On Spectral Asymptotics for Nonlinear Sturm–Liouville Problems

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By investigating the asymptotic properties of the eigenfunctions for a general class of nonlinear Sturm–Liouville problems, we shall establish a formula for spectral asymptotics.

1. INTRODUCTION

We consider the following nonlinear Sturm–Liouville problem,

\[
\begin{cases}
-u''(x) = f(u(x)) - \mu u(x), & x \in I = (0,1), \\
u(0) = u(1) = 0,
\end{cases}
\]

where \(f(u)\) is odd and \(C^1\).

The purpose of this paper is to establish a formula of spectral asymptotics from the standpoint of an \(L^2\)-theory. More precisely, let \((u_n(\mu, x), \mu) \in C^2(I) \times (-n\pi)^2, \infty)\) be a solution of (1.1), where \(n - 1\) is the number of the interior zeros of \(u_n(\mu, x)\), and

\[
\alpha_n(\mu) = \left( \int_I |u_n(\mu, x)|^2 \, dx \right)^{1/2}.
\]

In this paper we shall establish an asymptotic formula of \(\alpha_n(\mu)\) as \(\mu \to \infty\).

Related problems were investigated by Chiappinelli [5, 7] and Benguria and Depassier [1]. Chiappinelli [5] established the following asymptotic

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formula of $\mu = \mu_s(\alpha)$ as $\alpha \to 0$, that is, equivalent to considering the asymptotic behavior of $\alpha_s(\mu)$ as $\mu \to -(n\pi)^2$: under the general growth condition on $f$, namely, if $f$ satisfies

$$|f(u)| \leq a|u|^p,$$

where $p > 1$ and $a \geq 0$, then

$$|\mu_s(\alpha) + (n\pi)^2| = O(\alpha^{p-1}).$$

On the other hand, Benguria and Depassier [1] showed that for $f(u) = u^3$, the following asymptotic formula holds as $\mu \to \infty$:

$$\mu \sim \frac{1}{16} \alpha(\mu)^4.$$  \hspace{1cm} (1.3)

They obtained (1.3) by means of direct calculation of elliptic functions. Therefore, no transparent observation for (1.3) was given in [1]. We also refer to Heinz [8, 9], who treated related problems to (1.1) by using Ljusternik–Schnieriman theory.

Recently, motivated by the result of [1], Shibata [10] extended the formula (1.3) for pure power nonlinearity $f(u) = |u|^{p-1}u$ ($p > 1$): as $\mu \to \infty$, the following asymptotic formula holds,

$$\alpha_s(\mu)^2 = B_n \mu^{(5-p)/(2(p-1))} + o(\mu^{(5-p)/(2(p-1))}),$$  \hspace{1cm} (1.4)

where

$$B_n := \frac{\sqrt{\pi}}{2} \left( \frac{2}{p-1} \right)^{2/(p-1)} \frac{2^{p-1}}{\Gamma \left( \frac{p+3}{2p-1} \right) \Gamma \left( \frac{p+1}{2p-1} \right)}.$$  

If $f(u) = |u|^{p-1}u$, then the situation is simple and by virtue of simple scaling, (1.4) was obtained. However, it seems that no results such as (1.4) have been given for general nonlinearity $f$.

In this paper, by investigating the asymptotic behavior of eigenfunctions $u_s(\mu, x)$ as $\mu \to \infty$, we shall extend the formula (1.4) for a class of $f$ which satisfies the following conditions (A.1)–(A.3):

(A.1) (Existence) $f(0) = f'(0) = 0$, $g(t) := f(t)/t$ is positive and increasing for $t > 0$, $g(t) \to \infty$ as $t \to \infty$.

(A.2) (Asymptotics) Let $h(t) := f(t)/t^p$ ($t > 0$, $p > 1$). Then for any constant $C > 0$

$$\frac{h(t)}{h(s)} \to 1 \quad \text{as} \quad t, s \to \infty \quad \text{satisfying} \quad C \leq \frac{t}{s} \leq C^{-1}.$$
Furthermore, there exist constants $0 < \epsilon_1 \ll 1, A_1 \gg 1, C_1 \gg 1, C_2, C_3 > 0$ such that for $t, s \geq C_1$ satisfying $t/s \leq \epsilon_1 \,(\text{resp.} \, \geq A_1)$

$$\frac{h(t)}{h(s)} \leq C_2 \quad (\text{resp.} \, \geq C_3).$$

(A.3) (Monotonicity) $h$ satisfies one of the following conditions:

1. $h'(s) \geq 0$ for $s > 0$.
2. $h'(s) \leq 0$ for $s > 0$.

