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Monotonicity of CF-coefficients in Gauss-fractions

Haakon Waadeland*

Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

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Abstract

Monotonicity properties of coefficients in S-fraction expansions are often very useful in the computation of truncation error bounds for approximate function values. For hypergeometric functions ${}_2F_1$ with parameters such that the C-fraction expansion is an S-fraction, it turns out that the CF-coefficients essentially always have those properties. This is proved in the present paper.

Keywords: Hypergeometric function; Continued fraction; Monotonicity

Let $\mathbb{Z}$ be the set of integers, and

$$\mathbb{Z}^+ = \{x \in \mathbb{Z};\ x > 0\}, \qquad \mathbb{Z}_0^+ = \{x \in \mathbb{Z};\ x \geqslant 0\},$$
$$\mathbb{Z}^- = \{x \in \mathbb{Z};\ x < 0\}, \qquad \mathbb{Z}_0^- = \{x \in \mathbb{Z};\ x \leqslant 0\}.$$

The Gauss hypergeometric series ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \qquad a, b \in \mathbb{C}, \quad c \notin \mathbb{Z}_0^-.$$

* Corresponding author. Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim N-7491, Norway.

E-mail address: haakonwa@math.ntnu.no (H. Waadeland).

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It converges in the open unit disk $|z| < 1$ to an analytic function. The sum is also denoted ${}_2F_1(a, b; c; z)$ and is called the Gauss hypergeometric function. This also extends to possible analytic continuations. In the following, we leave out the subscripts.

Let $a, b, c \in \mathbb{R}$. Moreover, we assume that $a, c, c - b \notin \mathbb{Z}_0^-$ and $b, c - a \notin \mathbb{Z}^-$. Then we have the S-fraction expansion

$$z \cdot \frac{F(a, b+1; c+1; -z)}{F(a, b; c; -z)} = \mathop{\mathrm{K}}_{v=1}^{\infty} \left(\frac{c_v z}{1}\right), \tag{0}$$

where $c_1 = 1$ and

$$c_{2m+2} = \frac{(a+m)(c-b+m)}{(c+2m)(c+2m+1)}, \quad m \in \mathbb{Z}_0^+, \tag{1}$$

$$c_{2m+1} = \frac{(b+m)(c-a+m)}{(c+2m-1)(c+2m)}, \quad m \in \mathbb{Z}^+. \tag{2}$$

The equality in (0) means on the one hand, correspondence between the Taylor series expansion at $z = 0$ of the left-hand side and the C-fraction on the right-hand side. On the other hand, it means equality in a disk around $z = 0$ where both converge. Since the left-hand side is meromorphic and has a derivative $= 1$ at the origin, the series must converge in a disk centered at $z = 0$. The continued fraction is known to converge in $\mathbb{C} \backslash (-\infty, 1]$.

The variable z is generally complex. In view of the most frequent application we shall here mostly think of z as being real, $z = x$. We recall that $c_n \to \frac{1}{4}$ when $n \to \infty$. For simplicity we shall here in addition assume $c > 0$.

The background and motivation for this article is the question about the best possible upper bounds for truncation errors of S-fraction approximants *generally*. A strong tool in this is the oval sequence theorem by Lisa Lorentzen [1,2], also in the complex case. In the real case, ovals and disks are replaced by intervals. Possible monotonicity properties of the sequences of coefficients may enhance the method substantially by leading to very good (small) upper truncation error bounds. S-fractions of $\ln(1 + z)$ and $\arctan z$ are examples with such monotonicity properties.

In this article observations on monotonicity properties of the sequences $\{c_{2m}\}$ and $\{c_{2m+1}\}$ are presented. These observations extend substantially the range of applicability of the combination real oval sequence theorem/monotonicity properties of coefficients.

Theorem 1. *Let the sequences* $\{c_{2m+2}\}_0^{\infty}$ *and* $\{c_{2m+1}\}_1^{\infty}$ *be as above with the given conditions on* a, b, c. *If* $a = b + \frac{1}{2}$, *then the two sequences merge by interlacing to one sequence* $\{c_k\}_{k=2}^{\infty}$, *where*

$$c_k = \frac{(2b-1+k)(2c-2b-2+k)}{4(c+k-2)(c+k-1)}, \quad k \geqslant 2 \tag{3}$$

and the following holds: If $(2b - c + 1)(2b - c) \neq 0$, *then the sequence is monotone. The sequence is increasing if* $(2b - c + 1)(2b - c) > 0$ *and decreasing if* $(2b - c + 1)(2b - c) < 0$. *If* $a \neq b + \frac{1}{2}$ *and* $(2b - c + 1)(2b - c)(2a - c)(2a - c - 1) \leqslant 0$, *both sequences are monotone. If* $a < b + \frac{1}{2}$ *the sequence* $\{c_{2m}\}$ *is increasing, the sequence* $\{c_{2m+1}\}$ *is decreasing. If* $a > b + \frac{1}{2}$ *the sequence* $\{c_{2m}\}$ *is decreasing, the sequence* $\{c_{2m+1}\}$ *is increasing.*

