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A Sufficient Condition for a Projective Variety to Be the Proj of a Gorenstein Graded Ring

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We give a sufficient condition for a normal projective variety X to be isomorphic to Proj R for a normal Gorenstein graded ring R, based on a criterion of K.-i. Watanabe. When X is a Gorenstein variety, this condition for X is necessary and sufficient.

INTRODUCTION

Let X be a normal projective variety over an algebraically closed field k, and D an *ample* $\mathbb{Q}$*-divisor* on X, i.e., a rational coefficient Weil divisor whose multiple rD for some $r \in \mathbb{N}$ is an ample Cartier divisor. We consider a normal graded ring $R(X, D)$ defined by

$$R(X, D) = \bigoplus_{n=0}^{+\infty} H^0(X, \mathcal{O}_X(nD))\, T^n,$$

where $\mathcal{O}_X(nD)$ is the divisorial sheaf associated with a $\mathbb{Q}$-divisor nD (see (0.1)). We are interested in finding a criterion for a normal projective variety X over k to have an ample $\mathbb{Q}$-divisor D with $R(X, D)$ Gorenstein. Since $X = \operatorname{Proj} R(X, D)$, thanks to a theorem of Demazure [1], it is equivalent to asking when a normal projective variety over k is the Proj of a Gorenstein normal graded k-algebra. When D is an ample Cartier divisor, Goto and Watanabe [2] obtained a criterion for $R(X, D)$ to be Gorenstein. Using this, our problem for ample Cartier divisors D is answered satisfactorily. In the case of ample $\mathbb{Q}$-divisors, Watanabe [7] has established a criterion for $R(X, D)$ to be Gorenstein, in terms of D and the canonical divisor K_X of X (see (0.3)). But from this, much was not known about our problem for ample $\mathbb{Q}$-divisors D.

The purpose here is to solve our problem, at least when X is Gorenstein, based on the criterion of Watanabe [7].

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Our main result of this paper is the following:

THEOREM. *Let X be a Gorenstein normal projective variety of dimension N over an algebraically closed field k.*

(a) *Suppose that $H^i(X, \mathcal{O}_X)=0$ for $0<i<N$. Then, for every positive odd integer a, there exists an ample $\mathbb{Q}$-divisor D on X such that $R(X, D)$ is a Gorenstein graded ring with $a(R(X, D))=a$. (See* (0.3.2) *for the definition of $a(R(X, D))$.) In particular, X is the* Proj *of a Gorenstein normal graded k-algebra.*

(b) *Suppose furthermore that there exists a Cartier divisor F, with $H^i(X, \mathcal{O}_X(F))=0$ for $0<i<N$, such that $2F$ is linearly equivalent to a canonical divisor K_X. Then, for every positive even integer a, there exists an ample $\mathbb{Q}$-divisor D on X such that $R(X, D)$ is a Gorenstein graded ring with $a(R(X, D))=a$.*

It is well-known that the vanishing of cohomology groups of the structure sheaf, assumed in (a) of the Theorem, is necessary for the Cohen-Macaulay property of $R(X, D)$ (see (0.3.1)). On the other hand, the existence of a *Weil* divisor F with the property above, assumed in (b) as well as the Cartier property of F, is necessary for the Gorenstein property of $R(X, D)$ with even $a(R(X, D))$, as we show in the Lemma (Section 1). Therefore, by the theorem of Demazure [1, (3.5)], the Theorem implies:

COROLLARY. *Let X be a normal projective variety of dimension N over an algebraically closed field k.*

(a) *Suppose that X is Gorenstein. Let a be a positive odd integer. Then X is isomorphic to* Proj(R) *for a Gorenstein normal graded k-algebra R with $a(R)=a$ if and only if $H^i(X, \mathcal{O}_X)=0$ for $0<i<\dim X$.*

(b) *Suppose that X is a Cohen–Macaulay locally factorial scheme. Let a be a positive even integer. Then X is isomorphic to* Proj(R) *for a Gorenstein normal graded k-algebra R with $a(R)=a$ if and only if*

(b1) *$H^i(X, \mathcal{O}_X)=0$ for $0<i<\dim X$, and*

(b2) *there exists a divisor F on X, with $H^i(X, \mathcal{O}_X(F))=0$ for $0<i<\dim X$, such that $2F$ is linearly equivalent to the canonical divisor K_X.*