Typical examples of $f(u)$ satisfying (A.1)–(A.3) are collected below $(p > 1)$:

$$f(u) = |u|^{p-1}u,$$
$$f(u) = |u|^{p-1}u \log(|u| + 1),$$
$$f(u) = |u|^{p-1}ue^{-|u|/|u|+1}.$$

Before stating our results, let us define the ground state $w$ of Eq. (1.5),

$$\begin{cases}
-w''(t) = w(t)^p - w(t), & t \in \mathbb{R}, \\
\quad w(t) > 0, & t \in \mathbb{R}, \\
\quad \lim_{t \to \pm \infty} w(t) = 0.
\end{cases}$$

Then there exists a unique solution $w$ of (1.5). We call it the ground state. For the existence and uniqueness of the ground state, we refer the reader to Berestycki and Lions [3].

Now we state our results.

**Theorem 1.1.** As $\mu \to \infty$, the following asymptotic formula holds,

$$\alpha_n(\mu)^2 = B_n(g^{-1}(\mu))^2 \mu^{-1/2} + o\left((g^{-1}(\mu))^2 \mu^{-1/2}\right),$$

where $g^{-1}$ is the inverse function of $g$.

**Theorem 1.2.** Let

$$w_{j,\mu}(t) := \begin{cases}
(g^{-1}(\mu))^{-1}u_n(\mu, x), & t \in I_{n,\mu} := \left(-\frac{1}{2n} \mu^{1/2}, \frac{1}{2n} \mu^{1/2}\right), \\
0 & t \notin I_{n,\mu},
\end{cases}$$

where $t = \mu^{1/2}(x - j/n - 1/2n)$ for $x \in (j/n, (j + 1)/n)$ ($j = 0, 1, \ldots, n - 1$). Then $|w_{j,\mu}| \to w$ as $\mu \to \infty$ not only uniformly on any compact subsets in $\mathbb{R}$, but also in $L^2(\mathbb{R})$. 
The remainder of this paper is organized as follows. In Section 2, we give an asymptotic formula of the maximum norm of the solutions as \( \mu \to \infty \), which is the basis for further considerations. In Section 3, we shall prove Theorem 1.2 for \( n = 1 \). Section 4 is devoted to the proof of Theorem 1.1 for \( n = 1 \). Finally, we prove Theorem 1.1 and Theorem 1.2 for \( n \geq 2 \) in Section 5.

2. PRELIMINARIES

In Sections 2–4, let \( n = 1 \). We use the following notations. Let \( X := W^{1,2}_0(I) \) be the usual real Sobolev space. \( \| \cdot \|_q \) denotes the usual \( L^q \) norm \((q \geq 1)\) and

\[
F(u) := \int_0^u f(s) \, ds, \quad u_\mu(x) := u_\mu(\mu, x).
\]

For simplicity, a subsequence will be denoted by the same notation as that of the original sequence.

Here, let us recall some useful properties of \( u_\mu(x) \) (see [2] for example). Since Eq. (1.1) is odd, we may assume without loss of generality that \( u_\mu(x) > 0 \) for \( x \in I \) in what follows. Then

\[
u_\mu'(x) > 0 \quad \text{for} \ x \in \left(0, \frac{1}{2}\right), \quad u_\mu'(x) < 0 \quad \text{for} \ x \in \left(\frac{1}{2}, 1\right). \tag{2.1}
\]

Hence,

\[
\sigma_\mu := \max_{x \in I} u_\mu(x) = u_\mu\left(\frac{1}{2}\right). \tag{2.2}
\]

The aim of this section is to prove:

**Proposition 2.1.** As \( \mu \to \infty \)

\[
\mu = \frac{2}{p+1} g(\sigma_\mu) + o(g(\sigma_\mu)). \tag{2.3}
\]

To prove Proposition 2.1, we shall prepare some lemmas.

**Lemma 2.2.** Let \( C > 0 \) be an arbitrary constant. If \( 0 < t/s < C \), then \( h(t)/h(s) \) is bounded for \( t, s \gg 1 \).

**Proof.** We see from (A.2) that if \( \epsilon < t/s < C \), then

\[
\frac{h(t)}{h(s)} \leq 2.
\]
Furthermore, if $0 < t/s \leq \varepsilon$, then we see from (A.2) that
\[
\frac{h(t)}{h(s)} \leq C_2.
\]
Now our assertion follows from these inequalities.  

**Lemma 2.3.** For $x \in \bar{I}$,
\[
\frac{1}{2}u'_\mu(x)^2 + F(u_\mu(x)) - \frac{1}{2}\mu u_\mu(x)^2 = F'(\sigma_\mu) - \frac{1}{2}\mu \sigma_\mu^2 = \frac{1}{2}u'_\mu(0)^2 > 0.
\]
\[(2.4)\]

**Proof.** Multiplying $u'_\mu(x)$ by (1.1) we obtain
\[
u_\mu(x)u'_\mu(x) + f(u_\mu(x))u'_\mu(x) - \mu u_\mu(x)u'_\mu(x) = 0,
\]
which implies that for $x \in \bar{I}$
\[
\frac{d}{dx} \left( \frac{1}{2}u'_\mu(x)^2 + F(u_\mu(x)) - \frac{1}{2}\mu u_\mu(x)^2 \right) = 0.
\]
So we obtain
\[
\frac{1}{2}u'_\mu(x)^2 + F(u_\mu(x)) - \frac{1}{2}\mu u_\mu(x)^2 = \text{constant.}
\]
\[(2.5)\]
Now we put $x = 0$ and $x = 1/2$ in (2.5) and use (2.2) to obtain (2.4).