Proof. Using the fact that both c_{2m+2} and c_{2m+1} tend to $\frac{1}{4}$ when $m \to \infty$ we find, when we pretend that m is a positive real number:

$$[(c+2m)(c+2m+1)]^2 \frac{\mathrm{d}}{\mathrm{d}m}(c_{2m+2}) =: T_{\mathrm{ev}}(m),$$

where $T_{\mathrm{ev}}(m)$ is the following polynomial:

$$T_{\mathrm{ev}}(m) = P_{\mathrm{ev}} m^2 + Q_{\mathrm{ev}} m + R_{\mathrm{ev}},$$

where

$$P_{\mathrm{ev}} = (-4a+4b+2), \quad Q_{\mathrm{ev}} = (2c^2 - 8ac + 8ab + 2c),$$

$$R_{\mathrm{ev}} = (c^3 + 4abc - 3ac^2 + c^2 - bc^2 - bc - ac + 2ab).$$

For $a = b + \frac{1}{2}$ the polynomial $T_{\mathrm{ev}}(m)$ reduces to the linear expression

$$\left(\tfrac{1}{2}\right)((2c+1)(2b-c+1)(2b-c) + 4(2b-c+1)(2b-c)m), \tag{4}$$

which is 0 for $m = -(\frac{1}{2})(c+\frac{1}{2}) < 0$.

Doing the same for the *odd* ordered sequence we find, with self-explaining notations

$$T_{\mathrm{od}}(m) = P_{\mathrm{od}} m^2 + Q_{\mathrm{od}} m + R_{\mathrm{od}}.$$

Here

$$P_{\mathrm{od}} = (4a-4b-2), \quad Q_{\mathrm{od}} = (2c^2 - 8bc + 8ab - 2c),$$

$$R_{\mathrm{od}} = (c^3 + 4abc - ac^2 - c^2 - 3bc^2 + bc + ac - 2ab).$$

For $a = b + \frac{1}{2}$ the polynomial $T_{\mathrm{od}}(m)$ reduces to the linear expression

$$\left(\tfrac{1}{2}\right)((2c-1)(2b-c+1)(2b-c) + 4(2b-c+1)(2b-c)m), \tag{5}$$

which is 0 for $m = -(\frac{1}{2})(c-\frac{1}{2}) < \frac{1}{4}$.

From the two linear expressions, (4), (5), both with the same coefficient for m, follows immediately the monotonicity statement for the case $a = b + \frac{1}{2}$ in the theorem. Moreover, when $a = b + \frac{1}{2}$ is inserted into (1) and (2) we immediately get (3). The first part of the theorem is thus proved.

For $a \neq b + \frac{1}{2}$ the polynomials $T_{\mathrm{ev}}(m)$ and T_{od} are of degree 2, and the zeros are determined in the standard elementary way. The discriminants in both cases are equal and equal to

$$4(2b-c+1)(2b-c)(2a-c)(2a-c-1). \tag{6}$$

If the discriminant is $\leqslant 0$, the polynomials $T_{\mathrm{ev}}(m)$ and T_{od} do not change sign. This implies monotonicity of the sequences of even order and of odd order coefficients. The *types* of monotonicity are determined from the coefficient of m^2, which is $-4a+4b+2$ in the even case and $4a-4b-2$ in the odd case. From this the rest of the theorem follows immediately.

The remaining case is when discriminant (6) is positive,

$$4(2b-c+1)(2b-c)(2a-c)(2a-c-1) > 0,$$

in which case the quadratic equation has two distinct real roots. Let m_ℓ and m_r be the roots, $m_\ell < m_\mathrm{r}$. Usually they are different in the even and odd cases. Then the corresponding sequence of elements (even order sequence or odd order sequence) is monoton for all integer $m > m_\mathrm{r}$, increasing if the coefficient of m^2 is positive, decreasing if it is negative. Expressed in terms of the parameters this is as follows:

The case of real roots. In both cases we find

$$m_\ell, m_\mathrm{r} = \frac{-Q \pm \sqrt{Q^2 - 4PR}}{2P}. \tag{7}$$

By inserting the even and odd values for P, Q, R in (7) this leads to the following results:

For $a < b + \frac{1}{2}$ and

$$m \geqslant \frac{-c^2 + 4ac - 4ab - c + \sqrt{(2b-c+1)(2b-c)(2a-c)(2a-c-1)}}{-4a + 4b + 2}, \tag{8a}$$

the sequence $\{c_{2m+2}\}_{m=0}^{\infty}$ is increasing.

For $a > b + \frac{1}{2}$ and

$$m \geqslant \frac{-c^2 + 4ac - 4ab - c - \sqrt{(2b-c+1)(2b-c)(2a-c)(2a-c-1)}}{-4a + 4b + 2}, \tag{8b}$$

the sequence $\{c_{2m+2}\}_{m=0}^{\infty}$ is decreasing.