Our exposition proceeds as follows: First we make a remark on the necessary condition for X to have an ample $\mathbb{Q}$-divisor D such that $R(X, D)$ is a Gorenstein with even $a(R(X, D))$ (Section 1). We next discuss the condition for a $\mathbb{Q}$-Gorenstein projective variety X to have an ample $\mathbb{Q}$-divisor D with $R(X, D)$ quasi-Gorenstein (Section 2). Using this, we prove the Theorem (Section 3). Although the Gorenstein property of X plays an essential role in our proof, it seems likely that the assumption is somewhat redundant for our purpose. In Section 4, we give some examples of X, to which we cannot apply our theorem, having an ample $\mathbb{Q}$-divisor D with $R(X, D)$ Gorenstein.

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0. Notation and Preliminaries

(0.1) Let k be an algebraically closed field. Let X be a normal projective variety over k, where a *variety* over a field F means an integral separated scheme of finite type over F. A $\mathbb{Q}$-*divisor* on X is a $\mathbb{Q}$-linear combination of prime divisors on X. The $\mathbb{Q}$-divisors D_1, D_2 are *linearly equivalent*, denoted by $D_1 \sim D_2$, if $D_1 - D_2$ is a principal divisor on X. A $\mathbb{Q}$-divisor D is a $\mathbb{Q}$-*Cartier divisor* if some positive multiple rD is a Cartier divisor. A $\mathbb{Q}$-divisor D is *ample* if some positive multiple rD of D is an ample Cartier divisor in the usual sense. For a $\mathbb{Q}$-divisor $D = \sum_Y a_Y \cdot Y$ with Y running through the set of prime divisors on X, we define a divisorial sheaf $\mathcal{O}_X(D)$ by $\Gamma(U, \mathcal{O}_X(D)) := \{f \in K(X);\ \mathcal{V}_Y(f) + a_Y \geqslant 0$ for all prime divisors Y on X with $Y \cap U \neq \emptyset\}$ for each open set U of X. Here $K(X)$ is the rational function field of X and $\mathcal{V}_Y(f)$ is the value of f along Y. Hence $\mathcal{O}_X(D) = \mathcal{O}_X([D])$, where $[D] := \sum_Y [a_Y] \cdot Y$, i.e., the integral part of D.

(0.2) A canonical divisor K_X on X is a Weil divisor such that $\mathcal{O}_{X_{reg}}(K_X|_{X_{reg}}) = \Omega^{\dim X}_{X_{reg}}$, where X_{reg} is the nonsingular locus of X and $K_X|_{X_{reg}}$ is the restriction of K_X onto X_{reg}. The divisorial sheaf $\mathcal{O}_X(K_X)$ is called the *canonical sheaf* and is denoted by ω_X. Recall that X is a *Gorenstein* scheme if X is Cohen–Macaulay and if the canonical sheaf ω_X is locally free. Similarly, we say that X is a $\mathbb{Q}$-*Gorenstein* scheme if the canonical divisor K_X is a $\mathbb{Q}$-Cartier divisor.

(0.3) Given a normal projective variety X over k and an ample $\mathbb{Q}$-divisor D on X, we define a graded k-algebra $R(X, D)$ to be

$$R(X, D) = \bigoplus_{n=0}^{+\infty} H^0(X, \mathcal{O}_X(nD))\, T^n \subset K(X)[T],$$

where T is an indeterminate. Hence, it is easy to check that $R(X, D)$ is integrally closed in $K(X)(T)$. Since rD is an ample Cartier divisor for some $r \in \mathbb{N}$, X is isomorphic to Proj $R(X, D)$. Concerning the Cohen–Macaulay property and the Gorenstein property of the graded ring $R(X, D)$, we refer the reader to [7]. (See also [2].) The facts we need are the following:

(0.3.1) (See [7, (2.4)].) $R(X, D)$ is Cohen–Macaulay if and only if $H^i(X, \mathcal{O}_X(nD)) = 0$ for $0 < i < \dim X$ and for every $n \in \mathbb{Z}$.