**Lemma 2.4.** As $\mu \to \infty$, $\sigma_\mu \to \infty$.

**Proof.** We see by (2.4) that
\[
\frac{1}{2}\mu < \frac{F(\sigma_\mu)}{\sigma_\mu^2}.
\]
\[(2.6)\]
If there exists a subsequence of $\sigma_\mu$ such that $\sigma_\mu \to 0$ as $\mu \to \infty$, then we obtain by (A.1) that
\[
\lim_{\mu \to \infty} \frac{F(\sigma_\mu)}{\sigma_\mu^2} = \lim_{\mu \to \infty} \frac{f(\sigma_\mu)}{2\sigma_\mu} = \frac{f'(0)}{2} = 0.
\]
This contradicts (2.6). If there exists a subsequence of $\sigma_\mu$ such that $C \leq \sigma_\mu \leq C^{-1}$ as $\mu \to \infty$ for some constant $C > 0$, then the right hand side of (2.6) is bounded. This is a contradiction. Thus the proof is complete.
Lemma 2.5. Let \( \nu_{\mu} := u_{\mu}/\sigma_{\mu} \). Assume that there exists a constant \( \delta > 0 \) such that for \( \mu \gg 1 \)
\[
(1 + \delta) \mu < \frac{2}{p + 1} g(\sigma_{\mu}). \tag{2.7}
\]
Then \( \nu_{\mu}(x) \to 1 \) a.e. \( x \in I \) as \( \mu \to \infty \).

Proof. We have from (2.4) that
\[
u_{\mu}''(x)^2 = 2\left( F(\sigma_{\mu}) - F(u_{\mu}(x)) \right) - \mu\left( \sigma_{\mu}^2 - u_{\mu}(x)^2 \right);
\]
this along with (2.1) implies that for \( x \in (0, \frac{1}{2}) \)
\[
u_{\mu}''(x) = \sqrt{2\sigma_{\mu}^2 \left( F(\sigma_{\mu}) - F(u_{\mu}(x)) \right) - \mu \left( 1 - u_{\mu}(x)^2 \right)} \tag{2.8}
\]
We obtain by Lemma 2.2 and Lebesgue’s convergence theorem that for \( \mu \gg 1 \)
\[
F(u_{\mu}(x)) = \int_0^{\sigma_{\mu}(x)} t^p h(t) \, dt = \sigma_{\mu}^{r+1} \int_0^{u_{\mu}(x)} s^p h(\sigma_{\mu}) \, ds
= \sigma_{\mu}^{r+1} h(\sigma_{\mu}) \int_0^{u_{\mu}(x)} s^p \frac{h(\sigma_{\mu})}{h(\sigma_{\mu})} \, ds
= \sigma_{\mu}^{r+1} h(\sigma_{\mu}) \left( \frac{1}{p + 1} + o(1) \right) u_{\mu}(x)^{p+1}
= g(\sigma_{\mu}) \sigma_{\mu}^{2} \left( \frac{1}{p + 1} + o(1) \right) u_{\mu}(x)^{p+1}. \tag{2.9}
\]
By putting \( u_{\mu}(x) = 1 \) in (2.9) we also obtain
\[
F(\sigma_{\mu}) = g(\sigma_{\mu}) \sigma_{\mu}^{2} \left( \frac{1}{p + 1} + o(1) \right). \tag{2.10}
\]
Since \( 1 - u_{\mu}(x)^{p+1} \geq 1 - u_{\mu}(x)^2 \), it follows from (2.7)–(2.10) that
\[
\nu_{\mu}'(x) \geq \sqrt{\frac{2}{p + 1} g(\sigma_{\mu}) \left( 1 - u_{\mu}(x)^{p+1} \right) + o(1) g(\sigma_{\mu}) - \mu \left( 1 - u_{\mu}(x)^2 \right)}
\]
\[
= \sqrt{\frac{2}{p + 1} g(\sigma_{\mu}) \left( 1 - u_{\mu}(x)^{p+1} \right) + o(1) g(\sigma_{\mu}) - \mu \left( 1 - u_{\mu}(x)^2 \right)}
\]
\[
\geq \sqrt{\frac{2}{(p + 1)(1 + \delta)} g(\sigma_{\mu}) \left( 1 - u_{\mu}(x)^{p+1} \right) + o(1) g(\sigma_{\mu})}. \tag{2.11}
\]
Now assume that there exist \( x_0 \in (0, 1/2) \) and a constant \( 0 < \varepsilon < 1 \) such that for \( \mu \gg 1 \)
\[
v_\mu(x_0) < 1 - \varepsilon. \tag{2.12}
\]
Then it follows from (2.1) that for \( x \in (0, x_0) \)
\[
v_\mu(x) < 1 - \varepsilon. \tag{2.13}
\]
Hence, we obtain by (2.11) and (2.13) that there exists a constant \( \delta_1 > 0 \) such that for \( \mu \gg 1 \) and \( x \in (0, x_0) \)
\[
\frac{v_\mu'(x)}{x} \geq \sqrt{\left( \delta_1 + o(1) \right) g(\sigma_\mu)};
\]
integrate this over \([0, x_0]\) to obtain by (A.1) and Lemma 2.4
\[
1 - \varepsilon > \frac{v_\mu(x_0)}{x_0 \sqrt{\left( \delta_1 + o(1) \right) g(\sigma_\mu)}} \to \infty.
\]
This is a contradiction. Thus the proof is complete.

