that D_S is a directed graph whose underlying graph is a simple graph. Consequently, $|S| = \binom{d_1}{r-1}\binom{r-1}{t-1} + \binom{d_2}{r-1}\binom{r-1}{t-1} + \cdots + \binom{d_n}{r-1}\binom{r-1}{t-1}$. However, $\binom{d_1}{r-1} + \binom{d_2}{r-1} + \cdots + \binom{d_n}{r-1}$ is maximized when $d_1 = n-1, d_2 = n-2, \ldots, d_{n-r+1} = r-1$. Moreover, $\binom{n-1}{r-1} + \binom{n-2}{r-1} + \cdots + \binom{r-1}{r-1} = \binom{n}{r}$. Now choose a permutation $\sigma \in S_n$ such that $\sigma(i) = v_i$ for $i = 1, 2, \ldots, n-r+1$. Obviously, $S = S_\sigma$. ■

From the above discussion, the directed graph D_S is related to the independent set S of $U_t(n,r)$. Conversely, suppose that D is a directed graph on $[n]$ such that its underlying graph is a simple graph. Now, we want to construct an independent set I_D which is related to D. Set

$$I_D = \{(A,B) \mid \exists i \in [n]; A, B \subseteq N^+(i) \cup \{i\}, i \in A, A\cap B = \varnothing, |A| = t\ \ |B| = r-t\},$$

where $N^+(i) = \{j \mid (i,j) \in E(D)\}$. As the underlying graph of D is a simple graph, one can see that I_D is an independent set in $U_t(n,r)$. It is easy to see that for any maximum independent set S in $U_t(n,r)$ we have $I_{D_S} = S$.

Clearly, $f : U_t(n,r) \longrightarrow KG(n,t)$ is a homomorphism where $f((A,B)) \stackrel{\text{def}}{=} A$. Therefore, $\chi(U_t(n,r)) \le n - 2t + 2$. The chromatic number of local complete graphs has been investigated in [4].

Theorem B ([4]). *Let n and r be positive integers where $n \ge r$. We have $\chi(U(n,r)) \le r2^r \log_2\log_2 n$.*

Here we introduce an upper bound for the chromatic number of local Kneser graphs.

Theorem 2. *If n, r and t are positive integers where $n \ge r \ge 2t$, then*

$$\chi(U_t(n,r)) \le \left\lceil \frac{r^2}{t}\ln\left(\frac{en}{r}\right) + r\ln\left(\frac{er}{t}\right)\right\rceil.$$

Proof. If $r = 2t$, then $U_t(n, r)$ is a matching which implies that $\chi(U_t(n, r)) = 2$ and the assertion follows. Thus, suppose that $r \geq 2t + 1$. Assume that $\sigma_1, \sigma_2, \ldots, \sigma_l$ are l random permutations of S_n such that they have been chosen independently and uniformly. For each vertex $(A, B) \in V(U_t(n, r))$, define $\mathcal{E}_{(A,B)}$ to be the event that $(A, B) \notin \bigcup S_{\sigma_i}$. Obviously, $(A, B) \in S_\sigma$ if and only if there exists $a \in A$ such that a precedes all elements of $A \cup B \setminus \{a\}$ in σ. Clearly, $Pr(\mathcal{E}_{(A,B)}) = (1 - \frac{t}{r})^l$. Consider a random variable X where $X(\sigma_1, \ldots, \sigma_l) \stackrel{\text{def}}{=} |V(U_t(n, r)) \setminus \bigcup S_{\sigma_i}|$. Clearly, $E(X) = \binom{r}{t}\binom{n}{r}(1 - \frac{t}{r})^l$. It is well-known that $1 + x \leq e^x$ and $\binom{p}{q} \leq (\frac{ep}{q})^q$, consequently,

$$E(X) = \binom{r}{t}\binom{n}{r}\left(1 - \frac{t}{r}\right)^l \leq \left(\frac{er}{t}\right)^t \left(\frac{en}{r}\right)^r e^{-\frac{tl}{r}}.$$

If $l \stackrel{\text{def}}{=} \lceil \frac{r^2}{t} \ln(en) - \frac{r^2}{t} \ln r + r \ln(er) - r \ln t \rceil$, then one can check that $E(X) < 1$. Hence, $\chi(U_t(n, r)) \leq \lceil \frac{r^2}{t} \ln(\frac{en}{r}) + r \ln(\frac{er}{t}) \rceil$. ∎

Theorem 2 immediately yields the following corollary.

Corollary 1. *Let n and r be positive integers where $n \geq r$. We have*

$$\chi(U(n, r)) \leq \left\lceil r^2 \ln\left(\frac{en}{r}\right) + r \ln(er) \right\rceil.$$

In other words, the previous corollary shows if we have a proper coloring for a graph G with n colors which assigns at most r colors in the closed neighborhood of every vertex, then $\chi(G) \leq \lceil r^2 \ln(\frac{en}{r}) + r \ln(er) \rceil$. The two upper bounds in Theorem B and Corollary 1 are complementary.

Note that $KG(r, t)$ is a subgraph of $U_t(n, r)$, consequently, $r - 2t + 2$ is a lower bound for the chromatic number of $U_t(n, r)$ while here we show that $r - 2t + 2$ is an upper bound for the local chromatic number of $U_t(n, r)$.

Lemma 2. *Assume that n, r and t are positive integers where $n \geq r \geq 2t$. Then $\psi(U_t(n, r)) \leq r - 2t + 2$.*

Proof. Let $(A, B) \in V(U_t(n, r))$, $A = \{a_1, a_2, \ldots, a_t\}$ and $B = \{b_1, b_2, \ldots, b_{r-t}\}$ such that $a_1 < a_2 < \cdots < a_t$ and $b_1 < b_2 < \cdots < b_{r-t}$. Now, we show that there exists a graph homomorphism from $U_t(n, r)$ to $U(n - t + 1, r - 2t + 2)$. To see this, define $f((A, B)) \stackrel{\text{def}}{=} (\min A, B^*)$ where $\min A = a_1$ and $B^* \stackrel{\text{def}}{=} \{b_1, b_2, \ldots, b_{r-2t+1}\}$. If (A, B) and (C, D) are adjacent in $U_t(n, r)$, then obviously $\min A \in D^*$, $\min C \in B^*$ and $\min A \neq \min C$. Therefore, f is a graph homomorphism, as desired. ∎

The aforementioned lemma motivates us to propose the following question.

Question 1. *Assume that n, r and t are positive integers where $n \geq r \geq 2t$. Is it true that $\psi(U_t(n, r)) = r - 2t + 2$?*

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