For $a > b + \frac{1}{2}$ and

$$m \geqslant \frac{-c^2 + 4bc - 4ab + c + \sqrt{(2b-c+1)(2b-c)(2a-c)(2a-c-1)}}{4a - 4b - 2}, \tag{8c}$$

the sequence $\{c_{2m+1}\}_{m=1}^{\infty}$ is increasing.

For $a < b + \frac{1}{2}$ and

$$m \geqslant \frac{-c^2 + 4bc - 4ab + c - \sqrt{(2b-c+1)(2b-c)(2a-c)(2a-c-1)}}{4a - 4b - 2}, \tag{8d}$$

the sequence $\{c_{2m+1}\}_{m=1}^{\infty}$ is decreasing. □

Example 1. Take $a = 1, b = 2, c = 3$. Here $a < b + \frac{1}{2}$. Moreover, we find that the value of the root expression in (8a) is -1. In (8d) it is -2. This leads to the following:

For $m > -1$, i.e. for all m-values $\geqslant 0$, the sequence $\{c_{2m+2}\}$ is *increasing*.

For $m > -2$, i.e. for all m-values > 0, the sequence $\{c_{2m+1}\}$ is *decreasing*.

In this example

$$c_{2m+2} = \frac{(1+m)^2}{(3+2m)(4+2m)}, \quad z \in \mathbb{Z}_0^+,$$

$$c_{2m+1} = \frac{(2+m)^2}{(2+2m)(3+2m)}, \quad z \in \mathbb{Z}^+.$$

The established monotonicity properties here are easily verified.

In some cases the monotonicity can be seen in a simpler way. One such example is when the roots of $T(m)=0$ in even or odd case are negative. In such a case we have monotonicity of the even or odd coefficient sequence. Necessary and sufficient condition for the largest root to be negative and hence both roots to be negative is that their product is positive and their sum is negative. Stated in terms of the coefficients, this is

$$\frac{R}{P} > 0, \quad \frac{Q}{P} > 0.$$

Expressed by the parameters, and with the convenient change from quotients to products, this takes the form

$$(c^3 + 4abc - 3ac^2 + c^2 - bc^2 - bc - ac + 2ab)(-4a + 4b + 2) > 0, \tag{9a}$$

$$(2c^2 - 8ac + 8ab + 2c)(-4a + 4b + 2) > 0 \tag{9b}$$

in the even order case. The sequence $\{c_{2m+2}\}$ is increasing for $a < b + \frac{1}{2}$ and decreasing for $a > b + \frac{1}{2}$.

In the odd order case we get the condition for two negative roots as follows:

$$(c^3 + 4abc - ac^2 - c^2 - 3bc^2 + bc + ac - 2ab)(4a - 4b - 2) > 0, \tag{10a}$$

$$(2c^2 - 8bc + 8ab - 2c)(4a - 4b - 2) > 0. \tag{10b}$$

The sequence $\{c_{2m+1}\}$ is increasing for $a > b + \frac{1}{2}$ and decreasing for $a < b + \frac{1}{2}$.

In Example 1 we had this situation, as we already have seen, since the two larger roots are -1 and -2 in even and odd case, respectively. We have in this example in the even case

$$P = 6, \quad Q = 16, \quad R = 10$$

and thus $R/P > 0$, $Q/P > 0$ and $P > 0$.

In the odd case we have

$$P = -6, \quad Q = -20, \quad R = -16$$

and thus $R/P > 0$, $Q/P > 0$ and $P < 0$.

Example 2. Take $a = 1, b = 2, c = \frac{1}{2}$. Here $a < b + \frac{1}{2}$. Moreover, the value of the root expression in (8a), i.e. the larger root in the even case is $-0.552\ldots$, whereas in (8d) the root, the larger root in the odd case is $0.052\ldots$. From this follows that the even order sequence and the odd order sequence both are monotone. The sequence $\{c_{2m+2}\}_0^\infty$ is increasing, whereas $\{c_{2m+1}\}_1^\infty$ is decreasing. In this example,

$$c_{2m+2} = \frac{(m+1)\left(m - \frac{3}{2}\right)}{\left(2m + \frac{1}{2}\right)\left(2m + \frac{3}{2}\right)}, \quad z \in \mathbb{Z}_0^+$$

and

$$c_{2m+1} = \frac{(m+2)\left(m - \frac{1}{2}\right)}{\left(2m - \frac{1}{2}\right)\left(2m + \frac{1}{2}\right)} \quad z \in \mathbb{Z}^+.$$

The stated monotonicity properties are easily established.

In Example 2 the remarks after Example 1 were not applicable, since in the odd case, i.e. in (8d), the largest root was positive.

Remark. Related results, even simpler, may be obtained in the same way for *confluent* hypergeometric functions.

References

[1] L. Lorentzen, A priori truncation error bounds for continued fractions, Rocky Mountain J. Math., to appear.

[2] L. Lorentzen, H. Waadeland, Continued fractions with applications, in: C. Brezinski, L. Wuytack (Eds.), Studies in Computational Mathematics, vol. 3, North-Holland, Amsterdam, London, New York, Tokyo, 1992.