(0.3.2) (See [7, (2.9) and (2.10)].) Recall that a Noetherian ring R with the canonical module K_R is *quasi-Gorenstein* if the canonical module K_R is a locally free R-module. Suppose that $D = \sum_Y (p_Y/q_Y) \cdot Y$ with Y running through the set of prime divisors on X, where $p_Y, q_Y \in \mathbb{Z}$, $q_Y > 0$, and $(p_Y, q_Y) = 1$ for each Y. Then $R(X, D)$ is a quasi-Gorenstein ring if and only if there exist an integer a and a rational function f on X such that $K_X + D' - aD = \mathrm{div}_X(f)$, where $D' := \sum_Y \{(q_Y - 1)/q_Y\} \cdot Y$ and $\mathrm{div}_X(f)$ is the divisor of f. Then the integer a coincides with the integer $a(R(X, D)) = -\min\{m \in \mathbb{Z}: (K_{R(X,D)})_m \neq 0\}$. By definition, $R(X, D)$ is Gorenstein if and only if $R(X, D)$ is Cohen–Macaulay and quasi-Gorenstein.

1. A Remark on the Necessary Condition for X to Have D Such That $R(X, D)$ Is a Gorenstein Ring with Even $a(R(X, D))$

In the following lemma, we do not assume that k is algebraically closed, since (0.3.1) and (0.3.2) are valid over a field k (see [7]).

LEMMA. *Let X be a normal projective variety over a field k, and D an ample $\mathbb{Q}$-divisor on X. Suppose that $R(X, D)$ is a quasi-Gorenstein ring with even $a(R(X, D))$. Then there exists a Weil divisor F on X such that $2F$ is linearly equivalent to the canonical divisor K_X. Furthermore, if $R(X, D)$ is Gorenstein, the Weil divisor F satisfies the condition that $H^i(X, \mathcal{O}_X(F)) = 0$ for $0 < i < \dim X$.*

Proof. By (0.3.2), we have $K_X + D' - aD = \mathrm{div}_X(f)$ for some $f \in K(X)$ and $a = a(R(X, D))$. Suppose that $K_X - \mathrm{div}_X(f) = \sum_Y b_Y \cdot Y$ with Y running through the prime divisors. Note that every b_Y is an integer. Looking at each coefficient of Y in $\{K_X - \mathrm{div}_X(f)\} + D' = aD$, we have $b_Y + \{(q_Y - 1)/q_Y\} = a(p_Y/q_Y)$ and, therefore, $(b_Y + 1)\, q_Y - 1 = ap_Y$. Since a is even, b_Y is even. Set $c_Y := (b_Y/2) \in \mathbb{Z}$, and $F := \sum_Y c_Y \cdot Y$. Then $2F = K_X - \mathrm{div}(f)$ and $2F + D' = aD$. Since $[D'] = 0$ and F is a Weil divisor, we have $[(a/2)\, D] = F$. Hence, if $R(X, D)$ is Gorenstein and a is even, then $H^i(X, \mathcal{O}_X(F)) = H^i(X, \mathcal{O}_X((a/2)\, D)) = 0$ for $0 < i < \dim X$, by (0.3.1). Q.E.D.

EXAMPLES. (1) Let $X = \mathbb{P}^n$ be an even-dimensional projective space over a field k. Then there exists no ample $\mathbb{Q}$-divisor D on X such that $R(X, D)$ is a quasi-Gorenstein ring with even $a(R(X, D))$.

(2) Let X be a smooth projective variety. Let $\pi: \tilde{X} \to X$ be the

blowing-up of X along a smooth subvariety of even-codimension $r \geqslant 2$. Then there exists no ample $\mathbb{Q}$-divisor D on $\tilde{X}$ such that $R(\tilde{X}, D)$ is a quasi-Gorenstein ring with even $a(R(\tilde{X}, D))$.

2. A Sufficient Condition for X to Have D with $R(X, D)$ Quasi-Gorenstein

Proposition. *Let X be a $\mathbb{Q}$-Gorenstein normal projective variety of dimension N over an algebraically closed field k.*

(a) *For every positive odd integer a, there exists an ample $\mathbb{Q}$-divisor D on X such that $R(X, D)$ is a quasi-Gorenstein graded ring with $a(R(X, D)) = a$.*

(b) *Let a be a positive even integer. Then there exists an ample $\mathbb{Q}$-divisor D on X such that $R(X, D)$ is a quasi-Gorenstein graded ring with $a(R(X, D)) = a$ if and only if there exists a Weil divisor F such that $2F$ is linearly equivalent to the canonical divisor K_X.*

