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# Confluent operator algebras and the closability property ☆

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Dedicated to the memory of our good friends and mentors, Paul R. Halmos and Béla Szőkefalvi-Nagy

**Abstract**

Certain operator algebras $\mathcal{A}$ on a Hilbert space have the property that every densely defined linear transformation commuting with $\mathcal{A}$ is closable. Such algebras are said to have the *closability property*. They are important in the study of the transitive algebra problem. More precisely, if $\mathcal{A}$ is a two-transitive algebra with the closability property, then $\mathcal{A}$ is dense in the algebra of all bounded operators, in the weak operator topology. In this paper we focus on algebras generated by a completely nonunitary contraction, and produce several new classes of algebras with the closability property. We show that this property follows from a certain strict cyclicity property, and we give very detailed information on the class of completely nonunitary contractions satisfying this property, as well as a stronger property which we call *confluence*.


*Keywords:* Confluent algebra; Closability property; Completely nonunitary contraction; Rationally strictly cyclic vector; Quasisimilarity

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## 1. Introduction

Probably the best known problem in operator theory is the question of whether every bounded linear operator on a complex, separable, infinite dimensional Hilbert space $\mathcal{H}$ has a nontrivial invariant subspace. Despite considerable effort by many researchers for more than half a century, the general problem remains open. A generalization, still unresolved, asks whether every transitive algebra of operators must be dense in the weak operator topology. (Recall that an algebra is said to be transitive if there is no common nontrivial invariant subspace for the operators in it.)

In the sixties, Arveson approached this problem iteratively, starting from an observation going back essentially to von Neumann. Namely, assume that $\mathcal{A}$ is an algebra of operators on a Hilbert space $\mathcal{H}$, and $n \geqslant 1$ is an integer. The algebra $\mathcal{A}$ is said to be $n$-transitive if every invariant subspace for

$$\mathcal{A}^{(n)} = \bigl\{X^{(n)} = \underbrace{X \oplus X \oplus \cdots \oplus X}_{n \text{ times}}\colon\ X \in \mathcal{A}\bigr\}$$

is invariant for every operator of the form $Y^{(n)}$ where $Y$ is an operator on $\mathcal{H}$. Then $\mathcal{A}$ is dense, in the weak operator topology, if and only if it is $n$-transitive for every $n \geqslant 1$. Arveson observed that 2-transitivity is equivalent to the following statement: every *closed* linear transformation commuting with $\mathcal{A}$ is a scalar multiple of the identity operator. For $n \geqslant 3$, $n$-transitivity is implied by a similar statement for *densely defined* linear transformations commuting with $\mathcal{A}$. Thus, provided that every densely defined linear transformation commuting with $\mathcal{A}$ is closable, 2-transitivity implies $n$-transitivity for all $n$. As a consequence, Arveson was able to prove that transitive algebras containing certain kinds of subalgebras are indeed dense in the weak operator topology. His results apply to algebras on an $L^2$-space, containing the algebra $L^\infty$ of all bounded measurable multipliers, and to algebras on the Hardy space $H^2(\mathbb{D})$, containing the algebra $H^\infty(\mathbb{D})$. A few similar results were obtained by others for closely related algebras in the following years; see for instance [19, Chapter 8].

A year ago, Haskell Rosenthal became interested in the question of which algebras of operators on Hilbert space had what he called the *closability property* which means that every densely defined linear transformation in its commutant is closable. A key step in Arveson's proofs was to show that the algebras $L^\infty$ acting on $L^2$, and $H^\infty(\mathbb{D})$ acting on $H^2(\mathbb{D})$, have the closability property. Rosenthal showed that various algebras have the closability property and asked the authors a specific followup question. In finding the answer, the question piqued our interest and resulted in a series of questions related to this topic. Our investigation took us in some unexpected directions, making surprising connections with other topics in operator theory.

After some preliminaries in Section 2, we begin in Section 3 by investigating the closability property and determining some algebras which have it, as well as some that do not. In Section 4 we introduce the concept of a rationally strictly cyclic vector, and show that the existence of such a vector for a commutative algebra $\mathcal{A}$ implies the closability property. In Section 5 we discuss the invariance of the closability property, and of the existence of rationally strictly cyclic vectors, under an appropriate notion of quasisimilarity. We deduce, for instance, that the commutant of any contraction of class $C_0$ has the closability property. In the course of our study, the importance of something like a functional calculus for quotients became clear. To make this idea precise, in Section 6 we study the related notion of confluence (introduced in Section 4) as it applies to the algebra obtained by applying the $H^\infty$ functional calculus to a completely nonunitary contraction. Confluence implies the existence of a rationally strictly cyclic vector, and therefore the closability property as well. Section 7 contains a thorough study of confluence in the context of functional models for contractions. In particular, a characterization is obtained for those contractions which are quasisimilar to the unilateral shift of multiplicity one. This characterization involves the 'size' of the analytic functions in the reproducing kernel representative for the operator.

The analysis of confluence is somewhat subtle and rests on the harmonic analysis of contractions [23], the theory of the class $C_0$ [4], the theory of dual algebras [5], and results about the class $\mathcal{B}_1(\mathbb{D})$ [10].

We thank Haskell Rosenthal for the questions which led to this research. We are also grateful to the referee, who pointed out several errors and numerous misprints in our original manuscript.

## 2. Preliminaries

We will work with operators on Hilbert spaces over the complex numbers $\mathbb{C}$. The algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ is denoted $\mathcal{L}(\mathcal{H})$. Given $T \in \mathcal{L}(\mathcal{H})$, $\mathcal{P}_T$ denotes the smallest unital algebra containing $T$; that is, the set of all polynomials in $T$. The closure of $\mathcal{P}_T$ in the weak operator topology (also known as WOT) is denoted $\mathcal{W}_T$. The norm closure of a subset $\mathcal{M} \subset \mathcal{H}$ is denoted $\overline{\mathcal{M}}$. The orthogonal projection of $\mathcal{H}$ onto a closed linear subspace $\mathcal{M} \subset \mathcal{H}$ is denoted $P_{\mathcal{M}}$.

Several special operators play an important role. The space $L^2$ is the space of functions defined on the unit circle $\mathbb{T}$ which are square integrable relative to arclength measure. The bilateral shift $U \in \mathcal{L}(L^2)$ is the unitary operator defined by $(Uf)(\zeta) = \zeta f(\zeta)$ for $f \in L^2$ and a.e. $\zeta \in \mathbb{T}$. The Hardy space $H^2 \subset L^2$ is the cyclic subspace for $U$ generated by the constant function 1, and $S \in \mathcal{L}(H^2)$ is the unilateral shift of multiplicity 1 defined as $S = U \mid H^2$. More generally, denote by $H^\infty = H^\infty(\mathbb{D})$, the algebra of bounded analytic functions in the unit disk $\mathbb{D}$. For each $u \in H^\infty$ one defines an analytic Toeplitz operator $T_u \in \mathcal{L}(H^2)$ as the operator of pointwise multiplication by $u$. Here one takes advantage of the fact that functions in $H^\infty$ have a.e. defined radial limits on $\mathbb{T}$.

Given a subset $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$, $\mathcal{A}'$ denotes the set of operators commuting with every element of $\mathcal{A}$. The set $\mathcal{A}'$ is called the commutant of $\mathcal{A}$, and it is a unital algebra, closed in the weak operator topology. An important example is

$$\{S\}' = \mathcal{W}_S = \{T_u \colon u \in H^\infty\}.$$

A function $m \in H^\infty$ is inner if $|m(\zeta)| = 1$ for a.e. $\zeta \in \mathbb{T}$. For every inner function $m \in H^\infty$, the space $mH^2 = T_m H^2$ is closed and invariant for $S$. The compression of $S$ to $\mathcal{H}(m) = H^2 \ominus mH^2$ is denoted $S(m)$. In other words, $S(m) = P_{\mathcal{H}(m)} S \mid \mathcal{H}(m)$ or, equivalently, $S(m)^* = S^* \mid \mathcal{H}(m)$. Another important example of a commutant is

$$\{S(m)^*\}' = \mathcal{W}_{S(m)^*} = \{T_u^* \mid \mathcal{H}(m) \colon u \in H^\infty\}.$$

This was proved by Sarason [20].

An operator $T \in \mathcal{L}(\mathcal{H})$ is a contraction if $\|T\| \leqslant 1$. A contraction $T$ is completely nonunitary if it has no invariant subspace on which it acts as a unitary operator. For completely nonunitary contractions $T$, there is a homomorphism $u \mapsto u(T) \in \mathcal{L}(\mathcal{H})$ which extends the polynomial functional calculus to functions $u \in H^\infty$. This is called the Sz.-Nagy–Foias functional calculus. For instance, $u(S) = T_u$, and $u(S(m)) = P_{\mathcal{H}(m)} T_u \mid \mathcal{H}(m)$.

A completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$ is of class $C_0$ if $u(T) = 0$ for some $u \in H^\infty \setminus \{0\}$. The ideal $\{u \in H^\infty\colon\ u(T) = 0\} \subset H^\infty$ is principal, and it is generated by an inner function, uniquely determined up to a constant factor of absolute value 1. This function is called the minimal function of $T$. The most basic example is $S(m)$, whose minimal function is $m$.

We refer the reader to [23] for further background on the analysis of contractions, to [5] for dual algebras, and to [4] for the class $C_0$. We will refer as needed to these and other original sources for specific results.

## 3. The closability property

Consider a unital subalgebra $\mathcal{A}$ of the algebra $\mathcal{L}(\mathcal{H})$ of bounded operators on the complex Hilbert space $\mathcal{H}$. The algebra $\mathcal{A}$ is not assumed to be norm closed.

**Definition 3.1.** A linear transformation $X : \mathcal{D}(X) \to \mathcal{H}$ is said to *commute* with $\mathcal{A}$ if for every $h \in \mathcal{D}(X)$ and every $T \in \mathcal{A}$ we have $Th \in \mathcal{D}(X)$ and

$$XTh = TXh.$$

We define now the main concept we study in this paper.

**Definition 3.2.** The algebra $\mathcal{A}$ is said to have the *closability property* if every linear transformation $X$ which commutes with $\mathcal{A}$, and whose domain $\mathcal{D}(X)$ is dense in $\mathcal{H}$, is closable.

We recall that a linear transformation $X$ is *closable* if the closure of its graph

$$\mathcal{G}(X) = \bigl\{h \oplus Xh\colon\ h \in \mathcal{D}(X)\bigr\}$$

is again the graph of a linear transformation, usually denoted $\overline{X}$ and called the closure of $X$. Equivalently, $X$ is closable if given a sequence $h_n \in \mathcal{D}(X)$ such that $\lim_{n\to\infty} \|h_n\| = 0$ and the limit $k = \lim_{n\to\infty} Xh_n$ exists, it follows that $k = 0$.

The following observation is a trivial consequence of the fact that a linear transformation commuting with an algebra also commutes with smaller algebras.

**Lemma 3.3.** *Assume that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{L}(\mathcal{H})$ are unital algebras. If $\mathcal{A}$ has the closability property then so does $\mathcal{B}$. In particular, if $\mathcal{A}$ is commutative and has the closability property, then its commutant $\mathcal{A}'$ also has the closability property.*

We start with some examples of algebras which do not have the closability property. The arguments are based on the following simple fact.

**Lemma 3.4.** *Let $\mathcal{A}$ be a unital subalgebra of $\mathcal{L}(\mathcal{H})$. Assume that there exist linear manifolds $\mathcal{M}, \mathcal{N} \subset \mathcal{H}$ such that*

(1) *$T\mathcal{M} \subset \mathcal{M}$ and $T\mathcal{N} \subset \mathcal{N}$ for every $T \in \mathcal{A}$;*
(2) *$\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\overline{\mathcal{M} + \mathcal{N}} = \mathcal{H}$;*
(3) *$\overline{\mathcal{M}} \cap \overline{\mathcal{N}} \neq \{0\}$.*

*Then $\mathcal{A}$ does not have the closability property.*

**Proof.** Define a linear transformation with domain $\mathcal{D}(X) = \mathcal{M} + \mathcal{N}$ by setting $Xh = 0$ for $h \in \mathcal{M}$ and $Xh = h$ for $h \in \mathcal{N}$. If $X$ were closable, its closure would satisfy $\overline{X}h = 0$ and $\overline{X}h = h$ for any $h \in \overline{\mathcal{M}} \cap \overline{\mathcal{N}}$, and this is absurd for $h \neq 0$. $\square$

**Proposition 3.5.** *The following algebras do not have the closability property*:

(1) *The algebra $\mathcal{P}_S$ generated by the unilateral shift $S$.*
(2) *The algebra $\mathcal{P}_{S(m)}$, where $m$ is an inner function which is not a finite Blaschke product.*
(3) *The WOT-closed algebra $\mathcal{W}_{S^*}$.*
(4) *The WOT-closed algebra $\mathcal{W}_U$ generated by the bilateral shift $U$ on $L^2$.*
(5) *Any algebra of the form $\mathcal{A} \otimes I_{\mathcal{K}}$, where $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a unital algebra, and $\mathcal{K}$ is an infinite dimensional Hilbert space.*

**Proof.** For the first example, choose an outer function $f \in H^2$ which is not rational, and define $\mathcal{M}$ to consist of all polynomials and $\mathcal{N} = \{pf\colon\ p \text{ a polynomial}\}$. The hypotheses of Lemma 3.4 are verified trivially since both of these spaces are dense in $H^2$.

Next, assume that $m$ is an inner function but not a finite Blaschke product, and consider a factorization $m = m_1 m_2$ such that the inner functions $m_j$ are not finite Blaschke products. We can define then subspaces $\mathcal{M}, \mathcal{N} \subset \mathcal{H}(m)$ by $\mathcal{M} = \{P_{\mathcal{H}(m)} p\colon\ p \text{ a polynomial}\}$ and $\mathcal{N} = \{P_{\mathcal{H}(m)}(pm_2)\colon\ p \text{ a polynomial}\}$. The space $\mathcal{M}$ is dense in $\mathcal{H}(m)$, so to verify the hypotheses of Lemma 3.4 it suffices to show that $\mathcal{M} \cap \mathcal{N} = \{0\}$. Consider indeed two polynomials $p, q$ such that $P_{\mathcal{H}(m)} p = P_{\mathcal{H}(m)}(qm_2)$. In other words, we have $p = qm_2 + rm_1 m_2$ for some $r \in H^2$. If $p \neq 0$, this equality implies that the inner factor of $p$ (obviously a finite Blaschke product) is divisible by $m_2$, contrary to our choice of factors.

For example (3), we choose $\mathcal{M} = \{p\colon\ p \text{ a polynomial}\} \subset H^2$, and we denote by $\mathcal{N}$ the linear manifold generated by the functions $k_\lambda(z) = (1 - \lambda z)^{-1}$, $\lambda \in \mathbb{D} \setminus \{0\}$. These spaces are dense in $H^2$, and the identities

$$\big(S^* p\big)(z) = \frac{p(z) - p(0)}{z}, \qquad S^* k_\lambda = \lambda k_\lambda$$

easily imply that they are invariant under $\mathcal{W}_{S^*}$. Finally, a function $p$ in their intersection must be both a polynomial, and a rational function vanishing at $\infty$, hence $p = 0$.