Now we are ready to prove Proposition 2.1.

**Proof of Proposition 2.1.** We have by (2.6) and (2.10) that
\[
\mu < g(\sigma) \left( \frac{2}{p+1} + o(1) \right).
\]
Hence, to prove Proposition 2.1, we assume that there exists a subsequence of \( \{ \mu \} \) \( \mu \to \infty \) which satisfies (2.7) and derive a contradiction. Multiply (1.1) by \( u_\mu \). Then integration by parts yields
\[
\|u'_\mu\|_2^2 = \int_0^1 f(u_\mu(x)) u_\mu(x) \, dx - \int_0^1 \mu \|u_\mu\|_2^2.
\tag{2.14}
\]
Integrate (2.4) over \( I \) to obtain
\[
\frac{1}{2} \|u'\|_2^2 + \int_0^1 F(u_\mu(x)) \, dx = \frac{1}{2} \mu \|u_\mu\|_2^2 = F(\sigma_\mu) - \frac{1}{2} \mu \sigma_\mu^2. \tag{2.15}
\]
It follows from (2.14) and (2.15) that
\[
\frac{1}{2} \int_0^1 f(u_\mu(x)) u_\mu(x) \, dx + \int_0^1 F(u_\mu(x)) \, dx - \mu \|u_\mu\|_2^2 = F(\sigma_\mu) - \frac{1}{2} \mu \sigma_\mu^2. \tag{2.16}
\]
Use Lemma 2.2, Lemma 2.4, Lemma 2.5, and Lebesgue’s convergence theorem to obtain for $\mu \gg 1$

$$\int_0^1 f(u_\mu(x))u_\mu(x) \, dx = \int_0^1 u_\mu(x)^{p+1} h(u_\mu(x)) \, dx$$

$$= h(\sigma_\mu)^{p+1} \int_0^1 v_\mu(x)^{p+1} \frac{h(\sigma_\mu v_\mu(x))}{h(\sigma_\mu)} \, dx$$

$$= h(\sigma_\mu)^{p+1}(1 + o(1)), \quad (2.17)$$

$$\int_0^1 F(u_\mu(x)) \, dx = \int_0^1 \int_0^1 u_\mu(x) f(s) \, ds = \int_0^1 \int_0^{u_\mu(x)} s^p h(s) \, ds$$

$$= h(\sigma_\mu)^{p+1} \int_0^1 dx \int_0^{v_\mu(x)} t^p \frac{h(\sigma_\mu t)}{h(\sigma_\mu)} \, dt$$

$$= h(\sigma_\mu)^{p+1} \left( \frac{1}{p+1} \int_0^1 v_\mu(x)^{p+1} \, dx + o(1) \right)$$

$$= h(\sigma_\mu)^{p+1} \left( \frac{1}{p+1} + o(1) \right). \quad (2.18)$$

Combining (2.10) and (2.16)–(2.18) we obtain

$$\frac{1}{2} h(\sigma_\mu)^{p+1}(1 + o(1)) + h(\sigma_\mu)^{p+1} \left( \frac{1}{p+1} + o(1) \right)$$

$$\quad - h(\sigma_\mu)^{p+1} \left( \frac{1}{p+1} + o(1) \right)$$

$$\quad = \mu \sigma_\mu^2 (1 + o(1)) - \frac{1}{2} \mu \sigma_\mu^2 = \frac{1}{2} \mu \sigma_\mu^2 (1 + o(1));$$

this along with (2.6) and (2.10) implies that

$$\frac{1}{2} g(\sigma_\mu)(1 + o(1)) = \frac{1}{2} h(\sigma_\mu)^{p-1}(1 + o(1)) = \frac{1}{2} \mu (1 + o(1))$$

$$< g(\sigma_\mu) \left( \frac{1}{p+1} + o(1) \right).$$

This is a contradiction. Thus the proof is complete.
3. PROOF OF THEOREM 1.2 FOR n = 1

We put
\[ C_\mu := g^{-1}(\mu), \quad \xi_\mu := C_\mu^{-1} u_\mu, \]
\[ h_1(t, s) := h(t) - h(s) \quad \text{for } t, s \in R. \]