Proof. (a) Thanks to (0.3.2), we have only to find out an ample $\mathbb{Q}$-divisor D such that $K_X + D' - aD$ is linearly equivalent to 0. Let L be a very ample Cartier divisor on X such that $K_X + L$ is an ample $\mathbb{Q}$-divisor and that $\mathcal{O}_X(K_X + L)|_U$ is a very ample invertible sheaf on U, where U is the open subset of X on which K_X is a Cartier divisor. Since $U \supseteq X_{reg}$ and X is normal, by Bertini's theorem [8, p. 30, Theorem I.6.3], there exist prime divisors $Y_1 \neq Y_2$ on X such that $Y_1 \sim K_X + 2L$ and $Y_2 \sim L$. In fact, let us define the prime divisor Y_1 as follows. By Bertini's theorem, there exist prime divisor $Z_1 \sim (K_X + 2L)|_U$ on U. Define Y_1 to be the closure of Z_1 on X. Then $Y_1 \sim K_X + 2L$. (Note that the Weil divisors E_1 and E_2 on a normal variety X are linearly equivalent, if $E_1 \cap U \sim E_2 \cap U$ as divisors on $U \supseteq X_{reg}$.) Fix integers $p_i > 0$, $q_i > 4$ $(i = 1, 2)$ such that $2q_1 - 1 = ap_1$, $q_2 + 1 = ap_2$. Set $D := (p_1/q_1)\, Y_1 - (p_2/q_2)\, Y_2$. Then D satisfies the required condition. Indeed, D is $\mathbb{Q}$-Cartier, since Y_1 and Y_2 are $\mathbb{Q}$-Cartier divisors. Since D is numerically equivalent to $(p_1/q_1)(K_X + L) + \{(p_1/q_1) - (p_2/q_2)\}\, L$ and since $K_X + L$ and L are ample, if $(p_1/q_1) > 0$ and $(p_1/q_1) - (p_2/q_2) > 0$, then D is ample. But we have $(p_1/q_1) = (2/a) - (1/aq_1) > 7/4a > 0$, and $(p_1/q_1) - (p_2/q_2) = (1/a) - (1/aq_1) - (1/aq_2) > 1/2a > 0$, as required. On the other hand, we note that $D' = \{(q_1 - 1)/q_1\}\, Y_1 + \{(q_2 - 1)/q_2\}\, Y_2$ and that $K_X \sim Y_1 - 2Y_2$. Hence we have $K_X + D' - aD \sim (1/q_1)(2q_1 - 1 - ap_1)\, Y_1 + (1/q_2)(-q_2 - 1 + ap_2)\, Y_2 = 0$.

(b) The "only if" part was already shown in the Lemma. To prove the "if" part, as in the proof of (a), we have only to find out an ample $\mathbb{Q}$-divisor D such that $K_X + D' - aD \sim 0$. Let L be a very ample Cartier divisor on X such that $F+L$ is an ample $\mathbb{Q}$-divisor and that $\mathcal{O}_X(F+L)|_V$ is a very ample invertible sheaf on V, where V is the open subset of X on which F is Cartier divisor. Let Y_1, Y_2, and Y_3 be mutually distinct prime divisors such that $Y_1, Y_2 \sim F+2L$ and $Y_3 \sim 2L$. Fix integers $s>0$, $p_i>0$, $q_i>4$ ($i=1, 2, 3$) such that $(2s+3)q_1-1=ap_1$, $(2s-1)q_2+1=ap_2$, and $q_3+1=ap_3$. (Since a is even, it is easily seen that such integers actually exist.) Set $D:=(p_1/q_1)Y_1-(p_2/q_2)Y_2-(p_3/q_3)Y_3$. Then D is $\mathbb{Q}$-Cartier. Since D is numerically equivalent to the $\mathbb{Q}$-divisor $\{(p_1/q_1)-(p_2/q_2)\}(F+L)+\{(p_1/q_1)-(p_2/q_2)-2(p_3/q_3)\}L$ and since $F+L$ and L are ample, if $(p_1/q_1)-(p_2/q_2)>0$ and $(p_1/q_1)-(p_2/q_2)-2(p_3/q_3)>0$, then D is ample. But we have $(p_1/q_1)-(p_2/q_2)=\{(2s+3)/a-1/aq_1\}-\{(2s-1)/a+1/aq_2\}>7/2a$, and, $(p_1/q_1)-(p_2/q_2)-2(p_3/q_3)>7/2a-2\{1/a+1/aq_3\}>1/a$, as required. On the other hand, since $D'=\{(q_1-1)/q_1\}Y_1+\{(q_2-1)/q_2\}Y_2+\{(q_3-1)/q_3\}Y_3$ and $K_X\sim(2s+2)Y_1-2sY_2-2Y_3$, we have $K_X+D'-aD\sim(1/q_1)\{(2s+3)q_1-1-aq_1\}Y_1+(1/q_2)\{(-2s+1)q_2-1+ap_2\}Y_2+(1/q_3)\{-q_3-1+aq_3\}Y_3=0$. Q.E.D.