For example (4), define two sets $\omega_\pm = \{e^{\pm it}\colon\ 0 < t < 3\pi/2\} \subset \mathbb{T}$, denote by $\chi_\pm$ their characteristic functions, and set $\mathcal{M} = \chi_+ H^2$ and $\mathcal{N} = \chi_- H^2$. Since $\overline{\mathcal{M}} = \chi_+ L^2$ and $\overline{\mathcal{N}} = \chi_- L^2$, we clearly have $\overline{\mathcal{M} + \mathcal{N}} = L^2$ and $\overline{\mathcal{M}} \cap \overline{\mathcal{N}} = \chi_{\omega_+ \cap \omega_-} L^2$. The fact that $\mathcal{M} \cap \mathcal{N} = \{0\}$ follows easily from the F. and M. Riesz theorem.

Finally, assume that $\mathcal{K}$ is an infinite dimensional Hilbert space, and let $\mathcal{M}_0, \mathcal{N}_0 \subset \mathcal{K}$ be two dense linear manifolds such that $\mathcal{M}_0 \cap \mathcal{N}_0 = \{0\}$. Then $\mathcal{M} = \mathcal{H} \otimes \mathcal{M}_0$ and $\mathcal{N} = \mathcal{H} \otimes \mathcal{N}_0$ will satisfy the hypotheses of Lemma 3.4 for the algebra $\mathcal{A} \otimes I_{\mathcal{K}}$. $\square$

The first two examples above indicate that an algebra with the closability property must be reasonably large, while the last one shows that it should not have uniform infinite multiplicity. In this paper we will focus on algebras which have multiplicity one. The first example of an algebra with the closability property was of this kind: any maximal abelian selfadjoint subalgebra of $\mathcal{L}(\mathcal{H})$ has the closability property, as shown in [3]. This, along with the examples described in Proposition 3.7 (the first of which also appeared in [3], while the second was proved independently in [21]), will be treated in a unified manner in Section 4. The proofs of these particular cases do in fact suggest the more general methods.

First, a useful observation about bounded outer functions. This is certainly known, but we could not find a reference. We use the notation $\|u\|_2$ for the norm of a function $u \in L^2$.

**Lemma 3.6.** *Let $v \in H^\infty$ be an outer function. There exist outer functions $(w_n)_{n=1}^\infty \subset H^\infty$ with the property that $\lim_{n\to\infty} \|u - w_n vu\|_2 = 0$ for every $u \in L^2$. In particular, if $T$ is a completely nonunitary contraction, the sequence $(w_n(T)v(T))_{n=1}^\infty$ converges to $I$ in the strong operator topology.*

**Proof.** The functions $w_n$ will be specified by the requirements that $(w_n v)(0) > 0$, and

$$\left|(w_n v)(\zeta)\right| = \begin{cases} 1 & \text{if } |v(\zeta)| \geqslant 1/n, \\ |v(\zeta)| & \text{if } |v(\zeta)| < 1/n \end{cases}$$

for a.e. $\zeta \in \mathbb{T}$, so that $\|w_n\|_\infty \leqslant n$. Observe that

$$(w_n v)(0) = \frac{1}{2\pi} \int\limits_{|v(\zeta)|<1/n} \log\left|v(\zeta)\right| |d\zeta| \to 1$$

as $n \to \infty$, and it follows easily that $(w_n v)(\lambda) \to 1$ uniformly on every compact subset of $\mathbb{D}$. This also implies that $w_n v \to 1$ in the weak* topology of $H^\infty$ (given by its duality with $L^1/H_0^1$). Fix now $u \in L^2$ of unit norm, and use this weak convergence to deduce that

$$\begin{aligned} \limsup_{n\to\infty} \|u - w_n vu\|_2^2 &= \limsup_{n\to\infty} \|w_n vu\|_2^2 - 2\,\mathrm{Re} \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} w_n v|u|^2 \, |d\zeta| + 1 \leqslant 0, \\ &= \limsup_{n\to\infty} \|w_n vu\|_2^2 - 1 \leqslant 0. \end{aligned}$$

The lemma follows. □

**Proposition 3.7.** *The algebras $\mathcal{W}_S$ and $\mathcal{W}_{S(m)}$ have the closability property.*

**Proof.** Recall first that every function in $H^2$ is the quotient of two bounded functions in $H^\infty$. For instance, given a nonzero function $f \in H^2$, denote by $v_f$ the unique outer function defined by the requirements that $v_f(0) > 0$ and

$$\left|v_f(\zeta)\right| = \min\left\{1, \frac{1}{|f(\zeta)|}\right\}$$

for almost every $\zeta \in \mathbb{T}$. The functions $v_f$ and $u_f = f v_f$ belong to $H^\infty$, and in fact

$$\left|u_f(\zeta)\right| = \min\left\{1, \left|f(\zeta)\right|\right\} \quad \text{a.e. } \zeta \in \mathbb{T}. \tag{3.1}$$

Consider first the algebra $\mathcal{W}_S$ which consists precisely of the analytic Toeplitz operators $T_u$ with $u \in H^\infty$. Let $X$ be a densely defined linear transformation commuting with this algebra, and let $f, g \in \mathcal{D}(X)$. Observe first that $u_f = v_f f = T_{v_f} f \in \mathcal{D}(X)$, and therefore we can write

$$\begin{aligned} v_g u_f X g &= T_{v_g u_f} X g = X T_{v_g u_f} g = X(v_g u_f g) = X(u_f u_g) \\ &= X T_{u_g} u_f = T_{u_g} X u_f = u_g X u_f. \end{aligned}$$

Let now $g_n \in \mathcal{D}(X)$ be a sequence converging to zero such that the limit $h = \lim_{n\to\infty} X g_n$ exists. Passing if necessary to a subsequence, we may assume that $g_n(\zeta) \to 0$ for almost every $\zeta \in \mathbb{T}$. By virtue of (3.1) we also have $|v_{g_n}(\zeta)| \to 1$ and $u_{g_n}(\zeta) \to 0$ for a.e. $\zeta$, and therefore

$$\|u_{g_n} X u_f\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} \left|u_{g_n}\left(e^{it}\right)\right|^2 \left|(X u_f)\left(e^{it}\right)\right|^2 dt \to 0$$

as $n \to \infty$ by the dominated convergence theorem. The identity

$$v_{g_n} u_f X g_n = u_{g_n} X u_f$$

proved earlier, along with the fact that $|v_{g_n}| \to 1$ a.e., implies that $u_f h = 0$ for every $f \in \mathcal{D}(X)$. Choosing a function $f$ which is not identically zero, we deduce that $h = 0$, thus proving that $X$ is closable.

Consider now an inner function $m$, and define a map $J : \mathcal{H}(m) \to \mathcal{H}(m)$ by setting

$$(Jf)(\zeta) = \overline{\zeta f(\zeta)} m(\zeta), \quad \zeta \in \mathbb{T}. \tag{3.2}$$

This is a conjugate linear surjective isometry on $\mathcal{H}(m)$ satisfying the equation $JS(m) = S(m)^* J$. More generally, we have the identity

$$JA = A^* J, \quad A \in \mathcal{W}_{S(m)}.$$

This identity is easily verified when $A$ is a polynomial in $S(m)$, and it extends to arbitrary $A$ using the continuity properties of the functional calculus with $H^\infty$ functions.

Let us denote by $\xi = 1 - \overline{m(0)} m \in \mathcal{H}(m)$ the projection of 1 onto $\mathcal{H}(m)$. This is a separating vector for $\mathcal{H}(m)$. That is, the equality $A\xi = 0$ implies $A = 0$ for $A \in \mathcal{H}(m)$. Indeed, if $A = u(S(m))$, we have $A\xi = P_{\mathcal{H}(m)} u$, and this function is zero if and only if $u$ divides $m$, in which case $u(S(m)) = 0$.

Consider now a densely defined linear transformation $X$ commuting with $\mathcal{W}_{S(m)}$. We will show that $X$ is closable by proving the identity

$$\langle X h_1, J h_2 \rangle = \langle h_1, J X h_2 \rangle, \quad h_1, h_2 \in \mathcal{D}(X),$$

which shows that $JX \subset X^* J$, and hence $X^*$ is densely defined. Indeed, fix $h_1, h_2 \in \mathcal{D}(X)$, and choose an outer function $v \in H^\infty$ such that the functions

$$a_1 = v h_1, \qquad a_2 = v h_2, \qquad b_1 = v X h_1, \qquad b_2 = v X h_2$$

are bounded. Set $V = v(S(m))$, $A_j = a_j(S(m))$, $B_j = b_j(S(m))$ so that $Vh_j = A_j\xi$ and $VXh_j = B_j\xi$ for $j = 1, 2$. Observe first that

$$A_1B_2\xi = A_1VXh_2 = XA_1Vh_2 = XA_1A_2\xi,$$

and a similar calculation shows that $A_1B_2\xi = A_2B_1\xi$. We conclude that $A_1B_2 = A_2B_1$ because $\xi$ is separating. Next we use Lemma 3.6 to find operators $W_n \in \mathcal{W}_{S(m)}$ such that $W_nV$ converges to $I$ in the strong operator topology. We have then

$$\begin{aligned}
\langle Xh_1, Jh_2\rangle &= \lim_{n\to\infty} \langle W_nVXh_1, JW_nVh_2\rangle \\
&= \lim_{n\to\infty} \langle W_nB_1\xi, JW_nA_2\xi\rangle \\
&= \lim_{n\to\infty} \bigl\langle W_n\xi, B_1^*JW_nA_2\xi\bigr\rangle \\
&= \lim_{n\to\infty} \langle W_n\xi, JW_nB_1A_2\xi\rangle \\
&= \lim_{n\to\infty} \langle W_n\xi, JW_nA_1B_2\xi\rangle \\
&= \lim_{n\to\infty} \bigl\langle W_n\xi, A_1^*JW_nB_2\xi\bigr\rangle \\
&= \lim_{n\to\infty} \langle W_nA_1\xi, JW_nB_2\xi\rangle = \langle h_1, JXh_2\rangle,
\end{aligned}$$

where we used the identity $A_1B_2 = A_2B_1$ and the fact that $W_n$ commutes with $A_j$ and $B_j$. The theorem is proved. □

Note incidentally that the example of $\mathcal{W}_S$ shows that closability of an algebra $\mathcal{A}$ is not generally inherited by the algebra $\{T^*\colon\ T \in \mathcal{A}\}$.

We conclude this section with a simple fact which will be used in the study of closability for quasisimilar algebras. Let $\mathcal{A}_i \subset \mathcal{L}(\mathcal{H}_i)$, $i \in I$, be algebras. The algebra $\bigoplus_{i\in I} \mathcal{A}_i \subset \mathcal{L}(\bigoplus_{i\in I} \mathcal{H}_i)$ consists of those operators of the form $\bigoplus_{i\in I} T_i$, where $T_i \in \mathcal{A}_i$ for each $i$, and $\sup\{\|T_i\|\colon i \in I\} < \infty$.

**Lemma 3.8.** *A direct sum $\mathcal{A} = \bigoplus_{i\in I} \mathcal{A}_i$ has the closability property if and only if $\mathcal{A}_i$ has this property for every $i \in I$.*

**Proof.** Assume first that $\mathcal{A}$ has the closability property, and $X_{i_0}$ is a densely defined linear transformation on $\mathcal{H}_{i_0}$ commuting with $\mathcal{A}_{i_0}$ for some $i_0 \in I$. We define a linear transformation $X$ with dense domain $\mathcal{D}(X) = \bigoplus_{i\in I} \mathcal{D}_i$, where $\mathcal{D}_{i_0} = \mathcal{D}(X_{i_0})$, $\mathcal{D}_i = \mathcal{H}_i$ for $i \neq i_0$, and $X(\bigoplus h_i) = \bigoplus k_i$, where $k_{i_0} = X_{i_0}h_{i_0}$ and $k_i = 0$ for $i \neq i_0$. The linear transformation $X$ commutes with $\mathcal{A}$, hence it is closable. It follows that $X_{i_0}$ must be closable as well. Conversely, assume that each $\mathcal{A}_i$ has the closability property, and let $X$ be a densely defined linear transformation commuting with $\mathcal{A}$. If $P_j \in \mathcal{A}$ denotes the orthogonal projection onto the $j$th component of $\bigoplus_{i\in I} \mathcal{H}_i$, we have then $P_jX \subset XP_j$, and the linear transformation $X_j : \mathcal{D}_j = P_j\mathcal{D}(X) \to \mathcal{H}_j$ defined by $X_j = X \mid \mathcal{D}_j$ commutes with $\mathcal{A}_j$. It follows that each $X_j$ is closable, and then it is easy to verify that $X$ is closable as well. □

## 4. Rationally strictly cyclic vectors and confluence

The examples in Proposition 3.7, as well as maximal abelian selfadjoint subalgebras (also known as *masa*s), can actually be treated in a unified manner. For this purpose we need a new concept.

**Definition 4.1.** Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a unital algebra. A vector $h_0 \in \mathcal{H}$ is called a *rationally strictly cyclic* vector for $\mathcal{A}$ if for every $h \in \mathcal{H}$ there exist $A, B \in \mathcal{A}$ such that $Bh = Ah_0$ and $\ker B = \{0\}$.

Recall that $h_0$ is said to be *strictly cyclic* for $\mathcal{A}$ if $\mathcal{A}h_0 = \mathcal{H}$. Thus, a strictly cyclic vector is rationally strictly cyclic, but not conversely. The examples considered in this paper do not have strictly cyclic vectors except in trivial cases.

**Lemma 4.2.** *The following algebras have rationally strictly cyclic vectors*:

(1) $\mathcal{W}_S$.
(2) $\mathcal{W}_{S(m)}$.
(3) *Any masa on a separable Hilbert space. More generally, any masa with a cyclic vector.*

**Proof.** The vector $1 \in H^2$ is rationally strictly cyclic for $\mathcal{W}_S$, while $1 - \overline{m(0)}m = P_{\mathcal{H}(m)}1$ is rationally strictly cyclic for $\mathcal{W}_{S(m)}$. For (3), we may assume that $\mathcal{H} = L^2(\mu)$, where $\mu$ is a Borel probability measure on some compact topological space, and $\mathcal{A} = \{M_u\colon u \in L^\infty(\mu)\}$, where

$$M_u f = uf, \quad u \in L^\infty(\mu),\ f \in L^2(\mu).$$

Since every function in $L^2(\mu)$ is the quotient of two bounded functions, the constant function 1 is rationally strictly cyclic for $\mathcal{A}$. □

Here are two useful properties of algebras with rationally strictly cyclic vectors.

**Lemma 4.3.** *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a unital algebra with a rationally strictly cyclic vector $h_0$.*

(1) *If $T \in \mathcal{A}' \setminus \{0\}$ then $Th_0 \neq 0$.*
(2) *If $\mathcal{A}$ is commutative and $\mathcal{D} \subset \mathcal{H}$ is a dense linear manifold, invariant for $\mathcal{A}$, then*

$$\bigcap \{\ker T\colon T \in \mathcal{A},\ Th_0 \in \mathcal{D}\} = \{0\}.$$

**Proof.** Assume that $T \in \mathcal{A}'$ and $Th_0 = 0$. Given $x \in \mathcal{H}$, choose $A_x, B_x \in \mathcal{A}$ such that $B_x x = A_x h_0$ and $\ker B_x = \{0\}$. We have then

$$B_x Tx = TB_x x = TA_x h_0 = A_x Th_0 = 0,$$

and therefore $Tx = 0$. This implies that $T = 0$ since $x$ is arbitrary.