Then we obtain by (1.1) that
\[
\begin{aligned}
-\xi_\mu''(x) &= \mu(\xi_\mu'(x)^{p} - \xi_\mu'(x)) + \frac{h_1(C_\mu \xi_\mu(x), C_\mu \xi_\mu(x))}{h(C_\mu)} \xi_\mu'(x)^p, \quad x \in I, \\
\xi_\mu(0) &= \xi_\mu(1) = 0.
\end{aligned}
\]  

(3.1)

Furthermore, put
\[ I_\mu := \left(-\frac{1}{2} \mu^{1/2} \frac{1}{2} \mu^{1/2}\right), \quad t = \mu^{1/2} \left(x - \frac{1}{2}\right) \quad \text{for } x \in I, \ w_\mu(t) = \xi_\mu(x). \]

Then we obtain by (3.1) that
\[
\begin{aligned}
-w_\mu''(t) &= w_\mu(t)^p - w_\mu(t) + \frac{h_1(C_\mu w_\mu(t), C_\mu w_\mu(t))}{h(C_\mu)} w_\mu(t)^p, \quad t \in I_\mu, \\
w_\mu \left(\pm \frac{1}{2} \mu^{1/2}\right) &= 0.
\end{aligned}
\]  

(3.2)

Let
\[ \xi = \left(\frac{p + 1}{2}\right)^{1/(p-1)}, \quad \xi_\mu = \max_{t \in I_\mu} w_\mu(t) = C_\mu^{-1} \sigma_\mu. \] (3.3)

**Lemma 3.1.** As \( \mu \to \infty, \xi_\mu \to \xi. \)

**Proof.** We obtain by Proposition 2.1 that
\[
C_\mu^{p-1} h(C_\mu) = g(C_\mu) = \mu = \frac{2}{p + 1} (1 + o(1)) \sigma_\mu \]
\[
= \frac{2}{p + 1} (1 + o(1)) \sigma_\mu^{p-1} h(\sigma_\mu). \]
this implies that
\[
\frac{p + 1}{2(1 + o(1))} = \left( \frac{\sigma_{\mu}}{C_{\mu}} \right)^{p-1} \frac{h(\sigma_{\mu})}{h(C_{\mu})},
\] (3.4)

We shall show that there exists a constant \( C > 0 \) such that for \( \mu \gg 1 \)
\[
C \leq \frac{\sigma_{\mu}}{C_{\mu}} \leq C^{-1}.
\] (3.5)

To this end, we use a reduction to absurdity. There are two cases to consider:

1. Assume that for any small \( \varepsilon > 0 \), there exists a subsequence of \( (\mu) \) such that \( \sigma_{\mu}/C_{\mu} \leq \varepsilon \). Then we obtain by (A.2) and (3.4) that for \( \mu \gg 1 \)
\[
\frac{p + 1}{2(1 + o(1))} \leq C_2 \varepsilon^{p-1}.
\]
This is a contradiction.

2. If for any \( A \gg 1 \), there exists a subsequence of \( (\mu) \) such that \( \sigma_{\mu}/C_{\mu} \geq A \), then we obtain by (A.2) and (3.4) that
\[
\frac{p + 1}{2(1 + o(1))} \geq C_3 A^{p-1}.
\]
This is a contradiction. Hence, we obtain (3.5). Now, we can choose a subsequence of \( (\sigma_{\mu}/C_{\mu}) \) such that \( \xi_{\mu} = \sigma_{\mu}/C_{\mu} \rightarrow M \) as \( \mu \rightarrow \infty \) for some constant \( M > 0 \). Then it follows from (A.2), (3.4), and (3.5) that \( M = \zeta \).
Since the same argument as that just above can be applicable for any subsequences of \( (\mu) \), we obtain our conclusion.

**Lemma 3.2.** There exists \( y \in L^2(\mathbb{R}) \) such that \( y(t) > 0 \) and \( w_{\mu}(t) \leq y(t) \) for \( t \in R \).

**Proof.** By the same methods as those used to obtain (2.4), we obtain by (3.2) that for \( t \in I_{\mu} \)
\[
\left\{ w_{\mu}''(t) + w_{\mu}(t)^p - w_{\mu}(t) + \left( \frac{h(C_{\mu}w_{\mu}(t))}{h(C_{\mu})} - 1 \right) w_{\mu}(t)^p \right\} w_{\mu}'(t) = 0,
\]
so
\[
\left\{ w_{\mu}''(t) + w_{\mu}(t)^p - w_{\mu}(t) + \left( \frac{f(C_{\mu}w_{\mu}(t))}{h(C_{\mu})C_{\mu}^p} - w_{\mu}(t)^p \right) \right\} w_{\mu}'(t) = 0.
\]
this implies that for $t \in I_\mu$

$$\frac{d}{dt}\left(\frac{1}{2}(w'_\mu(t))^2 + \frac{1}{p+1}w_\mu(t)^{p+1} - \frac{1}{2}w_\mu(t)^2 \right) + \left(\frac{F(C_\mu w_\mu(t))}{h(C_\mu)/C_\mu^{p+1}} - \frac{1}{p+1}w_\mu(t)^{p+1}\right) = 0. \quad (3.6)$$