3. Proof of the Theorem

(a) We proceed in two steps. Note that the assumption, that K_X is Cartier and that X is Cohen–Macaulay, is required in Step I.

Step I. *There exists a very ample Cartier divisor L on X such that $L+K_X$ is very ample and that $H^i(X, \mathcal{O}_X(xL+yK_X))=0$ for $0<i<N$ and for $(x, y)\in S:=\{(x, y)\in\mathbb{Z}^2;\ 2x\geqslant y\geqslant 0\}\cup\{(x, y)\in\mathbb{Z}^2;\ 1\geqslant y\geqslant 2x+1\}$.*

Proof. Since X is projective, there exists a very ample invertible sheaf $\mathcal{M}$ such that $\mathcal{M}\otimes\omega_X$ is very ample (e.g., [4, p. 169, Exercise 7.5]). Let $\mathcal{E}$ be the vector bundle $(\mathcal{M}\otimes\omega_X)\oplus\mathcal{M}$ of rank 2 over X. Since $\mathcal{M}\otimes\omega_X$ and $\mathcal{M}$ are ample, the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$ is ample [3, Proposition 2.2]. Here, by $\mathbb{P}(\mathcal{E})$, we mean the projective space bundle defined by **Proj**(Symm($\mathcal{E}$)). Then it follows from Serre's vanishing theorem (e.g., [4, p. 229, Proposition 5.3]) that there exists an integer $t\geqslant 2$ such that $H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(tx))=0$ for all integers $i>0$ and $x>0$. On the other hand, by well-known facts about the projective space bundles (e.g., [4, p. 253, Exercise 8.4]), we have

$$\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(d))=\mathrm{Symm}^d(\mathcal{E}) \qquad \text{for every integer } d\geqslant 0, \text{ and}$$

$$R^1\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(d))=0 \qquad \text{for every integer } d\geqslant -1,$$

where π is the structure morphism $\mathbb{P}(\mathcal{E})\to X$ and $\mathrm{Symm}^d(\mathcal{E})$ denotes the

dth symmetric product of $\mathscr{E}$. Therefore, by a degenerate case of the Leray spectral sequence, we have

$$H^i(\mathbb{P}(\mathscr{E}), \mathcal{O}_{\mathbb{P}(\mathscr{E})}(d)) = H^i(X, \mathrm{Symm}^d(\mathscr{E}))$$
$$= \bigoplus_{y=0}^{d} H^i(X, \mathscr{M}^{\otimes d} \otimes \omega_X^{\otimes y}) \qquad \text{for all} \quad d \geqslant 0 \text{ and } i \geqslant 0.$$

Thus, for all integers $x > 0$, $0 \leqslant y \leqslant xt$, and $i > 0$, we have $H^i(X, \mathscr{M}^{\otimes xt} \otimes \omega_X^{\otimes y}) = 0$. For our purpose, let L be a Cartier divisor with $\mathcal{O}_X(L) \simeq \mathscr{M}^{\otimes t}$. Then L satisfies the required condition, since $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < N$ by the assumption and since $H^i(X, \mathcal{O}_X(xL + yK_X)) \simeq H^{N-i}(X, \mathcal{O}_X(-xL - (y-1) K_X))^*$ for each $i > 0$ and any integers x, y by Serre duality (e.g., [4, p. 244, Corollary 7.7]). Q.E.D.