Assume now that $\mathcal{A}$ is commutative and $\mathcal{D} \subset \mathcal{H}$ is a dense linear manifold, invariant for $\mathcal{A}$. Let $h \in \mathcal{H}$ be a vector such that $Ah = 0$ whenever $A \in \mathcal{A}$ and $Ah_0 \in \mathcal{D}$. Using the notation above, we have $A_x h_0 = B_x x \in \mathcal{D}$ whenever $x \in \mathcal{D}$, and therefore $A_x h = 0$ for $x \in \mathcal{D}$. Thus

$$\begin{aligned} 0 &= B_h A_x h = A_x B_h h = A_x A_h h_0 = A_h A_x h_0 \\ &= A_h B_x x = B_x A_h x \end{aligned}$$

for $x \in \mathcal{D}$, which implies $A_h x = 0$ for such vectors $x$. From the density of $\mathcal{D}$ we deduce that $A_h = 0$, and thus $B_h h = A_h h_0 = 0$ and $h = 0$, as desired. □

We can now prove a generalization of Proposition 3.7.

**Theorem 4.4.** *Any unital commutative algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ which has a rationally strictly cyclic vector has the closability property.*

**Proof.** Let $h_0 \in \mathcal{H}$ be a rationally strictly cyclic vector for the algebra $\mathcal{A}$, and let $X$ be a linear transformation with dense domain $\mathcal{D}(X)$ commuting with $\mathcal{A}$. Consider a sequence $\{x_n\} \subset \mathcal{D}(X)$ such that $x_n \to 0$ and $Xx_n \to h$ as $n \to \infty$. By Lemma 4.3(2), it will suffice to show that $Th = 0$ whenever $T \in \mathcal{A}$ and $Th_0 \in \mathcal{D}(X)$. Fix such an operator $T$, and choose operators $A_n$, $B_n$, $B, A \in \mathcal{A}$ satisfying $B_n x_n = A_n h_0$, $BXTh_0 = Ah_0$ and $\ker B_n = \ker B = \{0\}$ for all $n \geqslant 1$. Using the commutativity of $\mathcal{A}$, and the fact that $X$ commutes with $\mathcal{A}$ we deduce that

$$\begin{aligned} B_n(BTXx_n - Ax_n) &= BTXB_n x_n - AB_n x_n \\ &= BTXA_n h_0 - AA_n h_0 \\ &= BXA_n Th_0 - AA_n h_0 \\ &= A_n(BXTh_0 - Ah_0) = 0. \end{aligned}$$

Since $B_n$ is one-to-one, we have

$$BTXx_n = Ax_n.$$

Letting $n \to \infty$ we obtain $BTh = 0$ and hence $Th = 0$, as desired. □

Observe that if an algebra $\mathcal{B} \subset \mathcal{L}(\mathcal{H})$ contains a unital commutative algebra $\mathcal{A}$ with the closability property, then $\mathcal{B}$ also has the closability property. Therefore Theorem 4.4 and the result of Arveson mentioned in the introduction have the following consequence.

**Corollary 4.5.** *There exists no proper subalgebra of $\mathcal{L}(\mathcal{H})$ which is 2-transitive, closed in the strong operator topology and contains a unital commutative subalgebra with a rationally strictly cyclic vector.*

The calculations in the preceding proof can be used to relate closed, densely defined linear transformations commuting with $\mathcal{A}$ with linear transformations of the form $B^{-1}A$ with $A, B \in \mathcal{A}$ and $\ker B = \{0\}$. Note that

$$\mathcal{G}\big(B^{-1}A\big) = \{h \oplus k \in \mathcal{H} \oplus \mathcal{H}\colon\ Ah = Bk\},$$

and this is generally larger than

$$\mathcal{G}\big(AB^{-1}\big) = \{Bh \oplus Ah\colon\ h \in \mathcal{H}\}.$$

Also observe that two linear transformations of this form, say $B^{-1}A$, $B_1^{-1}A_1$, which agree on a dense linear manifold $\mathcal{D}$, must in fact be equal. Indeed, the equality on $\mathcal{D}$ implies that $BA_1h = B_1Ah$ for $h \in \mathcal{D}$, and therefore $B_1A = BA_1$. Thus for $h \oplus k \in \mathcal{G}(B^{-1}A)$ we have

$$B(B_1k - A_1h) = B_1(Bk - Ah) = 0,$$

and hence $h \oplus k \in \mathcal{G}(B_1^{-1}A_1)$ because $B$ is injective.

**Proposition 4.6.** *Let $\mathcal{A}$ be a commutative algebra with a rationally strictly cyclic vector $h_0$. For every densely defined linear transformation $X$ commuting with $\mathcal{A}$ such that $h_0 \in \mathcal{D}(X)$, there exist $A, B \in \mathcal{A}$ such that* $\ker B = \{0\}$ *and $X \subset B^{-1}A$. If $X \in \mathcal{L}(\mathcal{H})$, we have $X = B^{-1}A$. In particular, the commutant $\mathcal{A}'$ is a commutative algebra.*

**Proof.** As in the preceding proof, we choose for each $h \in \mathcal{H}$ operators $A_h, B_h \in \mathcal{A}$ such that $\ker B_h = \{0\}$ and $B_hh = A_hh_0$. Assume now that $h_0 \in \mathcal{D}(X)$ and $X$ commutes with $\mathcal{A}$. We have then for $h \in \mathcal{D}(X)$,

$$\begin{aligned} B_hB_{Xh_0}Xh &= B_{Xh_0}XB_hh = B_{Xh_0}XA_hh_0 \\ &= A_hB_{Xh_0}Xh_0 = A_hA_{Xh_0}h_0 \\ &= A_{Xh_0}B_hh = B_hA_{Xh_0}h, \end{aligned}$$

from which we conclude that $X \subset B_{Xh_0}^{-1}A_{Xh_0}$ because $B_h$ is injective. The remaining assertions follow easily from this. □

Sometimes an algebra with a rationally strictly cyclic vector has the stronger property defined below.

**Definition 4.7.** Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a unital algebra. We will say that $\mathcal{A}$ is *confluent* if for every two vectors $h_1, h_2 \in \mathcal{H} \setminus \{0\}$ there exist injective operators $B_1, B_2 \in \mathcal{A}$ such that $B_1h_1 = B_2h_2$.

**Proposition 4.8.** *For a commutative unital algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$, the following two assertions are equivalent*:

(1) *$\mathcal{A}$ has a rationally strictly cyclic vector and* $\ker B = \{0\}$ *for every $B \in \mathcal{A} \setminus \{0\}$*;
(2) *$\mathcal{A}$ is confluent.*

*If these equivalent conditions are satisfied, then every nonzero vector is rationally strictly cyclic for $\mathcal{A}$*; *moreover, every densely defined linear transformation commuting with $\mathcal{A}$ is contained in $B^{-1}A$ for some $A, B \in \mathcal{A}$ such that* $\ker B = \{0\}$.

**Proof.** Assume first that (1) holds, and $h_1, h_2 \in \mathcal{H} \setminus \{0\}$. With the notation used earlier, we have

$$A_{h_2}B_{h_1}h_1 = A_{h_2}A_{h_1}h_0 = A_{h_1}B_{h_2}h_2.$$

The operators $A_{h_1}, A_{h_2}$ are not zero, and therefore $A_{h_2}B_{h_1}, A_{h_1}B_{h_2}$ are injective by hypothesis.

Conversely, assume that $\mathcal{A}$ is confluent. Clearly, every nonzero vector is then rationally strictly cyclic. It remains to show that every $B \in \mathcal{A} \setminus \{0\}$ is injective. Assume to the contrary that $Bh_1 = 0$ for some $h_1 \neq 0$, and choose $h_2 \notin \ker B$. If $B_1, B_2$ are as in Definition 4.7, we obtain

$$0 = B_1 B h_1 = B B_1 h_1 = B B_2 h_2 = B_2 B h_2.$$

This implies $Bh_2 = 0$, contrary to the choice of $h_2$. The last assertion follows from Proposition 4.6. $\square$

As an application of the results in this section, we show that some other algebras of Toeplitz operators have the closability property. Consider a bounded, connected open set $\Omega \subset \mathbb{C}$ bounded by $n+1$ analytic simple Jordan curves, and fix a point $\omega_0 \in \Omega$. The algebra $H^\infty(\Omega)$ consists of the bounded analytic functions on $\Omega$, while $H^2_{\omega_0}(\Omega)$ is defined as the space of analytic functions $f$ on $\Omega$ with the property that $|f|^2$ has a harmonic majorant in $\Omega$. The norm on $H^2_{\omega_0}(\Omega)$ is defined as

$$\|f\|_2^2 = \inf\bigl\{u(\omega_0)\colon u \text{ a harmonic majorant of } |f|^2\bigr\}, \quad f \in H^2_{\omega_0}(\Omega).$$

Multiplication by a function $u \in H^\infty(\Omega)$ determines a bounded operator $T_u$ on $H^2_{\omega_0}(\Omega)$.

**Proposition 4.9.** *The constant function $1 \in H^2_{\omega_0}(\Omega)$ is a rationally strictly cyclic vector for the algebra $\{T_u\colon u \in H^\infty(\Omega)\}$. In particular, this algebra has the closability property.*

The statement is equivalent to the following result. We refer to [1] and [13] for the function theoretical background.

**Lemma 4.10.** *For every function $f \in H^2_{\omega_0}(\Omega)$ there exist $u, v \in H^\infty(\Omega)$ such that $v \not\equiv 0$ and $vf = u$.*

**Proof.** Denote by $\pi : \mathbb{D} \to \Omega$ a (universal) covering map such that $\pi(0) = \omega_0$, and denote by $\Gamma$ the corresponding group of deck transformations. Thus, $\Gamma$ consists of those analytic automorphisms $\varphi$ of $\mathbb{D}$ with the property that $\pi \circ \varphi = \pi$. The map $f \mapsto f \circ \pi$ is an isometry from $H^2_{\omega_0}(\Omega)$ onto the space of those functions $g \in H^2$ such that $g \circ \varphi = g$ for every $\varphi \in \Gamma$.

Fix now $f \in H^2_{\omega_0}(\Omega) \setminus \{0\}$, and construct an outer function $w \in H^2$ such that $|w(\zeta)| = \min\{1, 1/|f \circ \pi(\zeta)|\}$ for almost every $\zeta \in \mathbb{T}$. The function $w$ is obviously modulus automorphic in the sense that $|w \circ \varphi| = |w|$ for every $\varphi \in \Gamma$. It follows that there is a group homomorphism $\gamma : \Gamma \to \mathbb{T}$ such that $w \circ \varphi = \gamma(\varphi) w$ for every $\varphi \in \Gamma$. Choose a modulus automorphic Blaschke product $b \in H^\infty$ such that $b \circ \varphi = \overline{\gamma(\varphi)} b$ for $\gamma \in \Gamma$; see [13, Theorem 5.6.1] for the construction of such products. Then there exist functions $u, v \in H^\infty(\Omega)$ such that $v \circ \pi = bw$ and $u \circ \pi = bw(f \circ \pi)$. These functions satisfy the requirements of the lemma. $\square$

## 5. Quasisimilar algebras

We will now study the effect of quasisimilarity on the closability property and the existence of rationally strictly cyclic vectors. Recall that an operator $Q \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is called a *quasiaffinity* if it is injective and has dense range.

**Definition 5.1.** An algebra $\mathcal{A}_1 \subset \mathcal{L}(\mathcal{H}_1)$ is a *quasiaffine transform* of an algebra $\mathcal{A}_2 \subset \mathcal{L}(\mathcal{H}_2)$ if there exists a quasiaffinity $Q \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that for every $T_2 \in \mathcal{A}_2$, we have $QT_1 = T_2 Q$ for some $T_1 \in \mathcal{A}_1$. We write $\mathcal{A}_1 \prec \mathcal{A}_2$ if $\mathcal{A}_1$ is a quasiaffine transform of $\mathcal{A}_2$.

The relation $\mathcal{A}_1 \prec \mathcal{A}_2$ can simply be written as $Q^{-1}\mathcal{A}_2 Q \subset \mathcal{A}_1$ for some quasiaffinity $Q$.

**Proposition 5.2.** *Assume that $\mathcal{A}_1 \subset \mathcal{L}(\mathcal{H}_1)$ and $\mathcal{A}_2 \subset \mathcal{L}(\mathcal{H}_2)$ are unital algebras such that $\mathcal{A}_1 \prec \mathcal{A}_2$. Then*

(1) *If $\mathcal{A}_1$ is commutative, then $\mathcal{A}_2$ is commutative as well.*
(2) *If $\mathcal{A}_2$ has the closability property, then so does $\mathcal{A}_1$.*
(3) *If $\mathcal{A}_2$ is confluent, then so is $\mathcal{A}_1$.*

**Proof.** Let $Q$ be as in Definition 5.1. Since the map $T \mapsto Q^{-1}TQ$ is obviously an injective algebra homomorphism on $\mathcal{A}_2$, part (1) is immediate.

To prove (2), let $X$ be a densely defined linear transformation commuting with $\mathcal{A}_1$. Define the linear transformation $Y = QXQ^{-1}$ on the dense subspace $\mathcal{D}(Y) = Q\mathcal{D}(X)$. Since all the operators $T_2 \in \mathcal{A}_2$ have the property that $Q^{-1}T_2 Q$ is in $\mathcal{A}_1$, it follows easily that $Y$ commutes with $\mathcal{A}_2$. Assume now that $\mathcal{A}_2$ has the closability property, so that $Y$ is closable. We will verify that $X$ is closable as well. Assume that $h_n \in \mathcal{D}(X)$ are such that $h_n \to 0$ and $Xh_n \to k$ as $n \to \infty$. Obviously then $\mathcal{D}(Y) \ni Qh_n \to 0$ and $YQh_n \to Qk$. We deduce that $Qk = 0$, and therefore $k = 0$ since $Q$ is a quasiaffinity.

Finally, assume that $\mathcal{A}_2$ is confluent and let $h_1, h_2 \in \mathcal{H}_1 \setminus \{0\}$. We choose injective $C_1, C_2 \in \mathcal{A}_2$ so that $C_1 Q h_1 = C_2 Q h_2$, and observe that $B_1 h_1 = B_2 h_2$, where $B_j = Q^{-1} C_j Q \in \mathcal{A}_1$ are injective. $\square$

**Definition 5.3.** An algebra $\mathcal{A}_1 \subset \mathcal{L}(\mathcal{H}_1)$ is *quasisimilar* to an algebra $\mathcal{A}_2 \subset \mathcal{L}(\mathcal{H}_2)$ if there exist quasiaffinities $Q \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $R \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ such that $Q^{-1}\mathcal{A}_2 Q \subset \mathcal{A}_1$, $R^{-1}\mathcal{A}_1 R \subset \mathcal{A}_2$, $QR \in \mathcal{A}_2'$, and $RQ \in \mathcal{A}_1'$. We write $\mathcal{A}_1 \sim \mathcal{A}_2$ if $\mathcal{A}_1$ is quasisimilar to $\mathcal{A}_2$.