Since

$$F(C_\mu w_\mu(t)) = \int_0^{C_\mu w_\mu(t)} f(s) \, ds = \int_0^{C_\mu w_\mu(t)} h(s)s^p \, ds$$

$$= h(C_\mu)/C_\mu^{p+1}w_\mu(t)^{p+1}\int_0^1 h(C_\mu w_\mu(t)s)/h(C_\mu) \, s^p \, ds,$$

$$\frac{1}{p+1}w_\mu(t)^{p+1} = w_\mu(t)^{p+1}\int_0^1 s^p \, ds,$$  

we obtain from (3.6) and (3.7) that for $t \in I_\mu$

$$\frac{1}{2}w'_\mu(t)^2 + \frac{1}{p+1}w_\mu(t)^{p+1} - \frac{1}{2}w_\mu(t)^2 + w_\mu(t)^{p+1}\int_0^1 h(C_\mu w_\mu(t)s)/h(C_\mu) \, s^p \, ds = \text{const.},$$

hence put $t = 0$ to obtain

$$\frac{1}{2}w'_\mu(t)^2 + \frac{1}{p+1}w_\mu(t)^{p+1} - \frac{1}{2}w_\mu(t)^2 + w_\mu(t)^{p+1}\int_0^1 h(C_\mu w_\mu(t)s)/h(C_\mu) \, s^p \, ds$$

$$\frac{1}{p+1}w_\mu(t)^{p+1} - \frac{1}{2}w_\mu(t)^2 + w_\mu(t)^{p+1}\int_0^1 h(C_\mu w_\mu(t)s)/h(C_\mu) \, s^p \, ds. \quad (3.8)$$

Then (3.8) along with (2.1) implies that $w_\mu$ satisfies

$$\begin{cases}
  w'_\mu(t) = -\sqrt{K_\mu(w_\mu(t))}, & 0 < t < \frac{1}{2}\mu^{1/2}, \\
  w_\mu(0) = \xi_\mu,
\end{cases} \quad (3.9)$$
where

\[
K_1(z) = \frac{2}{p+1} \left( \xi_\mu^{p+1} - z^{p+1} \right) - \left( \xi_\mu^2 - z^2 \right) \\
+ \frac{2}{h(C_\mu)} \int_0^1 s^p \left( h_1 \left( C_\mu \xi_\mu s, C_\mu \right) \xi_\mu^{p+1} - h \left( C_\mu zs, C_\mu \right) z^{p+1} \right) ds.
\]

(3.10)

Let \( y_1(t) = (t + 1)^{-2} \). Then \( y_1 \) satisfies

\[
\begin{cases}
y_1'(t) = -\sqrt{4y_1(t)^3}, & t > 0, \\
y_1(0) = 1.
\end{cases}
\]

(3.11)

We shall show that there exists a constant \( \epsilon > 0 \) such that for \( 0 < z < \epsilon \) and \( \mu \gg 1 \)

\[
K(z) := K_1(z) - 4z^3 > 0.
\]

(3.12)

Now there are two cases to consider:

Case 1. If (A.3×1) is satisfied, then \( h(s) \) is increasing for \( s \geq 0 \). Hence, for \( 0 \leq zs < 1 \) we have

\[
h_1(C_\mu zs, C_\mu) = h(C_\mu zs) - h(C_\mu) < 0.
\]

(3.13)

Then by (3.10) and (3.13) we obtain that if \( 0 < z < \epsilon \), then

\[
K(z) = K_1(z) - 4z^3 \geq K_2(z) - 4z^3 \\
:= \frac{2}{p+1} \left( \xi_\mu^{p+1} - z^{p+1} \right) - \left( \xi_\mu^2 - z^2 \right) \\
+ \frac{2}{h(C_\mu)} \int_0^1 s^p h_1 \left( C_\mu \xi_\mu s, C_\mu \right) \xi_\mu^{p+1} ds - 4z^3.
\]

(3.14)

Clearly, \( K_2(0) = K(0) > 0 \) and for \( 0 < z < \epsilon \)

\[
(K_2(z) - 4z^3)' = 2z - 2z^p - 12z^2 > 0;
\]

it is clear from this and (3.14) that (3.12) holds.
Case 2. If (A.3)(2) is satisfied, then

\[
(K_1(z) - 4z^3)' = 2z - 2z^p - 2(p + 1)z^p \int_0^1 s^p \frac{h_1(C_\mu z^s, C_\mu)}{h(C_\mu)} \, ds
\]

\[
- \frac{2}{h(C_\mu)} \int_0^1 C_\mu s^{p+1} h'(C_\mu z^s) z^p + 1 \, ds \leq 12z^2 \quad (3.15)
\]

\[
\geq 2z - 2z^p - 2(p + 1)z^p \int_0^1 s^p \frac{h_1(C_\mu z^s, C_\mu)}{h(C_\mu)} \, ds - 12z^2.
\]