STEP II. *Let L be the very ample Cartier divisor in Step* I. *Hence $K_X + L$ is a very ample Cartier divisor on X. Set D as in the proof of Proposition* (a). *Then D is an ample $\mathbb{Q}$-divisor such that $R(X, D)$ is a Gorenstein ring with $a(r(X, D)) = a$.*

Proof. It is shown that in the proof of Proposition (a) that D is an ample $\mathbb{Q}$-divisor on X such that $R(X, D)$ is a quasi-Gorenstein with $a(R(X, D)) = a$. Now we show that $R(X, D)$ is a Cohen–Macaulay ring. By (0.3.1), we have to show that, for all $n \in \mathbb{Z}$,

$$H^i(X, \mathcal{O}_X(nD)) = 0 \qquad \text{for} \quad 0 < i < N. \tag{$*$}$$

(Hence, if $N \leqslant 1$, then there is nothing to prove.) For $n = 0$, $(*)$ is included in our assumption.

Let us show $(*)$ when $n > 0$. Set $k_1(n) := [(p_1/q_1) n]$ and $k_2(n) := \lceil (p_2/q_2) n \rceil$, where $\lceil z \rceil := -[-z]$, i.e., the round up of a real number z. Since $\mathcal{O}_X(nD) = \mathcal{O}_X([nD]) = \mathcal{O}_X(k_1(n) Y_1 - k_2(n) Y_2) = \mathcal{O}_X((2k_1(n) - k_2(n)) L + k_1(n) K_X)$, thanks to the Step I, we have only to check that $(2k_1(n) - k_2(n), k_1(n)) \in S$. First note that $(p_1/q_1) n \geqslant k_1(n) > (p_1/q_1) n - 1$ and $(p_2/q_2) n \leqslant k_2(n) < (p_2/q_2) n + 1$, and therefore, $(2/a) n > k_1(n) > (7/4a) n - 1$ and $(1/a) n < k_2(n) < (5/4a) n + 1$. Hence we have $2\{2k_1(n) - k_2(n)\} - k_1(n) = 3k_1(n) - 2k_2(n) > 3\{(7/4a) n - 1\} - 2\{(5/4a) n + 1\} = (11/4a) n - 5$. If $n \geqslant 2a$, then $2\{2k_1(n) - k_2(n)\} \geqslant k_1(n) \geqslant 0$. For $0 < n < 2a$, by the above inequalities on $k_1(n)$ and $k_2(n)$, we have $0 \leqslant k_1(n) < 4$ and $1 \leqslant k_2(n) < (5/4a) 2a + 1 = 7/2$. Therefore $(2k_1(n) - k_2(n), k_1(n)) \in S$ for each $0 < n < 2a$.

Finally, let us show $(*)$ when $n = -m < 0$. Set $h_1(m) := \lceil (p_1/q_1) m \rceil$ and $h_2(m) := [(p_2/q_2) m]$. Since $\mathcal{O}_X(nD) = \mathcal{O}_X((-2h_1(m) + h_2(m)) L - h_1(m) K_X)$ and Step I, we have only to check that $(-2h_1(m) + h_2(m), -h_1(m)) \in S$ for each $n = -m < 0$. Since $(7/4a) m < h_1(m) < (2/a) m + 1$

and $(5/4a)m > h_2(m) > (1/a)m - 1$, it is easily checked that $-h_1(m) - 2\{-2h_1(m) + h_2(m)\} > (11/4a)m$ and that $h_1(m) > 0$. Hence $0 \geqslant -h_1(m) \geqslant 2\{-2h_1(m) + h_2(m)\} + 1$ for each $n = -m < 0$, as required. Q.E.D.

(b) As in (a), we proceed in two steps.

STEP I. *There exists a very ample Cartier divisor L on X such that $F + L$ is very ample and that $H^i(X, \mathcal{O}_X(xL + yF)) = 0$ for $0 < i < N$ and for $(x, y) \in T := \{(x, y) \in \mathbb{Z}^2;\ 2x \geqslant y \geqslant 0\} \cup \{(x, y) \in \mathbb{Z}^2;\ 2 \geqslant y \geqslant 2x + 2\} \cup \{(0, 1)\}$.*

Proof. The assertion follows from the same proof as that in Step I of (a), if we replace K_X by F. Q.E.D.