Using the proofs of parts (1) and (2) of the following result, it is easy to see that quasisimilarity is an equivalence relation.

**Proposition 5.4.** *Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are commutative quasisimilar algebras, and $Q, R$ satisfy the conditions of Definition 5.3. Then*

(1) *We have $Q^{-1}\mathcal{A}_2 Q = \mathcal{A}_1$ and $R^{-1}\mathcal{A}_1 R = \mathcal{A}_2$.*
(2) *The maps $T_2 \mapsto Q^{-1}T_2 Q$ and $T_1 \mapsto R^{-1}T_1 R$ are mutually inverse algebra isomorphisms between $\mathcal{A}_1$ and $\mathcal{A}_2$.*
(3) *The commutant $\mathcal{A}_1'$ is commutative if and only if $\mathcal{A}_2'$ is commutative.*
(4) *If $h_1 \in \mathcal{H}_1$ is rationally strictly cyclic for $\mathcal{A}_1$ then $Qh_1$ is rationally strictly cyclic for $\mathcal{A}_2$.*
(5) *The algebra $\mathcal{A}_1$ is confluent if and only if $\mathcal{A}_2$ is confluent.*
(6) *The algebra $\mathcal{A}_1'$ is confluent if and only if $\mathcal{A}_2'$ is confluent.*
(7) *The algebra $\mathcal{A}_1'$ has the closability property if and only if $\mathcal{A}_2'$ does.*

**Proof.** Define $\Phi : \mathcal{A}_2 \to \mathcal{A}_1$ and $\Psi : \mathcal{A}_1 \to \mathcal{A}_2$ by setting $\Phi(T_2) = Q^{-1}T_2Q$ and $\Psi(T_1) = R^{-1}T_1R$. We have

$$\Psi\big(\Phi(T_2)\big) = R^{-1}Q^{-1}T_2QR = R^{-1}Q^{-1}QRT_2 = T_2, \quad T_2 \in \mathcal{A}_2,$$

and similarly $\Phi(\Psi(T_1)) = T_1$ for $T_1 \in \mathcal{A}_1$. This proves (2), and (1) follows from (2).

Assume now that $\mathcal{A}_1'$ is commutative and $A, B \in \mathcal{A}_2'$. We claim that $RAQ$ and $RBQ$ belong to $\mathcal{A}_1'$. Indeed,

$$\begin{aligned} T_1RAQ &= R\big(R^{-1}T_1R\big)AQ = RA\big(R^{-1}T_1R\big)Q \\ &= RAR^{-1}T_1(RQ) = RAR^{-1}(RQ)T_1 = RAQT_1 \end{aligned}$$

for $T_1 \in \mathcal{A}_1$. We deduce that $RAQRBQ = RBQRAQ$ and hence $AQRB = BQRA$. Taking $A$ or $B$ to be the identity operator, we deduce that $QR$ commutes with $B$ and $A$, and therefore $QRAB = QRBA$, and finally the desired equality $AB = BA$.

To prove (4), assume that $h_1$ is rationally strictly cyclic for $\mathcal{A}_1$. Proposition 4.6 implies the existence of $A_1, B_1 \in \mathcal{A}_1$ such that $\ker B_1 = \{0\}$ and $RQ = B_1^{-1}A_1$. Set $A_2 = R^{-1}A_1R$, $B_2 = R^{-1}B_1R \in \mathcal{A}_2$, and observe that

$$B_2QR = R^{-1}(B_1RQ)R = R^{-1}A_1R = A_2,$$

that is $QR = B_2^{-1}A_2$. Since $QR$ is a quasiaffinity, it follows that $\ker A_2 = \{0\}$. To show that $Qh_1$ is rationally strictly cyclic for $\mathcal{A}_2$, fix a vector $h_2 \in \mathcal{H}_2$, and choose $S_1, T_1 \in \mathcal{A}_1$ such that $\ker T_1 = \{0\}$ and $T_1Rh_2 = S_1h_1$. Set now $T_2 = R^{-1}T_1R$, $S_2 = R^{-1}S_1R \in \mathcal{A}_2$, and note that $\ker T_2 = \{0\}$. We have

$$RQRT_2h_2 = RQT_1Rh_2 = RQS_1h_1 = S_1RQh_1 = RS_2Qh_1,$$

so that $QRT_2h_2 = S_2Qh_1$. Applying $B_2$ to both sides we obtain $A_2T_2h_2 = B_2S_2Qh_1$, and strict cyclicity follows because $A_2T_2, B_2S_2 \in \mathcal{A}_2$ and $\ker(A_2T_2) = \{0\}$.

Assertion (5) follows easily from (4) and Proposition 4.8, or directly from Proposition 5.2(3).

Assume now that $\mathcal{A}_1'$ is confluent, and let $h, k \in \mathcal{H}_2$ be two nonzero vectors. Then there exist injective operators $A_1, B_1 \in \mathcal{A}_1'$ such that $A_1Rh = B_1Rk$. Thus we have $A_2h = B_2k$, where $A_2 = QA_1R$ and $B_2 = QB_1R$ are injective operators in $\mathcal{A}_2'$. This proves (6).

Finally, assume that $\mathcal{A}_2'$ has the closability property, and let $X$ be a densely defined linear transformation commuting with $\mathcal{A}_1'$. As in the proof of Proposition 5.2(2), to prove (7) it will suffice to show that the linear transformation $Y_0 = QXQ^{-1}$ defined on the dense space $\mathcal{D}(Y_0) = Q\mathcal{D}(X)$ is closable. To show this, we will define a linear transformation $Y \supset Y_0$ which commutes with $\mathcal{A}_2'$. Its domain $\mathcal{D}(Y)$ consists of all the finite sums of the form $\sum_n T_nQh_n$, where $T_n \in \mathcal{A}_2'$ and $h_n \in \mathcal{D}(X)$, and

$$Y\sum_n T_nQh_n = \sum_n T_nQXh_n.$$

To show that $Y$ is well defined, it will suffice to prove that $\sum_n T_n Q h_n = 0$ implies $R\sum_n T_n Q X h_n = 0$. Indeed, since $RT_n Q \in \mathcal{A}_1'$, we have $RT_n Q h_n \in \mathcal{D}(X)$ and

$$\sum_n RT_n Q X h_n = \sum_n X R T_n Q h_n = XR\sum_n T_n Q h_n = 0.$$

The fact that $Y$ commutes with every $T \in \mathcal{A}_2'$ is easily verified. If $\sum_n T_n Q h_n \in \mathcal{D}(Y)$ then clearly $\sum_n T T_n Q h_n \in \mathcal{D}(Y)$, and

$$YT\sum_n T_n Q h = \sum_n T T_n Q X h_n = TY\sum_n T_n Q h_n.$$

The inclusion $Y \supset Y_0$ is obvious since $\mathcal{A}_2'$ is unital. $\square$

We will be using the results in this section for the special case of algebras generated by a completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$. For such a contraction we will write

$$H^\infty(T) = \{u(T): u \in H^\infty\}.$$

Parts (1) and (2) of the following lemma are easily verified; in fact Definition 5.3 was formulated so as to make part (2) correct.

**Lemma 5.5.** *Let $T_1$ and $T_2$ be two completely nonunitary contractions.*

(1) *If $T_1 \prec T_2$ then $H^\infty(T_1) \prec H^\infty(T_2)$.*
(2) *If $T_1 \sim T_2$ then $H^\infty(T_1) \sim H^\infty(T_2)$.*
(3) *If $H^\infty(T_1) \sim H^\infty(T_2)$ and $T_1$ is of class $C_0$, then $T_2$ is also of class $C_0$.*
(4) *If $H^\infty(T_1) \sim H^\infty(T_2)$ and $T_1$ is not of class $C_0$, then $T_1 \sim \varphi(T_2)$ for some conformal automorphism $\varphi$ of $\mathbb{D}$.*

**Proof.** To prove (3), observe that $H^\infty(T_1) \sim H^\infty(T_2)$ implies that $H^\infty(T_2)$ is isomorphic to $H^\infty(T_1)$. Assume that $T_1$ is of class $C_0$. If $T_1$ is a scalar multiple of the identity, then $H^\infty(T_1) = \mathbb{C}I$, and therefore $H^\infty(T_2) = \mathbb{C}I$ and then $T_2$ must be a scalar multiple of the identity, hence of class $C_0$. If $T_1$ is not a scalar multiple of the identity, then $H^\infty(T_1)$ has zero divisors. Indeed, in this case the minimal function $m$ of $T_1$ can be factored into a product $m = m_1 m_2$ of two nonconstant inner functions, and then $m_1(T_1) \neq 0 \neq m_2(T_1)$, while $m_1(T_1)m_2(T_1) = 0$. We conclude that $H^\infty(T_2)$ must also have zero divisors, and this obviously implies that $T_2$ is of class $C_0$.

Finally, assume that $H^\infty(T_1) \sim H^\infty(T_2)$ and $T_1$ (as well as $T_2$ by part (3)) is not of class $C_0$. Let $Q$ and $R$ be quasiaffinities satisfying the conditions of Definition 5.3 for the algebras $\mathcal{A}_1 = H^\infty(T_1)$ and $\mathcal{A}_2 = H^\infty(T_2)$. The hypothesis implies that the maps $u \mapsto u(T_1)$ and $u \mapsto u(T_2)$ are algebra isomorphisms from $H^\infty$ to $H^\infty(T_1)$ and $H^\infty(T_2)$, respectively. Thus, for every $u \in H^\infty$ there exists a unique $v \in H^\infty$ satisfying $v(T_2) = R^{-1}u(T_1)R$. The map $\Phi : u \mapsto v$ is an algebra automorphism of $H^\infty$. In particular, the function $\varphi = \Phi(\mathrm{id}_{\mathbb{D}})$ must have spectrum (in $H^\infty$) equal to $\overline{\mathbb{D}}$, so that $\varphi(\mathbb{D}) = \mathbb{D}$. We claim that $\Phi(u) = u \circ \varphi$ for every $u \in H^\infty$. Indeed, given $\lambda \in \mathbb{D}$, we can factor $u(z) - u(\varphi(\lambda)) = (z - \varphi(\lambda))w$ for some $w \in H^\infty$, so that $\Phi(u) - u(\varphi(\lambda)) = (\varphi - \varphi(\lambda))\Phi(w)$. The equality $(\Phi(u))(\lambda) = u(\varphi(\lambda))$ follows immediately.

Since $\Phi$ is an automorphism, it follows that $\varphi$ is a conformal automorphism of $\mathbb{D}$, and clearly $T_1 \sim \varphi(T_2)$. $\square$

**Corollary 5.6.** *Let $T$ be a completely nonunitary contraction. If $T \sim S$ then $H^\infty(T)$ is confluent. If $T \sim S(m)$ then $H^\infty(T)$ has a rationally strictly cyclic vector.*

**Proof.** It suffices to observe that $H^\infty(S) = \mathcal{W}_S$, $H^\infty(S(m)) = \mathcal{W}_{S(m)}$, and to apply Proposition 5.4(2), (5) and (4). $\square$

For operators of class $C_0$, the converse of the preceding result is also true. The case of confluent algebras of the form $H^\infty(T)$ will be discussed more thoroughly in the remaining two sections of the paper.

**Proposition 5.7.** *Assume that $T$ is a completely nonunitary contraction such that $H^\infty(T)$ has a rationally strictly cyclic vector.*

(1) *If there exists $f \in H^\infty \setminus \{0\}$ such that $\ker f(T) \neq \{0\}$, then $T$ is of class $C_0$ and $T \sim S(m)$, where $m$ is the minimal function of $T$.*
(2) *If $\ker f(T) = \{0\}$ for every $f \in H^\infty \setminus \{0\}$, then $H^\infty(T)$ is confluent.*

**Proof.** Part (2) follows immediately from Proposition 4.8. To verify (1), assume that $f \in H^\infty \setminus \{0\}$, $\ker f(T) \neq \{0\}$, and $H^\infty(T)$ has a rationally strictly cyclic vector $h_0 \in \mathcal{H}$. Choose a nonzero vector $h_1 \in \ker f(T)$, and functions $u_1, v_1 \in H^\infty$ such that $v_1(T)$ is injective and $v_1(T)h_1 = u_1(T)h_0$. The function $u_1$ is not zero since $v_1(T)h_1 \neq 0$. We claim that $f(T)u_1(T) = 0$. Indeed, let $h$ be an arbitrary vector in $\mathcal{H}$. Choose $u, v \in H^\infty$ such that $v(T)$ is injective and $v(T)h = u(T)h_0$. We have then

$$\begin{aligned} v(T)\big[f(T)u_1(T)h\big] &= f(T)u_1(T)\big[v(T)h\big] = f(T)u_1(T)u(T)h_0 \\ &= f(T)u(T)u_1(T)h_0 = f(T)u(T)v_1(T)h_1 = 0, \end{aligned}$$

and therefore $f(T)u_1(T)h = 0$. Thus $T$ is of class $C_0$ because $(fu_1)(T) = 0$ and $fu_1 \in H^\infty \setminus \{0\}$.

Finally, let $m$ be the minimal function of $T$, denote by $\mathcal{M}$ the cyclic space for $T$ generated by $h_0$, and set $\mathcal{N} = \mathcal{M}^\perp$. Let $T' = P_{\mathcal{N}}T \mid \mathcal{N}$ be the compression of $T$ to $\mathcal{N}$. Clearly we have $m(T') = 0$. Now let $h \in \mathcal{N}$ be a vector, and pick $u, v \in H^\infty$ such that $v(T)$ is injective and $v(T)h = u(T)h_0$. In particular, we have $v(T')h = 0$. The injectivity of $v(T)$ is equivalent to the condition $v \wedge m = 1$, and this implies that $v(T')$ is injective as well, so that $h = 0$ We proved therefore that $\mathcal{M} = \mathcal{H}$. In other words, $T$ has a cyclic vector, and thus $T \sim S(m)$ by the results of [25] (see also [4, Theorem III.2.3]). $\square$

We conclude this section with a result about arbitrary operators of class $C_0$.

**Proposition 5.8.** *For any operator $T$ of class $C_0$, the commutant $\{T\}'$ has the closability property.*

**Proof.** The operator $T$ is quasisimilar to an operator of the form $T' = \bigoplus_{i\in I} S(m_i)$, where each $m_i$ is an inner function; see [4, Theorem III.5.1]. By Proposition 5.4(7), it suffices to show that

$\{T'\}'$ has the closability property. Now, $\{T'\}' \supset \bigoplus_{i\in I}\{S(m_i)\}'$, and Lemma 3.8 shows that it suffices to show that $\{S(m)\}'$ has the closability property for each inner function $m$. This follows from Proposition 3.7 because $\{S(m)\}' = \mathcal{W}_{S(m)}$. $\square$

## 6. Confluent algebras of the form $H^\infty(T)$

Consider a completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$ such that $H^\infty(T)$ has a rationally strictly cyclic vector. According to Proposition 5.7, we have $T \sim S(m)$ if some nonzero operator in $H^\infty(T)$ has nonzero kernel. Therefore we will restrict ourselves now to operators $T$ such that $f(T)$ is injective for every nonzero element of $H^\infty$. In other words, we will assume that $H^\infty(T)$ is a confluent algebra (cf. Proposition 4.8) and $\dim\mathcal{H} > 1$. In this case, the space $\mathcal{H}$ can be identified with a space of meromorphic functions. Let us denote by $N$ the Nevanlinna class consisting of those meromorphic functions in $\mathbb{D}$ which can be written as $u/v$, with $u, v \in H^\infty$.