Since we obtain by (A.2) that for \( 0 < z < \epsilon \)

\[
0 \leq \frac{h_1(C_\mu z^s, C_\mu)}{h(C_\mu)} = \frac{h(C_\mu z^s)}{h(C_\mu)} - 1 \leq C_2 - 1, 
\]

then we obtain by (3.15) and (3.16) that for \( 0 < z < \epsilon \)

\[
(K_1(z) - 4z^3)' \geq 2z - 2z^p - 2(C_2 - 1)z^p - 12z^2 > 0.
\]

Hence, we find that (3.12) holds.

Let \( I_\epsilon := (\epsilon^{-1/2} - 1, \infty) \). If there exists an interval \([t_1, t_2] \subseteq I_\epsilon\) such that

\[
w'(t) = y'(t_1) = z_1, \quad w(t) > y(t) \quad \text{for} \quad t \in (t_1, t_2),
\]

then we obtain that \( w'(t) \geq y'(t_1) \); this implies along with (3.9) and (3.11) that \( K(z_1) < 0 \); however, this is impossible, since \( z_1 < \epsilon \). Therefore, we find that

\[
w'(t) = y'(t) \quad \text{for} \quad t \in I_\epsilon.
\]

Now we put

\[
y(t) = \begin{cases} 2\xi, & \text{if} \ |t| \leq \epsilon^{-1/2} - 1, \\
y(t_1), & \text{if} \ |t| > \epsilon^{-1/2} - 1. 
\end{cases}
\]

Then we obtain our conclusion by Lemma 3.1 and (3.17). Thus the proof is complete.

Now we are in the position to prove Theorem 1.2 for \( n = 1 \).

\textbf{Proof of Theorem 1.2 for} \( n = 1 \). We know from Lemma 2.2, Lemma 3.1, (3.2) and (3.8) that \((w_\mu), (w'_\mu), (w''_\mu)\) are bounded. Moreover, it is clear that \((w_\mu)\) and \((w'_\mu)\) are equicontinuous. Hence, we can apply the
Ascoli–Arzela theorem to choose a subsequence of \((w_\mu)\) such that as \(\mu \to \infty\)
\[
w_\mu \to w_1, \quad w'_\mu \to w_2 \tag{3.18}
\]
on any compact subsets in \(R\). Then for \(s \in R\)
\[
w_\mu(s) - w_\mu(0) = \int_0^s w'_\mu(t) \, dt,
\]
by letting \(\mu \to \infty\), we obtain by Lemma 3.1 and (3.18) that
\[
w_1(s) - \zeta = \int_0^s w_2(t) \, dt;
\]
this implies that
\[
w_1'(s) = w_2(s). \tag{3.19}
\]
For \(\psi \in C_0^\infty(R)\) and \(\mu \gg 1\), it follows from (3.2) that
\[
\int_R w'_\mu(t) \psi'(t) \, dt = \int_R \left( w_\mu(t)^p - w_\mu(t) \right) \psi(t) \, dt
+ \int_R h_1 \left( \frac{C_\mu w_\mu(t)}{h(C_\mu)} \right) w_\mu(t)^p \psi(t) \, dt.
\tag{3.20}
\]
Clearly, we have by Lemma 2.1 and Lemma 3.1 that for \(\mu \gg 1\)
\[
\left| h_1 \left( \frac{C_\mu w_\mu(t)}{h(C_\mu)} \right) \right| \leq \frac{h(C_\mu w_\mu(t))}{h(C_\mu)} + 1 \leq C + 3. \tag{3.21}
\]
Then by (A.2) and (3.21) we can apply Lebesgue’s convergence theorem to (3.18) and obtain by letting \(\mu \to \infty\) in (3.18)
\[
\int_R w_1'(t) \psi'(t) \, dt = \int_R \left( w_2(t)^p - w_2(t) \right) \psi(t) \, dt. \tag{3.22}
\]
Moreover, it follows from Lemma 3.1, (3.18), and (3.19) that
\[
w_1(0) = \lim_{\mu \to \infty} w_\mu(0) = \zeta, \quad w'_1(0) = \lim_{\mu \to \infty} w'_\mu(0) = 0. \tag{3.23}
\]
Thus we find that \(w_1\) is the ground state of Eq. (1.5). Combining this fact and Lemma 3.2, we obtain by Lebesgue’s convergence theorem that \(w_\mu \to w_1\) in \(L^2(R)\). Finally, our conclusion follows from a standard compactness argument. Thus the proof is complete. \(\blacksquare\)
4. PROOF OF THEOREM 1.1 FOR n = 1