STEP II. *Let L be the very ample Cartier divisor in Step* I. *Hence $F + L$ is a very ample Cartier divisor on X. Set D as in the proof of Proposition* (b). *Then D is an ample $\mathbb{Q}$-divisor such that $R(X, D)$ is a Gorenstein ring with $a(R(X, D)) = a$.*

Proof. It is shown that in the proof of Proposition (b) that D is an ample $\mathbb{Q}$-divisor on X such that $R(X, D)$ is a quasi-Gorenstein with $a(R(X, D)) = a$. Now we show that $R(X, D)$ is a Cohen–Macaulay ring. By (0.3.1), we have to show that, for all $n \in \mathbb{Z}$,

$$H^i(X, \mathcal{O}_X(nD)) = 0 \qquad \text{for} \quad 0 < i < N. \tag{$**$}$$

(Hence, if $N \leqslant 1$, then there is nothing to prove.) For $n = 0$, $(**)$ is the assumption.

Let us show $(**)$ when $n > 0$. Set $k_1(n) := [(p_1/q_1)n]$, $k_2(n) := \lceil (p_2/q_2)n \rceil$, and $k_3(n) := \lceil (p_3/q_3)n \rceil$. Since Step I and $\mathcal{O}_X(nD) = \mathcal{O}_X(k_1(n)Y_1 - k_2(n)Y_2 - k_3(n)Y_3) = \mathcal{O}_X(2\{k_1(n) - k_2(n) - k_3(n)\}L + \{k_1(n) - k_2(n)\}F)$, we have only to check that $(2\{k_1(n) - k_2(n) - k_3(n)\}, k_1(n) - k_2(n)) \in T$ for each $n > 0$. First note that $\{(2s+3)/a\}n > k_1(n) > \{(8s+11)/4a\}n - 1$, $\{(2s-1)/a\}n < k_2(n) < \{(8s-3)/4a\}n + 1$, and $(1/a)n < k_3(n) < (5/4a)n + 1$. It is easily checked that $(4/a)n > k_1(n) - k_2(n) > (7/2a)n - 2$, and, $4\{k_1(n) - k_2(n) - k_3(n)\} - \{k_1(n) - k_2(n)\} = 3\{k_1(n) - k_2(n)\} - 4k_3(n) > (11/2a)n - 10$. If $n \geqslant 2a$, then $2\{2(k_1(n) - k_2(n) - k_3(n))\} \geqslant k_1(n) - k_2(n) \geqslant 0$. For $0 < n \leqslant a$ we have $-1 \leqslant k_1(n) - k_2(n) \leqslant 3$ and $1 \leqslant k_3(n) \leqslant 2$, and, therefore, $(2\{k_1(n) - k_2(n) - k_3(n)\}, k_1(n) - k_2(n)) \in T$. Similarly, for $a < n \leqslant 2a$, we have $2 \leqslant k_1(n) - k_2(n) \leqslant 7$ and $2 \leqslant k_3(n) \leqslant 3$. But in this case, it does not occur that $(2\{k_1(n) - k_2(n) - k_3(n)\}, k_1(n) - k_2(n)) = (0, 3)$. In fact, if $k_1(n) - k_2(n) = 3$, by the above inequalities on $k_1(n) - k_2(n)$, we have $(3/4)a < n < (10/7)a$, and, therefore, $k_3(n)=2$. This is a contradiction. Hence $(2\{k_1(n)-k_2(n)-k_3(n)\}, k_1(n)-k_2(n))\in T$ for each $a<n\leqslant 2a$.

Finally, let us show $(**)$ when $n=-m<0$. Set $h_1(m):=\lceil (p_1/q_1)m\rceil$, $h_2(m):=[(p_2/q_2)m]$, and $h_3(m):=[(p_2/q_3)m]$. Then $\mathcal{O}_X(nD)=\mathcal{O}_X(2\{-h_1(m)+h_2(m)+h_3(m)\}L+\{-h_1(m)+h_2(m)\}F)$. On the other hand, it is easily seen that $h_1(m)-h_2(m)-2h_3(m)>(1/a)m$ and that $h_1(m)-h_2(m)>(7/2a)m$. Hence $(2\{-h_1(m)+h_2(m)+h_3(m)\}, -h_1(m)+h_2(m))\in\{(x,y)\in\mathbb{Z}^2; 0\geqslant y\geqslant x, x\in 2\mathbb{Z}\}\subset T$, as required. Q.E.D.

4. Remark and Example

We want to determine the necessary and sufficient condition for a normal projective variety X to have an ample $\mathbb{Q}$-divisor D with $R(X,D)$ Gorenstein.