**Lemma 6.1.** *Assume that $T$ is a completely nonunitary contraction such that $H^\infty(T)$ is confluent. Let $h, h_0$ be two vectors such that $h_0 \neq 0$, and choose $u, v \in H^\infty$, $v \neq 0$, such that $v(T)h = u(T)h_0$. Then the function $u/v \in N$ is uniquely determined by $h$ and $h_0$. We have $u/v = 0$ if and only if $h = 0$.*

**Proof.** Choose another pair of functions $u_1, v_1 \in H^\infty$, $v_1 \neq 0$, satisfying $v_1(T)h = u_1(T)h_0$. We have

$$\big(v_1(T)u(T) - v(T)u_1(T)\big)h_0 = \big(v_1(T)v(T) - v(T)v_1(T)\big)h = 0,$$

and therefore $h_0 \in \ker(v_1u - vu_1)(T)$. The hypothesis implies that $v_1u = vu_1$ and hence $u/v = u_1/v_1$. $\square$

The function $u/v$ will be denoted $h/h_0$. It is clear that the map $h \mapsto h/h_0$ is an injective linear map from $\mathcal{H}$ to $N$, and $u(T)h/u(T)h_0 = h/h_0$ if $u \in H^\infty \setminus \{0\}$. We also have

$$\frac{h}{h_0} = \frac{h}{h_1}\cdot\frac{h_1}{h_0}$$

provided that $h_0, h_1 \in \mathcal{H}\setminus\{0\}$. Now let $h, h_0 \in \mathcal{H}\setminus\{0\}$. There exists a unique integer $n$ such that the nonzero function $h/h_0$ can be written as

$$\frac{h}{h_0}(z) = z^n\frac{u(z)}{v(z)}$$

with $u, v \in H^\infty$ and $u(0) \neq 0 \neq v(0)$. The number $n$ will be denoted $\mathrm{ord}_0(h/h_0)$. It will be convenient to write $\mathrm{ord}_0(h/h_0) = \infty$ if $h = 0$.

**Lemma 6.2.** *Let $T$ be a completely nonunitary contraction such that $H^\infty(T)$ is confluent. Then $0 \geqslant \inf\{\mathrm{ord}_0(h/h_0)\colon h \in \mathcal{H}\} > -\infty$ for every $h_0 \in \mathcal{H}\setminus\{0\}$.*

**Proof.** Clearly $\mathrm{ord}_0(h_0/h_0) = 0$. For each integer $n$, the set

$$\mathcal{H}_n = \big\{h \in \mathcal{H}\colon \mathrm{ord}_0(h/h_0) \geqslant -n\big\}$$

is a linear manifold, and $\bigcup_{n\geqslant 0}\mathcal{H}_n=\mathcal{H}$. Given integers $m,k$ such that $k\geqslant 1$, we denote by $\mathcal{D}_{m,k}$ the set of all vectors $h\in\mathcal{H}$ for which $h/h_0$ can be written as

$$\frac{h}{h_0}(z)=z^{-m}\frac{u(z)}{v(z)}$$

with $\|u\|_\infty,\|v\|_\infty\leqslant k$ and $|u(0)|,|v(0)|\geqslant 1$. Observe that

$$\bigcup_{m\leqslant n,k\geqslant 1}\mathcal{D}_{m,k}=\mathcal{H}_n\setminus\{0\}.$$

The proposition will follow if we can show that one of the sets $\mathcal{D}_{m,k}$ has an interior point, and this will follow from the Baire category theorem once we prove that each $\mathcal{D}_{m,k}$ is closed. Assume indeed that $(h_i)_{i=0}^\infty\subset\mathcal{D}_{m,k}$ is a sequence such that $h_i\to h$ as $i\to\infty$. For each $i$ write

$$\frac{h_i}{h_0}(z)=z^{-m}\frac{u_i}{v_i}$$

with $\|u_i\|_\infty,\|v_i\|_\infty\leqslant k$ and $|u_i(0)|,|v_i(0)|\geqslant 1$. By the Vitali–Montel theorem we can assume, after dropping to a subsequence, that there exist functions $u,v\in H^\infty$ such that $u_i(z)\to u(z)$ and $v_i(z)\to v(z)$ uniformly for $z$ in each compact subset of $\mathbb{D}$. Clearly $\|u\|_\infty,\|v\|_\infty\leqslant k$ and $|u(0)|,|v(0)|\geqslant 1$. Moreover, we have $u_i(T)h_0\to u(T)h_0$ and $v_i(T)h_i\to v(T)h$ in the weak topology. (For the second sequence we need to write

$$v_i(T)h_i-v(T)h=v_i(T)(h_i-h)+\big(v_i(T)-v(T)\big)h,$$

and use the fact that the first term tends to zero in norm, while the second tends to zero weakly by [23, Theorem III.2.1].) The identities $T^m v_i(T)h_i=u_i(T)h_0$ for $m\geqslant 0$ (resp., $v_i(T)h_i=T^{-m}u_i(T)h_0$ for $m<0$) therefore imply $T^m v(T)h=u(T)h_0$ (resp., $v(T)h=T^{-m}u(T)h_0$) so that $h/h_0=z^{-m}u/v$, and thus $h\in\mathcal{D}_{m,k}$, as desired. □

**Lemma 6.3.** *Let $T$ be a completely nonunitary contraction such that $H^\infty(T)$ is confluent. Then $T$ is injective and $T\mathcal{H}$ is a closed subspace of codimension* 1. *Thus $T$ is a Fredholm operator with index$(T)=-1$.*

**Proof.** The operator $T$ belongs to a confluent algebra, hence it is injective. Note next that

$$\mathrm{ord}_0(Th/h_0)=\mathrm{ord}_0(h/h_0)+1$$

and hence

$$\inf\big\{\mathrm{ord}_0(h/h_0)\colon\ h\in\mathcal{H}\big\}+1=\inf\big\{\mathrm{ord}_0(h/h_0)\colon\ h\in T\mathcal{H}\big\}.$$

Since these numbers are finite, we cannot have $T\mathcal{H}=\mathcal{H}$. To conclude the proof, it will suffice to show that $T\mathcal{H}$ has codimension one, since this implies that it is closed as well. Choose $h_0\in\mathcal{H}\setminus T\mathcal{H}$, and note that $\mathrm{ord}_0(h/h_0)\geqslant 0$ for every $h$. Indeed, $\mathrm{ord}_0(h/h_0)=-n<0$ implies an identity of the form

$$T^n v(T)h=u(T)h_0$$

with $u(0) \neq 0$. Factoring $u(z) - u(0) = zw(z)$, we obtain

$$h_0 = \frac{1}{u(0)} T\big(T^{n-1}v(T)h - w(T)h_0\big) \in T\mathcal{H},$$

a contradiction. Thus the function $h/h_0$ is analytic at 0, and we can therefore define a linear functional $\Phi : \mathcal{H} \to \mathbb{C}$ by setting $\Phi h = (h/h_0)(0)$. We will show that $\ker \Phi \subset T\mathcal{H}$. Indeed, $h \in \ker \Phi$ implies that $v(T)h = Tu(T)h_0$ for some $u, v \in H^\infty$ with $v(0) \neq 0$. Factoring again $v(z) - v(0) = zw(z)$, we obtain

$$h = \frac{1}{v(0)} T\big(u(T)h_0 - w(T)h\big) \in T\mathcal{H},$$

as claimed. Thus $T\mathcal{H}$ has codimension 1, and the lemma is proved. □

The preceding results allow us to describe completely the spectral picture of $T$, as well as its commutant. The argument for (3) already appears in [10], and is included for the reader's convenience.

**Theorem 6.4.** *Let $T \in \mathcal{L}(\mathcal{H})$ be a completely nonunitary contraction such that $H^\infty(T)$ is confluent. Then*

(1) *We have $\sigma(T) = \overline{\mathbb{D}}$ and $\sigma_{\mathrm{e}}(T) = \mathbb{T}$.*
(2) *For each $\lambda \in \mathbb{D}$, $\lambda I - T$ is injective and has closed range of codimension* 1.
(3) $\bigvee\{\ker(\lambda I - T^*) : \lambda \in \mathbb{D}\} = \mathcal{H}$. *More generally, $\bigvee\{\ker(\lambda I - T^*)\colon \lambda \in S\} = \mathcal{H}$ whenever the set $S \subset \mathbb{D}$ has an accumulation point in $\mathbb{D}$.*
(4) *For every nonzero invariant subspace $\mathcal{M}$ of $T$, there exists an inner function $m \in H^\infty$ such that $\overline{m(T)\mathcal{H}} = \mathcal{M}$ and the compression $T_{\mathcal{M}^\perp}$ of $T$ to $\mathcal{M}^\perp$ is quasisimilar to $S(m)$. Conversely, for every inner function $m$, the minimal function of $T_{(m(T)\mathcal{H})^\perp}$ is $m$.*
(5) $\{T\}' = H^\infty(T)$.
(6) *The operator $T$ is of class $C_{10}$. Thus, the powers $T^{*n}$ converge strongly to zero and* $\lim_{n\to\infty} \|T^n h\| \neq 0$ *for $h \in \mathcal{H} \setminus \{0\}$.*

*In particular, properties* (2) *and* (3) *say that $T^*$ belongs to the class $\mathcal{B}_1(\mathbb{D})$ defined in* [10].

**Proof.** For $\lambda \in \mathbb{D}$, the operator $T_\lambda = (I - \bar{\lambda}T)^{-1}(T - \lambda I)$ is also a completely nonunitary contraction, and $H^\infty(T_\lambda) = H^\infty(T)$ is confluent. Thus Lemma 6.3 implies immediately (2). In turn, (1) follows from (2) since $T$ is a contraction.

Next we prove (4). Let $\mathcal{M} \neq \{0\}$ be invariant for $T$, set $\mathcal{N} = \mathcal{M}^\perp$, and choose $h_0 \in \mathcal{M} \setminus \{0\}$. Denote by $T' = P_{\mathcal{N}}T \mid \mathcal{N}$ the compression of $T$ to $\mathcal{N}$. Given $h \in \mathcal{N}$, an equality of the form $v(T)h = u(T)h_0$ implies $v(T)h \in \mathcal{M}$, and therefore $v(T')h = 0$. The fact that $h_0$ is rationally strictly cyclic for $H^\infty(T')$ implies that $T'$ is locally of class $C_0$, and hence of class $C_0$ by [24] (see also [4, Theorem III.3.1]). Denote by $m$ the minimal function of $T'$. Note that in particular $P_{\mathcal{N}}m(T) \mid \mathcal{N} = m(T') = 0$, so that

$$m(T)\mathcal{N} \subset \mathcal{M}. \tag{6.1}$$

We show next that $T'$ has a cyclic vector, and hence it is quasisimilar to $S(m)$. Assume to the contrary that $T'$ does not have a cyclic vector, and let $\mathcal{N}_1, \mathcal{N}_2$ be cyclic spaces for $T'$ generated by two nonzero vectors $h_1, h_2$ such that $T' \mid \mathcal{N}_1 \sim S(m)$ and $\mathcal{N}_1 \cap \mathcal{N}_2 = \{0\}$ (see [25] or [4, Theorem III.2.13]). There exist nonzero functions $u_1, u_2 \in H^\infty$ such that $u_1(T)h_1 = u_2(T)h_2$. Dividing these functions by their greatest common inner divisor, we may assume that $u_1$ and $u_2$ do not have any (non constant) common inner factor. We also have $u_1(T')h_1 = u_2(T')h_2 \in \mathcal{N}_1 \cap \mathcal{N}_2$, and hence these vectors are equal to zero. We deduce that $m$ divides $u_1$, and hence $m \wedge u_2 = 1$. This last equality implies that $u_2(T')$ is a quasiaffinity, hence $u_2(T')h_2 \neq 0$, a contradiction. Thus $T'$ is indeed cyclic. Using (6.1), we observe that $m(T)\mathcal{H} = m(T)\mathcal{M} + m(T)\mathcal{N} \subset \mathcal{M}$. Denote now $\mathcal{M}_1 = \overline{m(T)\mathcal{H}}$, $\mathcal{N}_1 = \mathcal{M}_1^\perp$, and $T_1 = P_{\mathcal{N}_1} T \mid \mathcal{N}_1$. Clearly $m(T_1) = 0$, and $T'^* = T_1^* \mid \mathcal{N}$. It follows that the minimal function of $T_1$ is also $m$. Applying to $T_1$ the argument showing that $T'$ has a cyclic vector yields the same for $T_1$. Hence $\mathcal{M} = \mathcal{M}_1$ by the results of [25] (see also [4, Theorem III.2.13]).

We start next with a given inner function $m$, and denote by $m_1$ the minimal function of $T_{(m(T)\mathcal{H})^\perp}$. The function $m_1$ must divide $m$, so that $m = m_1 m_2$ for some other inner function $m_2$. With the notation $\mathcal{H}_1 = \overline{m_1(T)\mathcal{H}} = \overline{m(T)\mathcal{H}}$, $T_1 = T \mid \mathcal{H}_1$, the algebra $H^\infty(T_1)$ is confluent, and

$$\overline{m_2(T_1)\mathcal{H}_1} = \overline{m_2(T)m_1(T)\mathcal{H}} = \overline{m(T)\mathcal{H}} = \mathcal{H}_1,$$

so $m_2(T_1)$ has dense range. We claim that $\overline{m_2(T_1)\mathcal{M}} = \mathcal{M}$ for every invariant subspace $\mathcal{M}$ for $T_1$. Indeed, from the first part of (4) we know that $\mathcal{M} = \overline{m_3(T_1)\mathcal{H}_1}$ for some inner function $m_3$. Hence

$$\overline{m_2(T_1)\mathcal{M}} = \overline{m_2(T_1)m_3(T_1)\mathcal{H}_1} = \overline{m_3(T_1)\overline{m_2(T_1)\mathcal{H}_1}} = \overline{m_3(T_1)\mathcal{H}_1} = \mathcal{M},$$

as claimed. Since $H^\infty(T_1)$ is confluent, we have $\sigma(T_1) = \overline{\mathbb{D}}$ by part (1) of the theorem. This implies that $T_1$ belongs to the class $\mathbb{A}$ defined in [5]. By the results of [8], there exist vectors $x, y \in \mathcal{H}_1$ such that

$$\langle u(T_1)x, y \rangle = \frac{1}{2\pi} \int_0^{2\pi} \left(1 - m_2(0)\overline{m_2\left(e^{it}\right)}\right) u\left(e^{it}\right) dt$$

for all $u \in H^\infty$. In particular, $\langle v(T_1)m_2(T_1)x, y\rangle = 0$ for $v \in H^\infty$. Set $\mathcal{M} = \bigvee\{T_1^n x\colon n \geqslant 0\}$, and observe now that $y \perp m_2(T_1)\mathcal{M}$, and therefore $y \perp \mathcal{M}$ as well. In particular,

$$0 = \langle x, y \rangle = \frac{1}{2\pi} \int_0^{2\pi} \left(1 - m_2(0)\overline{m_2\left(e^{it}\right)}\right) dt = 1 - \left|m_2(0)\right|^2,$$

and this implies that $m_2$ is a constant function. We reach the desired conclusion that the minimal function of $T_{(m(T)\mathcal{H})^\perp}$ is $m$.