In order to prove Theorem 1.1 for \( n = 1 \), let us briefly recall the properties of the ground state \( w \) of Eq. (1.5). We know from Berestycki and Lions [3] that

\[
\begin{align*}
  w(0) &= \zeta, \quad w'(0) = 0. \\
  w(t) &= w(-t), \quad w'(t) < 0 \quad \text{for} \ t > 0. \\
  \frac{1}{2}w'(t)^2 + \frac{1}{p+1}w(t)^{p+1} - \frac{1}{2}w(t)^2 &= 0 \quad \text{for} \ t \in R.
\end{align*}
\]

**Lemma 4.1.** Let \( w \) be the ground state of (1.5). Then

\[
\int_R w(t)^2 \, dt = B_1. \tag{4.4}
\]

**Proof.** We see from (4.2) and (4.3) that for \( t \geq 0 \)

\[
w'(t) = -w(t)\sqrt{1 - \frac{2}{p+1}w(t)^{p-1}}.
\]

Put \( s = w(t) = r\zeta, \ r = \sin^{2/(p-1)}\theta \), and use (4.5) to obtain

\[
\int_0^\infty w(t)^2 \, dt = \int_0^\infty w(t) \cdot \frac{-w(t)}{\sqrt{1 - (2/(p+1))w(t)^{p-1}}} \, dt
\]

\[
= \int_0^{\zeta} \frac{s}{\sqrt{1 - (2/(p+1))s^{p-1}}} \, ds
\]

\[
= \zeta^2 \int_0^1 \frac{r}{\sqrt{1 - r^{p-1}}} \, dr
\]

\[
= \frac{2}{p-1} \zeta^2 \int_0^{\pi/2} \sin^{6-(p-1)(p-1)} \theta \, d\theta = \frac{B_1}{2}.
\]

Thus the proof is complete. \( \blacksquare \)

Now we are ready to prove Theorem 1.1 for \( n = 1 \).

**Proof of Theorem 1.1 for n = 1.** Let \( v_\mu \) and \( w_\mu \) be the functions defined at the beginning of Section 3. Then

\[
\alpha_1(\mu)^2 = \norm{u_\mu}^2_2 = C_\mu^2 \norm{v_\mu}^2_2 = C_\mu^2 \mu^{-1/2} \int_{I_\mu} w_\mu(t)^2 \, dt;
\]
this along with Theorem 1.2 for \( n = 1 \) and Lemma 4.1 that as \( \mu \rightarrow \infty \)

\[
\frac{\alpha_n(\mu)^2}{C_\mu^2 \mu^{-1/2}} = \int_{I_{n,\mu}} w(t)^2 \, dt \rightarrow \int_R w(t)^2 \, dt = B_1.
\]

Thus the proof is complete. 

5. PROOF OF THEOREM 1.1 AND THEOREM 1.2 FOR \( n \geq 2 \)

Since Eq. (1.1) is odd and autonomous, we can apply the argument of [5, Corollary 3.5(c)] to find that the interior zeros of \( u_n(\mu, x) \) are exactly \((1/n, 2/n, \ldots, (n - 1)/n)\). Furthermore, for \( j = 0, 1, \ldots, n - 1 \)

\[
u_n(\mu, x) = (-1)^j u_n\left(\mu, x - \frac{j}{n}\right) \quad \text{for } x \in I_j = \left(\frac{j}{n}, \frac{j + 1}{n}\right), \quad (5.1)
\]

We consider the following equation instead of (1.1):

\[
\begin{cases}
-u''(x) = f(u(x)) - \mu u(x), & x \in I_0, \\
u(0) = u\left(\frac{1}{n}\right) = 0.
\end{cases}
\]

(5.2)

Then by (5.1), \( |u_n(\mu, x + j/n)| (x \in I_0) (j = 0, 1, \ldots, n - 1) \) satisfies (5.2) and using the same arguments as those used in the proof of the case \( n = 1 \), we can also prove that for \( x \in I_j \)

\[
w_j, \mu (t) = \begin{cases}
C^{-1}_\mu |u_n(\mu, x)|, & t = \mu^{1/2} \left(x - \frac{j}{n} - \frac{1}{2n}\right) \in I_{n, \mu}, \\
0, & t \notin I_{n, \mu}
\end{cases}
\]

converges to the ground state \( w \) of (1.5), where \( I_{n, \mu} \) is the interval defined in Theorem 1.2. Therefore, we obtain

\[
\alpha_n(\mu)^2 = C_\mu^2 \mu^{-1/2} \sum_{j=1}^n \int_{I_{n,\mu}} w_j, \mu (t)^2 \, dt,
\]

which along with Lemma 3.2 and Lemma 4.1 implies that

\[
\frac{\alpha_n(\mu)^2}{C_\mu^2 \mu^{-1/2}} = \sum_{j=1}^n \int_{I_{n,\mu}} w_j, \mu (t)^2 \, dt \rightarrow n \int_R w(t)^2 \, dt = nB_1 = B_n.
\]

Thus the proof is complete.
REFERENCES