The most deficient aspect of our results is that the normal projective variety X is required to be Gorenstein. It seems likely that this assumption is somewhat redundant for our purpose. (Of course, we should assume that X is Cohen–Macaulay, since the Cohen–Macaulay property of $R(X,D)$ implies that X is a Cohen–Macaulay scheme.) In fact, the Gorenstein property of $R(X,D)$ does not necessarily imply that X is Gorenstein or even $\mathbb{Q}$-Gorenstein. For example, it is proved by the author [5, (2.6)] that every projective torus embedding X has an ample $\mathbb{Q}$-divisor D on X such that $R(X,D)$ is a Gorenstein ring with $a(R(X,D))=-1$. Note that a projective torus embedding is not necessarily Gorenstein nor $\mathbb{Q}$-Gorenstein.

On the other hand, it seems likely that the condition required in (b) of the Theorem, that is, F is a Cartier divisor, is also unnecessary for our purpose.

Indeed, we have:

Example. Concerning the torus embeddings, we refer the reader to [6]. Let T be a 2-dimensional algebraic torus defined over an algebraically closed field k and let $N\simeq\mathbb{Z}^2$ be the group of one-parameter subgroups of T with $\{n_1,n_2\}$ as a $\mathbb{Z}$-basis. Let Δ be the complete fan generated by one-dimensional cones $\rho_1:=\mathbb{R}_{\geqslant 0}n_1$, $\rho_2:=\mathbb{R}_{\geqslant 0}(n_1+2n_2)$, $\rho_3:=\mathbb{R}_{\geqslant 0}n_2$, $\rho_4:=\mathbb{R}_{\geqslant 0}(-n_1)$, and, $\rho_5:=\mathbb{R}_{\geqslant 0}(-n_2)$, where $\mathbb{R}_{\geqslant 0}:=\{x\in\mathbb{R}; x\geqslant 0\}$. Let X be the projective torus embedding $T_N\operatorname{emb}(\Delta)$ associated with the complete fan Δ. Let V_i $(i=1,\ldots,5)$ be the prime divisors, stable under the torus action, associated with the one-dimensional cones ρ_i. The canonical divisor $K_X=-(V_1+V_2+V_3+V_4+V_5)$ is linearly equivalent to $-2(V_2+V_3+V_4)$. Set $F:=-(V_2+V_3+V_4)$. Since $H^1(X,\mathcal{O}_X)=0$ and $V_2+V_3+V_4=-F$ is connected and reduced as a subscheme of X, we have $H^1(X,\mathcal{O}_X(F))=0$. Since F is a $\mathbb{Q}$-Cartier divisor but is *not* a Cartier divisor, we cannot apply our theorem to this case. Nevertheless, for a positive even integer a, there exists an ample $\mathbb{Q}$-divisor D such that $R(X, D)$ is a Gorenstein ring with $a(R(X, D)) = a$.

Indeed, the assumption that F is a Cartier divisor is required only in Step I of the proof of the Theorem. Thus, with notation as in (b) of the Proposition and the Theorem, we have only to prove that there exists a very ample Cartier divisor L such that $\mathcal{O}_X(F+L)|_V$ is a very ample line bundle on V and that $H^1(X, \mathcal{O}_X(xL+yF)) = 0$ for $(x, y) \in T$. Then the same proof of Step II is still valid in this case. Let L be a very ample Cartier divisor such that $F + (1/2) L$ is an ample $\mathbb{Q}$-divisor and that $\mathcal{O}_X(F+L)|_V$ is a very ample line bundle. Then, for each pair of integers x, y with $x > 0$ and $y \leqslant 2x$, $yF + xL$ is an ample $\mathbb{Q}$-divisor stable under the torus action. In fact, since $yF + xL = y(F + (1/2) L) + (x - (y/2)) L$, it is ample for $x > 0$ and $0 \leqslant y \leqslant 2x$. On the other hand, since $-2F$ is generated by its global sections and $yF + xL = xL + (-y)(-F)$, it is also ample for $x > 0$ and $y < 0$. By [5, Corollary 1.6], we have $H^1(X, \mathcal{O}_X(xL+yF)) = 0$ and $H^1(X, \mathcal{O}_X(-xL-yF)) = 0$ for $x > 0$ and $y \leqslant 2x$, as required.

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