To prove (3), assume that $S \subset \mathbb{D}$ has an accumulation point in $\mathbb{D}$, and note that the space $\mathcal{N} = \bigvee\{\ker(\lambda I - T^*)\colon \lambda \in S\}$ is invariant for $T^*$. Therefore $\mathcal{M} = \mathcal{N}^\perp$ is invariant for $T$. If $\mathcal{M} \neq \{0\}$, we have then $m(T)\mathcal{H} \subset \mathcal{M}$ for some inner function $m$, and therefore $\ker m(T)^* \supset \mathcal{N}$. Given $\lambda \in S$, choose a nonzero vector $f_\lambda \in \ker(\lambda I - T^*)$, and observe that $0 = m(T)^* f_\lambda = \overline{m(\bar{\lambda})} f_\lambda$.

Thus $m(\bar{\lambda}) = 0$ for $\lambda \in S$, and we conclude that $m = 0$, which is impossible. This contradiction implies that $\mathcal{M} = \{0\}$, thus verifying (6.1).

Consider next an operator $X \in \{T\}' = H^\infty(T)'$. By Proposition 4.6, there exist $u, v \in H^\infty$, $v \neq 0$, so that $v(T)X = u(T)$. With $f_\lambda$ as above, we have

$$\overline{v(\bar{\lambda})} X^* f_\lambda = \big(v(T)X\big)^* f_\lambda = u(T)^* f_\lambda = \overline{u(\bar{\lambda})} f(\lambda),$$

and thus

$$\left| \frac{u(\bar{\lambda})}{v(\bar{\lambda})} \right| = \frac{\|X^* f_\lambda\|}{\|f_\lambda\|} \leqslant \|X^*\|.$$

We deduce that $w = u/v \in H^\infty$ and $X = w(T)$.

The fact that the powers of $T^*$ tend strongly to zero follows from (3) because $T^{*n} f_\lambda = \lambda^n f_\lambda \to 0$ as $n \to \infty$ for $\lambda \in \mathbb{D}$. It remains to prove that the space

$$\mathcal{M} = \Big\{ h \in \mathcal{H}\colon \lim_{n\to\infty} \|T^n h\| = 0 \Big\}$$

is equal to $\{0\}$. Assume to the contrary that $\mathcal{M} \neq \{0\}$, and observe that $H^\infty(T \mid \mathcal{M})$ is also confluent. In particular, $\sigma(T \mid \mathcal{M}) = \overline{\mathbb{D}}$ and $T \mid \mathcal{M}$ is of class $C_{00}$. According to [7] and [5, Theorem 6.6], $T \mid \mathcal{M}$ belongs to the class $\mathbb{A}_{\aleph_0}$, and, by [5, Corollary 5.5], $T$ has a further invariant subspace $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{N} \ominus \overline{T\mathcal{N}}$ has infinite dimension. This space must however have dimension 1 because $H^\infty(T \mid \mathcal{N})$ is confluent. This contradiction shows that we must have $\mathcal{M} = \{0\}$, as claimed. $\square$

Recall that $N_+ \subset N$ denotes the collection of functions of the form $u/v$, where $u, v \in H^\infty$ and $v$ is outer.

**Corollary 6.5.** *Let $T \in \mathcal{L}(\mathcal{H})$ be a completely nonunitary contraction such that $H^\infty(T)$ is confluent, and fix a vector $h_0 \in \ker T^*$, $h_0 \neq 0$. Assume that $\mathcal{H} = \bigvee\{T^n h_0\colon n \geqslant 0\}$; that is, $h_0$ is cyclic for $T$. Then $h/h_0 \in N_+$ for every $h \in \mathcal{H}$.*

**Proof.** We can assume that $h \neq 0$. Now choose functions $u, u_0 \in H^\infty \setminus \{0\}$ such that $u_0/u = h/h_0$. Thus we have $u_0(T)h_0 = u(T)h$. Consider the factorizations $u = mv$ and $u_0 = m_0 v_0$, where $m, m_0$ are inner and $v, v_0$ are outer. By [23, Proposition III.3.1], the operator $v_0(T)$ is a quasiaffinity, and therefore

$$\overline{m_0(T)\mathcal{H}} = \bigvee_{n\geqslant 0} T^n v_0(T) m_0(T) h_0 = \bigvee_{n\geqslant 0} T^n v(T) m(T) h \subset \overline{m(T)\mathcal{H}}.$$

It follows that $(m(T)\mathcal{H})^\perp \subset (m_0(T)\mathcal{H})^\perp$, and thus $m$ divides $m_0$ by Theorem 6.4(4). It follows that

$$\frac{h}{h_0} = \frac{u_0}{u} = \frac{v_0(m_0/m)}{v} \in N_+,$$

as claimed. $\square$

We will denote by $A$ the *disk algebra*. This consists of those functions in $H^\infty$ which are restrictions of continuous functions on $\overline{\mathbb{D}}$. If $T$ is a completely nonunitary contraction, we set $A(T) = \{u(T)\colon u \in A\}$.

**Corollary 6.6.** *Consider an operator $T \in \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional Hilbert space. Then*

(1) *The algebra $\mathcal{P}_T$ is not confluent.*
(2) *If $T$ is a completely nonunitary contraction, then $A(T)$ is not confluent.*

**Proof.** In proving (1), there is no loss of generality in assuming that $\|T\| < 1$ since $\mathcal{P}_T = \mathcal{P}_{\alpha T}$ for any $\alpha > 0$. Under this assumption, we have $\mathcal{P}_T \subset A(T)$, so it suffices to prove part (2). Assume therefore that $T$ is a completely nonunitary contraction and $A(T)$ is confluent. The larger algebra $H^\infty(T)$ is confluent as well, and Proposition 4.6 implies that for every $f \in H^\infty$, the operator $f(T) \in \{T\}' = A(T)'$ can be written as $f(T) = v(T)^{-1}u(T)$ with $u, v \in A$, $v \neq 0$. We have then $v(T)f(T) = u(T)$, and thus $f = u/v$. It is known, however, that there are functions in $H^\infty$ which cannot be represented as quotients of elements of $A$. An example is provided by any singular inner function

$$f(\lambda) = e^{-\int_{\mathbb{T}} \frac{\zeta+\lambda}{\zeta-\lambda}\, d\mu(\zeta)}, \quad \lambda \in \mathbb{D},$$

such that the closed support of the singular measure $\mu$ is the entire circle $\mathbb{T}$. $\square$

The assertion in Proposition 4.6, concerning unbounded linear transformations can be improved when $H^\infty(T)$ is confluent.

**Proposition 6.7.** *Let $T \in \mathcal{L}(\mathcal{H})$ be a completely nonunitary contraction such that $H^\infty(T)$ is confluent. Then every closed, densely defined linear transformation commuting with $T$ is of the form $v(T)^{-1}u(T)$, where $u, v \in H^\infty$ and $v$ is an outer function.*

**Proof.** Let $X$ be a closed, densely defined linear transformation commuting with $T$. Since $X$ is closed, it must also commute with every operator in $H^\infty(T)$. By Proposition 4.6, there exist $u, v \in H^\infty$ such that $v \not\equiv 0$ and $X \subset v(T)^{-1}u(T)$. Let us set

$$T_1 = (T \oplus T) \mid \mathcal{G}\big(v(T)^{-1}u(T)\big),$$

and observe that the quasiaffinity $Q : h \oplus k \mapsto h$ from $\mathcal{G}(v(T)^{-1}u(T))$ to $\mathcal{H}$ satisfies $QT_1 = TQ$. Thus $H^\infty(T_1) \prec H^\infty(T)$, and therefore $H^\infty(T_1)$ is confluent by Proposition 5.2(3). The subspace $\mathcal{G}(X)$ is invariant for $T_1$, so

$$\mathcal{G}(X) = \overline{m(T_1)\mathcal{G}\big(v(T)^{-1}u(T)\big)}$$

for some inner function $m$. To prove the equality $X = v(T)^{-1}u(T)$, it suffices to show that $m$ is in fact constant. Indeed, we have

$$\begin{aligned}\overline{m(T)\mathcal{H}} &= \overline{m(T)Q\mathcal{G}\big(v(T)^{-1}u(T)\big)} = \overline{Qm(T_1)\mathcal{G}\big(v(T)^{-1}u(T)\big)} \\ &= \overline{Q\mathcal{G}(X)} = \overline{\mathcal{D}(X)} = \mathcal{H},\end{aligned}$$

and the desired conclusion follows from the second assertion in Theorem 6.4(4). There is no loss of generality in assuming that $u$ and $v$ do not have any nonconstant common inner divisor. We conclude the proof by showing that in this case $v$ must be outer. Let $m$ be an inner divisor of $v$, and note that for every $h \oplus k \in \mathcal{G}(X)$ we have

$$u(T)h = v(T)k \in \overline{m(T)\mathcal{H}},$$

and therefore $u(T)\mathcal{D}(X) \subset \overline{m(T)\mathcal{H}}$. Since $\mathcal{D}(X)$ is dense in $\mathcal{H}$, we conclude that $u(T_{(m(T)\mathcal{H})^\perp}) = 0$, and therefore $m$ divides $u$. Thus $m$ is constant, and hence $v$ is outer. $\square$

It follows from the results of [10] that the one-dimensional spaces $\ker(\bar{\lambda} I - T)^*$ depend analytically on $\lambda$ and, in fact, there exists an analytic function $f : \mathbb{D} \to \mathcal{H}$ such that $\ker(\bar{\lambda} I - T)^* = \mathbb{C} f(\lambda)$ for $\lambda \in \mathbb{D}$. A local version of this result is easily proved. Indeed, set $L = (T^*T)^{-1}T^*$. Given a unit vector $f_0 \in \ker T^*$, the function

$$f(\lambda) = \left(I - \lambda L^*\right)^{-1} f_0 = \sum_{n=0}^{\infty} \lambda^n L^{*n} f_0 \tag{6.2}$$

is analytic for $|\lambda| < 1/\|L\|$, and obviously $T^* f(\lambda) = \lambda f(\lambda)$. This calculation is valid for any left inverse of $T$. The operator $L$ has the advantage that $L^*\mathcal{H} = T\mathcal{H}$, and therefore $\langle T^n f_0, f_0\rangle = \langle f_0, L^{*n} f_0\rangle = 0$ for $n \geqslant 1$. These relations, along with $LT = I$, obviously imply

$$\left\langle T^n f_0, L^{*m} f_0\right\rangle = \delta_{nm}, \quad n, m \geqslant 0. \tag{6.3}$$

**Proposition 6.8.** *Let $T \in \mathcal{L}(\mathcal{H})$ be a completely nonunitary contraction such that $H^\infty(T)$ is confluent. Define $L = (T^*T)^{-1}T^*$ and fix a unit vector $f_0 \in \ker T^*$. Then*

(1) *The vector $f_0$ is cyclic for $L^*$.*
(2) $\bigcap\{T^n\mathcal{H}\colon n \geqslant 0\} = \{0\}$.
(3) $\bigcap\{L^{*n}\mathcal{H}\colon n \geqslant 0\} = \mathcal{H} \ominus [\bigvee\{T^n f_0\colon n \geqslant 0\}]$.

**Proof.** We have seen that $\ker(\lambda I - T^*) = \mathbb{C} f(\lambda)$ for $\lambda$ close to zero, where $f(\lambda)$ is given by (6.2) and belongs to $\bigvee\{L^{*n} f_0\colon n \geqslant 0\}$. Thus (1) follows from Theorem 6.4(3). To prove (2), let $f$ be a nonzero element in the intersection, and set

$$k = \inf\left\{\operatorname{ord}_0(h/f_0)\colon h \in \mathcal{H}\right\}, \qquad m = \operatorname{ord}_0(f/f_0) < \infty.$$

For each integer $n$ we can write $f = T^n g$ for some $g \neq 0$, and therefore

$$m = n + \operatorname{ord}_0(g/f_0) \geqslant n + k.$$

This yields a contradiction for large $n$.

The orthogonality relations (6.3) imply the inclusion

$$\bigcap_{n\geqslant 0} L^{*n}\mathcal{H} \subset \mathcal{H} \ominus \bigvee_{n\geqslant 0} T^n f_0.$$

Conversely, consider a vector $h \in \mathcal{H} \ominus [\bigvee\{T^n f_0\colon n \geqslant 0\}]$. Given $n \geqslant 1$, we have

$$h = L^{*n}T^{*n}h + \sum_{k=0}^{n-1} L^{*k}\big(I - L^*T^*\big)T^{*k}h.$$

Since $I - L^*T^*$ is the orthogonal projection onto $\mathbb{C}f_0$, and

$$\big\langle T^{*k}h, f_0\big\rangle = \big\langle h, T^k f_0\big\rangle = 0,$$

we deduce that $h = L^{*n}T^{*n}h \in L^{*n}\mathcal{H}$, thus proving the opposite inclusion. □

## 7. Confluence and functional models

The results in Section 6 show that completely nonunitary contractions $T$ for which $H^\infty(T)$ is confluent share many of the properties of the unilateral shift $S$. In this section we will describe some quasiaffine transforms of such operators $T$. These quasiaffine transforms are in fact functional models associated with purely contractive inner functions of the form

$$\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix},$$

where $\theta_1, \theta_2 \in H^\infty$. The condition that $\Theta$ be inner amounts to the requirement that

$$\big|\theta_1(\zeta)\big|^2 + \big|\theta_2(\zeta)\big|^2 = 1, \quad \text{a.e. } \zeta \in \mathbb{T},$$

while pure contractivity means simply that

$$\big|\theta_1(0)\big|^2 + \big|\theta_2(0)\big|^2 < 1.$$

We recall the construction of the functional model associated with such a function $\Theta$. The subspace

$$\big\{\theta_1 u \oplus \theta_2 u\colon u \in H^2\big\} \subset H^2 \oplus H^2$$

is obviously invariant for $S \oplus S$, and thus the orthogonal complement

$$\mathcal{H}(\Theta) = \big[H^2 \oplus H^2\big] \ominus \big\{\theta_1 u \oplus \theta_2 u\colon u \in H^2\big\}$$

is invariant for $S^* \oplus S^*$. The operator $S(\Theta) \in \mathcal{L}(\mathcal{H}(\Theta))$ is the compression of $S \oplus S$ to this space or, equivalently, $S(\Theta)^* = (S^* \oplus S^*) \mid \mathcal{H}(\Theta)$.

Observe that $I - S(\Theta)^*S(\Theta)$ has rank one, while $I - S(\Theta)S(\Theta)^*$ has rank two. It follows that $\sigma(S(\Theta)) = \overline{\mathbb{D}}$, and in particular $S(\Theta)$ is not of class $C_0$.

**Lemma 7.1.** *Let $\Theta = \left[\begin{smallmatrix} \theta_1 \\ \theta_2 \end{smallmatrix}\right]$ be a purely contractive inner function. The algebra $H^\infty(S(\Theta))$ is confluent if and only if the functions $\theta_1$ and $\theta_2$ do not have a nonconstant common inner factor.*

**Proof.** If either of the functions $\theta_j$ is equal to zero, the other one must be inner. The lemma is easily verified in this case. Indeed, assume that $\theta_1$ is inner and $\theta_2 = 0$. If $\theta_1$ is not constant then $\ker \theta_1(S(\Theta)) \neq \{0\}$, so that $H^\infty(S(\Theta))$ is not confluent. Also, $\theta_1$ is a common inner divisor of $\theta_1$ and $\theta_2$, so that both conditions in the statement are false. On the other hand, if $\theta_1$ is constant then $\Theta$ is not pure, so this case does not arise.

For the remainder of this proof, we consider the case in which both functions $\theta_j$ are different from zero. Assume first that $\theta_j = m\varphi_j$, where $m$ is a nonconstant inner function and $\varphi_j \in H^\infty$ for $j = 1, 2$. The nonzero vector $h \in \mathcal{H}(\Theta)$ defined by $h = P_{\mathcal{H}(\Theta)}(\varphi_1 \oplus \varphi_2)$ satisfies $m(S(\Theta))h = 0$, and therefore $m(S(\Theta))$ has nontrivial kernel. Thus $H^\infty(S(\Theta))$ is not confluent.

Assume now that $\theta_1$ and $\theta_2$ do not have a nonconstant common inner factor. We verify first that $\ker u(S(\Theta)) = \{0\}$ for $u \in H^\infty \setminus \{0\}$. It suffices to consider the case of an inner function $u$. A vector $f_1 \oplus f_2 \in \ker u(S(\Theta))$ must satisfy $uf_1 = \theta_1 g$ and $uf_2 = \theta_2 g$ for some $g \in H^2$. The fact that $\theta_1 \wedge \theta_2 = 1$ implies that $u$ divides $g$, and therefore $f_1 \oplus f_2 = \theta_1(g/u) \oplus \theta_2(g/u)$ belongs to $\mathcal{H}(\Theta)^\perp$; the equality $f_1 \oplus f_2 = 0$ follows. To conclude the proof, we will show that $h = P_{\mathcal{H}(\Theta)}(1 \oplus 0)$ is a rationally strictly cyclic vector for $H^\infty(S(\Theta))$. Indeed, assume that $f = f_1 \oplus f_2 \in \mathcal{H}(\Theta) \setminus \{0\}$, and write $f_1 = a_1/b$ and $f_2 = a_2/b$, where $a_1, a_2, b \in H^\infty$ and $b$ is outer. Define functions $u = -b\theta_2$, $v = \theta_1 a_2 - \theta_2 a_1$, and note that

$$\begin{aligned} v\big(S(\Theta)\big)h - u\big(S(\Theta)\big)f &= P_{\mathcal{H}(\Theta)}(v \oplus 0 - uf_1 \oplus uf_2) \\ &= P_{\mathcal{H}(\Theta)}(\theta_1 a_2 \oplus \theta_2 a_2) = 0. \end{aligned}$$

The lemma follows because $u \not\equiv 0$, and hence $u(S(\Theta))$ is injective. □

Let us remark that the condition $\theta_1 \wedge \theta_2 = 1$ is equivalent to the fact that the function $\Theta$ is $*$-outer. In other words, the operators $S(\Theta)$ described in the preceding lemma are of class $C_{10}$. This is in agreement with Theorem 6.4(6).

**Proposition 7.2.** *Assume that $T$ is a completely nonunitary contraction such that $H^\infty(T)$ is confluent. Then*

(1) *Either $S \prec T$ or there exists a purely contractive inner function $\Theta = \left[\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right]$ such that $S(\Theta) \prec T$ and $H^\infty(S(\Theta))$ is confluent.*
(2) *We have $S \prec T$ if and only if $T$ has a cyclic vector.*

**Proof.** Denote by $U_+ \in \mathcal{L}(\mathcal{K}_+)$ the minimal isometric dilation of $T$. Thus $\mathcal{H} \subset \mathcal{K}_+$ and $TP_\mathcal{H} = P_\mathcal{H} U_+$. Since $T \in C_{10}$, the operator $U_+$ is a unilateral shift. Let us set $\mathcal{M} = \bigvee\{T^n h_1\colon n \geqslant 0\}$, where $h_1 \in \mathcal{H} \setminus \{0\}$, and let $h_2 \in \mathcal{H} \ominus \mathcal{M}$ be a cyclic vector for the compression of $T$ to this subspace. Such a vector exists by Theorem 6.4(4). Observe that $\mathcal{H} = \bigvee\{T^n h_1, T^n h_2\colon n \geqslant 0\}$. We define now a space

$$\mathcal{E} = \bigvee\big\{U_+^n h_1, U_+^n h_2\colon n \geqslant 0\big\}$$

and an operator $Y \in \mathcal{L}(\mathcal{E}, \mathcal{H})$ by setting $Y = P_\mathcal{H} \mid \mathcal{E}$. The space $\mathcal{E}$ is invariant for $U_+$, $Y(U_+ \mid \mathcal{E}) = TY$, and $Y$ has dense range. Moreover, the restriction $U_+ \mid \mathcal{E}$ is a unilateral shift of multiplicity 1 or 2. Finally, set $\mathcal{H}' = \mathcal{E} \ominus \ker Y$, $X = Y \mid \mathcal{H}'$, and denote by $T'$ the compression of $U_+ \mid \mathcal{E}$ to the space $\mathcal{H}'$. Then $X\mathcal{H}' = Y\mathcal{E}$ so that $X$ is a quasiaffinity, and $XT' = TX$. Thus we have $T' \prec T$ and hence $H^\infty(T')$ is confluent by Proposition 5.2(3).

We will now prove that at least one of the following alternatives must hold: either $S \prec T'$ or $T'$ is unitarily equivalent to an operator of the form $S(\Theta)$, where $\Theta = \left[\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right]$ is a purely contractive inner function. For this, we first note that $U_+ \mid \mathcal{E}$ is the minimal isometric lifting of $T'$. Therefore $T'$ is of class $C_{\bullet 0}$ and its functional model is the compression of the canonical shift on $H^2(\mathcal{D}_{T'*})$ to

$$\mathcal{H}(\Theta_{T'}) = H^2(\mathcal{D}_{T'*}) \ominus \Theta_T H^2(\mathcal{D}_{T'}),$$

where

$$\Theta_T(z) : \mathcal{D}_{T'} \to \mathcal{D}_{T'*}, \quad z \in \mathbb{D},$$

is the characteristic function of $T'$. Therefore $\Theta_{T'}$ is a purely contractive inner function and according to [23, Chapter VI]

$$\dim \mathcal{D}_{T'} \leqslant \dim \mathcal{D}_{T'*} = \dim \mathcal{E}.$$

Thus we must consider the following possibilities:

(i) $\dim \mathcal{D}_{T'} = \dim \mathcal{D}_{T'*} = 2$;
(ii) $\dim \mathcal{D}_{T'} = 1$, $\dim \mathcal{D}_{T'*} = 2$;
(iii) $\dim \mathcal{D}_{T'} = 0$, $\dim \mathcal{D}_{T'*} = 2$;
(iv) $\dim \mathcal{D}_{T'} = \dim \mathcal{D}_{T'*} = 1$; and
(v) $\dim \mathcal{D}_{T'} = 0$, $\dim \mathcal{D}_{T'*} = 1$.

In cases (i) and (iv) $T'$ is of class $C_{00}$, hence of class $C_0$ (see [23, Proposition VI.3.5 and Theorem VI.5.2]), thus $H^\infty(T')$ is not confluent. In case (ii) $T'$ is unitarily equivalent to an operator of the form $S(\Theta)$, where $\Theta = \left[\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right]$ is a purely contractive inner function. In case (iii) $T'$ is unitarily equivalent to $S \oplus S$ and $H^\infty(S \oplus S)$ is not confluent; to see this consider the vectors $1 \oplus 0$ and $0 \oplus 1$. Finally, in case (v) $T'$ is unitarily equivalent to $S$, and thus $S \prec T$.

If $T$ has a cyclic vector $h_1$, we can take $h_2 = 0$, and then $U_+ \mid \mathcal{E}$ is a shift of multiplicity 1. In this case, we must have $\ker Y = \{0\}$ so that $U_+ \mid \mathcal{E} \prec T$. Conversely, $S \prec T$ implies that $T$ has a cyclic vector since $S$ has one. $\square$

The argument in the preceding proof appeared earlier in the classification of contractions of class $C_{\bullet 0}$ [26,27], and even earlier in [14] and in the study of the class $C_0$ [22].

When $T$ has a cyclic vector, it is natural to ask under what conditions we actually have $T \sim S$.

**Lemma 7.3.** *Assume that $T$ is a completely nonunitary contraction such that $H^\infty(T)$ is confluent. Then the following assertions are equivalent*:

(1) $T \prec S$.
(2) $T \mid \mathcal{M} \prec S$ *for some invariant subspace* $\mathcal{M}$ *of* $T$.
(3) $T \mid \mathcal{M} \prec S$ *for every nonzero invariant subspace* $\mathcal{M}$ *of* $T$.

**Proof.** The implications (3) $\Rightarrow$ (1) $\Rightarrow$ (2) are obvious. Next we show that $T \prec T \mid \mathcal{M}$ for every nonzero invariant subspace $\mathcal{M}$ of $T$. By Theorem 6.4(4), there is an inner function $m$ such that $\overline{m(T)\mathcal{H}} = \mathcal{M}$. Then the operator $X : \mathcal{H} \to \mathcal{M}$ defined by $Xh = m(T)h$, $h \in \mathcal{H}$, is a quasiaffinity and $XT = (T \mid \mathcal{M})X$. Using this fact, it is easy to show that (2) $\Rightarrow$ (1). Indeed, if (2) holds we have $T \mid \mathcal{M} \prec S$ for some $\mathcal{M}$, and the relations $T \prec T \mid \mathcal{M} \prec S$ imply the desired conclusion $T \prec S$. Finally, we prove that (1) $\Rightarrow$ (3). Assume that (1) holds, so that $YT = SY$ for some quasiaffinity $Y$. If $\mathcal{M}$ is a nonzero invariant subspace for $T$, the operator $Z = Y \mid \mathcal{M} : \mathcal{M} \to \overline{Y\mathcal{M}}$ is a quasiaffinity realizing the relation $T \mid \mathcal{M} \prec S \mid \overline{Y\mathcal{M}}$. We conclude that (3) is true since $S \mid \overline{Y\mathcal{M}}$ is unitarily equivalent to $S$. $\square$

We can now state some conditions equivalent to the relation $T \sim S$.

**Theorem 7.4.** *Assume that $T$ is a completely nonunitary contraction such that $H^\infty(T)$ is confluent and has a cyclic vector. Let $f : \mathbb{D} \to \mathcal{H}$ be an analytic function such that $\|f(0)\| = 1$ and $\ker(\lambda I - T^*) = \mathbb{C}f(\lambda)$ for every $\lambda \in \mathbb{D}$, and denote $\mathcal{H}_0 = \bigvee\{T^n f(0)\colon\ n \geqslant 0\}$. Then the following conditions are equivalent*:

(1) $T \sim S$.
(2) $T \mid \mathcal{H}_0 \prec S$.
(3) *There exists an outer function $b \in H^\infty$ such that $b(h/f(0)) \in H^2$ for every $h \in \mathcal{H}_0$.*
(4) *There exists an outer function $b \in H^\infty$ such that*

$$b\frac{\langle h, f(\bar{\lambda})\rangle}{\langle f(0), f(\bar{\lambda})\rangle} \in H^2$$

*for every $h \in \mathcal{H}_0$.*

**Proof.** Since $T$ has a cyclic vector, we have $S \prec T$ by Proposition 7.2(2). Therefore $T \sim S$ is equivalent to $T \prec S$, and this is equivalent to condition (2) by Lemma 7.3. This establishes the equivalence (1) $\Leftrightarrow$ (2).

For an arbitrary $h \in \mathcal{H} \setminus \{0\}$, write the function $h/f(0)$ as a quotient $u/v$ of functions in $H^\infty$. We have then

$$\langle v(T)h, f(\bar{\lambda})\rangle = \langle h, v(T)^* f(\bar{\lambda})\rangle = \langle h, \overline{v(\lambda)} f(\bar{\lambda})\rangle = v(\lambda)\langle h, f(\bar{\lambda})\rangle,$$

and analogously $\langle u(T)f(0), f(\bar{\lambda})\rangle = u(\lambda)\langle f(0), f(\bar{\lambda})\rangle$. Since $v(T)h = u(T)f(0)$, we conclude that

$$b(\lambda)\frac{h}{f(0)}(\lambda) = b(\lambda)\frac{\langle h, f(\bar{\lambda})\rangle}{\langle f(0), f(\bar{\lambda})\rangle} \tag{7.1}$$

for those $\lambda$ for which the denominators do not vanish. This proves the equivalence (3) $\Leftrightarrow$ (4). Note that the analytic function $\langle f(0), f(\bar{\lambda})\rangle$ cannot be identically zero since it takes the value 1 for $\lambda = 0$.

It remains to prove the equivalence (2) $\Leftrightarrow$ (3), and for this purpose we may as well assume that $\mathcal{H} = \mathcal{H}_0$. We apply the construction in the proof of Proposition 7.2 for this particular case. Thus, consider the minimal isometric dilation $U_+ \in \mathcal{L}(\mathcal{K}_+)$ of $T$, and denote $\mathcal{E} = \bigvee\{U_+^n f(0)\colon\ n \geqslant 0\}$.

Since $U_+^* f(0) = T^* f(0) = 0$, there exists a unitary operator $W : H^2 \to \mathcal{E}$ such that $W1 = f(0)$ and $WS = (U_+ \mid \mathcal{E})W$. One can then construct a quasiaffinity $Y : H^2 \to \mathcal{H}$, namely $Y = P_{\mathcal{H}} W$, such that $TY = YS$ and $Y1 = f(0)$. Since an equality of the form $v(S)x = u(S)1$ for $x \in H^2$ is equivalent to $v(T)Yx = u(T)f(0)$, we deduce that

$$\frac{Yx}{f(0)} = x, \quad x \in H^2,$$

and, conversely, that any vector $h \in \mathcal{H}$ such that $k = h/f(0) \in H^2$ must belong to $YH^2$; namely $h = Yk$.

With this preparation, assume that (2) holds, and let $X \in \mathcal{L}(\mathcal{H}, H^2)$ be a quasiaffinity such that $XT = SX$. Then the operator $XY$ is a quasiaffinity in the commutant of $S$, and therefore $XY = b(S)$ for some outer function $b \in H^\infty$. The equality

$$X\big(b(T) - YX\big) = b(S)X - (XY)X = 0$$

implies that we also have $YX = b(T)$. For any $h \in \mathcal{H} \setminus \{0\}$ we have then

$$b\frac{h}{f(0)} = \frac{b(T)h}{f(0)} = \frac{YXh}{Y1} = Xh \in H^2,$$

thus proving (3). Conversely, if (3) holds, we can define a linear map $X : \mathcal{H} \to H^2$ by setting $Xh = b(h/f(0))$ for $h \in \mathcal{H}$, and this map obviously satisfies $XT = SX$. Using (7.1) it is easy to verify that $X$ is a closed linear transformation, and hence $X$ is continuous. It is also immediate that $XY = b(S)$ and $YX = b(T)$, and this implies that $X$ is a quasiaffinity since $b$ is outer. □

**Corollary 7.5.** *Assume that $T \in \mathcal{L}(\mathcal{H})$ is a completely nonunitary contraction such that $T \sim S$. Let $f : \mathbb{D} \to \mathcal{H}$ be an analytic function such that $\|f(0)\| = 1$ and $\ker(\lambda I - T^*) = \mathbb{C}f(\lambda)$ for every $\lambda \in \mathbb{D}$, and assume that $\mathcal{H} = \bigvee\{T^n f(0)\colon n \geqslant 0\}$. Then $\langle f(0), f(\lambda)\rangle \neq 0$ for every $\lambda \in \mathbb{D}$.*

**Proof.** Let $b$ be an outer function satisfying condition (4) of Theorem 7.4. Assume that $\langle f(0), f(\bar\lambda)\rangle = 0$ for some $\lambda \in \mathbb{D}$. Since $b(\lambda) \neq 0$, it follows that $\langle h, f(\bar\lambda)\rangle = 0$ for every $h \in \mathcal{H}$, and therefore $f(\bar\lambda) = 0$, which is impossible since this vector generates $\ker(\lambda I - T)^*$. □

The relation $T \prec S$ can also be studied in terms of the minimal unitary dilation of $T$. We will denote by $R_* \in \mathcal{L}(\mathcal{R}_*)$ the $*$-residual part of this minimal unitary dilation; see [23, Section II.3] for the relevant definitions. The facts we require about this operator are as follows:

(a) $R_*$ is a unitary operator with absolutely continuous spectral measure relative to arclength measure on $\mathbb{T}$.
(b) There exists an operator $Z : \mathcal{H} \to \mathcal{R}_*$ (namely, the orthogonal projection onto $\mathcal{R}_*$) such that $ZT = R_*Z$ and

$$\|Zh\| = \lim_{n\to\infty} \big\|T^n h\big\|.$$

In particular, $Z$ is injective if and only if $T$ is of class $C_{1\cdot}$.
(c) The smallest reducing subspace for $R_*$ containing $Z\mathcal{H}$ is $\mathcal{R}_*$.

**Proposition 7.6.** *Assume that $T$ is a completely nonunitary contraction such that $H^\infty(T)$ is confluent. Then*

(1) *The $*$-residual part $R_*$ of the minimal unitary dilation of $T$ has spectral multiplicity at most* 1.
(2) *We have $T \prec S$ if and only if $R_*$ is a bilateral shift of multiplicity* 1.
(3) *We have $T \prec R_* \mid \overline{Z\mathcal{H}}$, and $T \prec S$ if and only if $\overline{Z\mathcal{H}} \neq \mathcal{R}_*$.*
(4) *$T^*$ has a cyclic vector.*

**Proof.** Given $h_1, h_2 \in \mathcal{H} \setminus \{0\}$, select $u_1, u_2 \in H^\infty \setminus \{0\}$ such that $u_1(T)h_1 = u_2(T)h_2$. Then we have $u_1(R_*)Zh_1 = u_2(R_*)Zh_2$. Since $u_1(\zeta)$ and $u_2(\zeta)$ are different from zero a.e. relative to the spectral measure of $R_*$, it follows that the vectors $Zh_1$ and $Zh_2$ generate the same reducing space for $R_*$. Therefore $R_*$ has a $*$-cyclic vector, and this implies (1).

Next we prove (3). The fact that $T \prec R_* \mid \overline{Z\mathcal{H}}$ is immediate. If $\overline{Z\mathcal{H}}$ is not reducing, then $R_* \mid \overline{Z\mathcal{H}}$ is unitarily equivalent to $S$ and hence $T \prec S$. Conversely, if $T \prec S$, let $W$ be a quasi-affinity such that $WT = SW$. For any $h \in \mathcal{H}$ we have

$$\|Wh\| = \lim_{n\to\infty} \left\|S^n Wh\right\| = \lim_{n\to\infty} \left\|WT^n h\right\| \leqslant \|W\| \|Zh\|,$$

so there exists an operator $X : \overline{Z\mathcal{H}} \to H^2$ such that $\|X\| \leqslant \|W\|$ and $XZ = W$. Since the range of $X$ contains the range of $W$, we have $X \neq 0$. Pick a vector $f \in H^2$ such that $X^* f \neq 0$, and observe that

$$\lim_{n\to\infty} \left\|(R_* \mid \overline{Z\mathcal{H}})^{*n} X^* f\right\| = \lim_{n\to\infty} \left\|X^* S^{*n} f\right\| = 0.$$

Therefore $R_* \mid \overline{Z\mathcal{H}}$ is not unitary, and consequently $\overline{Z\mathcal{H}} \neq \mathcal{R}_*$.

Assume now that $T \prec S$. The fact that $R_*$ is a bilateral shift follows from (3) because the only absolutely continuous unitary operator of multiplicity 1 which has nonreducing invariant subspaces is the bilateral shift. Conversely, if $R_*$ is a bilateral shift, the results of [17] imply the existence of an invariant subspace $\mathcal{M}$ for $T$ such that $T \mid \mathcal{M} \prec S$. We deduce that $T \prec S$ by Lemma 7.3. This proves (2).

Finally, (4) also follows from (3) because $(R_* \mid \overline{Z\mathcal{H}})^*$ has a cyclic vector. $\square$

**Corollary 7.7.** *Assume that $\Theta = \left[\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right]$ is inner, $*$-outer, and purely contractive. Then $S(\Theta) \prec S$. More precisely, the operator $Q : \mathcal{H}(\Theta) \to H^2$ defined by $Q(f_1 \oplus f_2) = \theta_1 f_2 - \theta_2 f_1$, $f_1 \oplus f_2 \in \mathcal{H}(\Theta)$, is a quasiaffinity and $QS(\Theta) = SQ$.*

**Proof.** We will show that $\overline{P_{\mathcal{R}_*}\mathcal{H}(\Theta)} \neq \mathcal{R}_*$. To do this, we observe first that the minimal unitary dilation of $S(\Theta)$ is the operator $U \oplus U$ on $L^2 \oplus L^2$. The space $\mathcal{R}_*$ is the orthogonal complement of the smallest reducing space for $U \oplus U$ containing $\{\theta_1 u \oplus \theta_2 u\colon u \in H^2\}$. Thus

$$\mathcal{R}_* = \left(L^2 \oplus L^2\right) \ominus \left\{\theta_1 u \oplus \theta_2 u\colon u \in L^2\right\},$$

and it follows that $P_{\mathcal{R}_*}$ is the operator of pointwise multiplication by the matrix

$$I - \Theta\Theta^* = \begin{bmatrix} |\theta_2|^2 & -\overline{\theta_2}\theta_1 \\ -\theta_2\overline{\theta_1} & |\theta_1|^2 \end{bmatrix}.$$

Finally, we have $P_{\mathcal{R}_*}\mathcal{H}(\Theta) = P_{\mathcal{R}_*}(H^2 \oplus H^2)$, and therefore $\overline{P_{\mathcal{R}_*}\mathcal{H}(\Theta)}$ is the invariant subspace for $U$ generated by $P_{\mathcal{R}_*}(1 \oplus 0)$ and $P_{\mathcal{R}_*}(0 \oplus 1)$. These two vectors are precisely

$$|\theta_2|^2 \oplus (-\theta_2\overline{\theta_1}) = (-\overline{\theta_2}u) \oplus \overline{\theta_1}u,$$
$$(-\overline{\theta_2}\theta_1) \oplus |\theta_1|^2 = (-\overline{\theta_2}v) \oplus \overline{\theta_1}v,$$

with $u = -\theta_2$ and $v = \theta_1$. Since $\theta_1$ and $\theta_2$ do not have nonconstant common inner divisors, the invariant subspace for $S$ they generate is the entire $H^2$. It follows that

$$\overline{P_{\mathcal{R}_*}\mathcal{H}(\Theta)} = \{(-\overline{\theta_2}w) \oplus \overline{\theta_1}w\colon\ w \in H^2\},$$

and $R_* \mid \overline{P_{\mathcal{R}_*}\mathcal{H}(\Theta)}$ is unitarily equivalent to $S$. The final assertion is verified by noting that (see (b) above)

$$Z(f_1 \oplus f_2) = P_{\mathcal{R}_*}(f_1 \oplus f_2) = \big(-\overline{\theta_2}Q(f_1 \oplus f_2)\big) \oplus \big(\overline{\theta_1}Q(f_1 \oplus f_2)\big)$$

for $f_1 \oplus f_2 \in \mathcal{H}(\Theta)$. □

The preceding result can be extended considerably. As seen in the proof below, the assumption that $I - T^*T$ has finite rank can be replaced by the requirement that $I - TT^*$ have finite rank.

**Corollary 7.8.** *Let $T$ be a completely nonunitary contraction such that $H^\infty(T)$ is confluent and $I - T^*T$ has finite rank. Then*

(1) *We have $T \prec S$.*
(2) *If in addition $T$ has a cyclic vector, then $T \sim S$.*

**Proof.** Denote by $n$ the rank of $I - T^*T$, and observe that the characteristic function $\Theta_T$ is inner, $*$-outer, and coincides with an $(n+1) \times n$ matrix over $H^\infty$. Indeed, $\Theta_T(0)$ is a Fredholm operator of index $-1$. It follows that $I - \Theta_T(\zeta)\Theta_T(\zeta)^*$ has rank 1 for a.e. $\zeta \in \mathbb{T}$, and therefore $R_*$ is a bilateral shift by [23, Section VI.6]. Thus (1) follows from Proposition 7.6(2). Part (2) follows from (1) and Proposition 7.2(2). □

The following result shows that there exist some purely contractive inner functions of the form $\Theta = \left[\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right]$ with the property that $S(\Theta) \prec S$. Thus part (1) of Proposition 7.2 could be restated as follows:

(1) If $T$ is a completely nonunitary contraction such that $H^\infty(T)$ is confluent, then there exists a purely contractive inner function $\Theta = \left[\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right]$ such that $S(\Theta) \prec T$, and $H^\infty(S(\Theta))$ is confluent.

**Corollary 7.9.** *Assume that $\Theta = \left[\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right]$ is purely contractive, inner and $*$-outer.*

(1) *If $f_1 \oplus f_2 \in \mathcal{H}(\Theta)$ is cyclic for $S(\Theta)$, then $\theta_1 f_2 - \theta_2 f_1$ is an outer function.*
(2) *Conversely, if $\theta_1 f_2 - \theta_2 f_1$ is outer for some $f_1, f_2 \in H^2$, then $P_{\mathcal{H}(\Theta)}(f_1 \oplus f_2)$ is cyclic for $S(\Theta)$.*

(3) *There exists $\Theta$ such that $S(\Theta)$ does not have a cyclic vector.*
(4) *We have $S(\Theta) \sim S$ if and only if $S(\Theta)$ has a cyclic vector.*

**Proof.** With the notation of Corollary 7.7, $Q(f_1 \oplus f_2)$ must be cyclic for $S$ if $f_1 \oplus f_2$ is cyclic for $S(\Theta)$. This proves (1).

Conversely, assume that $u = \theta_1 f_2 - \theta_2 f_1$ is outer for some $f_1, f_2 \in H^2$. Upon multiplying $f_1, f_2$ by some outer function, we may assume that $f_1, f_2 \in H^\infty$. Let $g_1 \oplus g_2 \in \mathcal{H}(\Theta)$ be a vector orthogonal to $\bigvee\{S(\Theta)^n P_{\mathcal{H}(\Theta)}(f_1 \oplus f_2)\colon n \geqslant 0\}$. We have then

$$\langle g_1 \oplus g_2, \theta_1 p \oplus \theta_2 p\rangle = \langle g_1 \oplus g_2, f_1 p \oplus f_2 p\rangle = 0$$

for every polynomial $p$. Equivalently, $\overline{\theta_1} g_1 + \overline{\theta_2} g_2$ and $\overline{f_1} g_1 + \overline{f_2} g_2$ belong to $L^2 \ominus H^2$, and therefore the functions

$$\bar{u} g_1 = \overline{f_2}(\overline{\theta_1} g_1 + \overline{\theta_2} g_2) - \overline{\theta_2}(\overline{f_1} g_1 + \overline{f_2} g_2),$$
$$\bar{u} g_2 = \overline{\theta_1}(\overline{f_1} g_1 + \overline{f_2} g_2) - \overline{f_1}(\overline{\theta_1} g_1 + \overline{\theta_2} g_2)$$

are also in $L^2 \ominus H^2$. Thus $\langle g_j, up\rangle = 0$ for all polynomials $p$, and hence $g_j = 0$, $j = 1, 2$, because $u$ is outer. Assertion (2) follows.

To prove (3), let $m_1$ and $m_2$ be two relatively prime inner functions, and set $\theta_1 = \frac{3}{5} m_1$ and $\theta_2 = \frac{4}{5} m_2$. Nordgren [18] showed that it is possible to choose $m_1$ and $m_2$ so that no function of the form $m_1 f_2 - m_2 f_1$ is outer if $f_1, f_2 \in H^2$. The corresponding operator $S(\Theta)$ does not have a cyclic vector. Finally (4) follows from Corollary 7.7 and Proposition 7.2(2). □

Let us also note a related result which follows easily from [28].

**Proposition 7.10.** *Assume that $\Theta = \left[\begin{smallmatrix}\theta_1 \\ \theta_2\end{smallmatrix}\right]$ is inner and $*$-outer. Then the operator $S(\Theta)$ is similar to $S$ if and only if there exist $f_1, f_2 \in H^\infty$ such that $\theta_1 f_2 - \theta_2 f_1 = 1$.*

**Proof.** It was shown in [28] that $S(\Theta)$ is similar to an isometry if and only if $\Theta$ is left invertible. To conclude, one must observe that the only possible isometry is a unilateral shift of multiplicity 1. □

The proof of the following proposition follows easily from the above arguments, along with the corresponding properties of $S$.

**Proposition 7.11.** *Let $T$ be a completely nonunitary contraction such that $T \sim S$. Then $T$ is of class $C_{10}$, both $T$ and $T^*$ have cyclic vectors, and the $*$-residual part $R_*$ of the minimal unitary dilation of $T$ is a bilateral shift of multiplicity* 1.

The converse of this proposition is not true. Indeed, it was shown in [6] (see also [16]) that there exist operators $T$ of class $C_{10}$, with a cyclic vector, such that $R_*$ is a bilateral shift of multiplicity 1, and $\sigma(T) \not\supset \mathbb{D}$. For such operators we will have $R_*^* \prec T^*$, so $T^*$ also has a cyclic vector, but $T \not\prec S$.

The following partial converse follows from Propositions 7.2(2) and 7.6(2).

**Proposition 7.12.** *Let $T$ be a completely nonunitary contraction such that $H^\infty(T)$ is confluent, $T$ has a cyclic vector, and the $*$-residual part $R_*$ of the minimal unitary dilation of $T$ is a bilateral shift of multiplicity* 1. *Then $T \sim S$.*

**Remark 7.13.** For more information about which operators are or can be quasisimilar to the unilateral shift, see [2,9,11,12,15,29] and the references therein.

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