Asymptotically Optimal Weighted Numerical Integration

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We study numerical integration of Hölder-type functions with respect to weights on
the real line. Our study extends previous work by F. Curbera (J. Complexity 14(1), (1998)
and relies on a connection between this problem and the approximation of distribution
functions by empirical ones. As an application we reproduce a variant of the well-known
result for weighted integration of Brownian paths. © 1998 Academic Press

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1. INTRODUCTION, PROBLEM FORMULATION

The present study is initiated by work of Curbera, [3], which was devoted to
asymptotically optimal numerical quadrature of Lipschitz functions with respect
to a Gaussian weight.

Let \( \varphi \) be an integrable continuous function on \( \mathbb{R} \). For a given function \( f: \mathbb{R} \rightarrow \mathbb{R} \) we let

\[
I_\varphi(f) := \int_{\mathbb{R}} f(x)\varphi(x) \, dx.
\]

We aim at approximating \( I_\varphi(f) \) by using a quadrature formula

\[
u(f) := \sum_{j=1}^{n} c_j f(x_j),
\]

where \( n \) is a number of knots, while \( c_j, j = 1, \ldots, n \) and \( x_j, j = 1, \ldots, n \) are
weights and knots, respectively. We shall apply quadrature rules to the integra-
tion of certain classes of Hölder functions, i.e., for \( 1 < q < \infty \) we denote

\[
\mathcal{F}_q(L) := \{f: \mathbb{R} \rightarrow \mathbb{R} \text{ absolutely continuous with derivative } f' \text{ and } \|f'\|_q \leq L\}.
\]

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In case $q = \infty$ this is identified with the space of Lipschitz functions $f$ satisfying $|f(y) - f(x)| \leq L|y - x|$, $x, y \in \mathbb{R}$. Thus, we are interested in the overall error of a quadrature rule $u$ given by

$$e(\mathcal{F}_q(L), u, \varphi) := \sup_{f \in \mathcal{F}_q(L)} |I\varphi(f) - u(f)|.$$ 

The important quantity under consideration is

$$e_n(\mathcal{F}_q(L), \varphi) := \inf_{u \in \mathcal{L}_n} e(\mathcal{F}_q(L), u, \varphi),$$

where the infimum is taken over all quadrature rules $u$ using at most $n$ knots. Without loss of generality we shall assume $\int_{\mathbb{R}} \varphi(x) \, dx = 1$ throughout, although the formulation of the results does not differ for any other normalization.

It is evident that additional assumptions have to be imposed on the weight $\varphi$ in order to make $I\varphi(f)$ finite and $e_n(\mathcal{F}_q(L), \varphi)$ converge to 0. This will be seen now when translating the problem of numerical integration to an approximation problem.

We first observe that any quadrature rule $u(f) = \sum_{j=1}^n c_j f(x_j)$ with finite error has to integrate constant functions exactly, which amounts to $\sum_{j=1}^n c_j = 1$. Thus to every quadrature rule a distribution function $Q$ can be assigned via

$$Q(x) := \sum_{j=1}^n c_j \chi(-\infty, x)(x_j), \quad x \in \mathbb{R}. \quad (1)$$

Moreover, we may rewrite for any function $f \in \mathcal{F}_q(L)$ and quadrature rule $u$ the respective error by

$$|I\varphi(f) - u(f)| = \left| \int_{\mathbb{R}} f \, dF - \int_{\mathbb{R}} f \, dQ \right|,$$

where $F$ is the distribution function corresponding to the weight $\varphi$. This yields, using integration by parts and letting $1/p = 1 - 1/q$, the relation

$$\sup_{f \in \mathcal{F}_q(L)} |I\varphi(f) - u(f)| = L \left( \int_{\mathbb{R}} |F(x) - Q(x)|^p \, dx \right)^{1/p}. \quad (2)$$

This is well known; see e.g., [8, Example 4.3.2].

Thus, we arrived at the announced approximation problem. Suppose we are given two distribution functions $F$ and $G$ on the real line possessing $p$th absolute moments. In this case the distance between these distributions can be measured in the $L_p$-sense (see [8, Chap. 3.2] for more details) by letting

$$\theta_p(F, G) := \left( \int_{\mathbb{R}} |F(x) - G(x)|^p \, dx \right)^{1/p}.$$
So we may ask for approximating a given distribution function $F$ by an empirical one, i.e., a step function $Q$ as in (1) with a finite number $n$ of steps. Thus, we ask for

$$e_n(F, p) := \inf[\theta_p(F, Q), \quad Q \text{ has at most } n \text{ steps}].$$

Although there is vast literature concerning probability metrics (see [8] for further references), this specific type of questions does not seem to be settled.

In view of relation (2) the problems of weighted numerical integration and approximation of distribution functions are closely connected. Moreover, each approximating step function can be assigned a quadrature rule and vice versa, which amounts to $e_n(F, Q) = L e_n(F, p)$, $n \in \mathbb{N}$. Thus, below we will focus on the problem of approximating distribution functions by step functions.

The plan of our considerations will be as follows: We first turn to an auxiliary problem. We then translate these results into statements for the distribution function approximation problem and for the integration problem. Finally we shall apply our results to integration of Brownian paths.

2. THE AUXILIARY PROBLEM

For technical reasons we shall turn to a further auxiliary problem. Given a strictly increasing distribution function $F$ and a step function $Q$ as above, and substituting $t := F(x)$, thus $t_j := F(x_j)$, $j = 1, \ldots, n$, we may rewrite

$$\theta_p(F, Q) = \left( \int_0^1 |t - Q(F^{-1}(t))|^p \frac{dt}{\varphi(F^{-1}(t))} \right)^{1/p}.$$  

Letting

$$\alpha(t) := \left( \varphi(F^{-1}(t)) \right)^{-1}, \quad t \in (0, 1),$$

and observing that $R(t) := Q(F^{-1}(t))$, $t \in (0, 1)$, is also an empirical distribution function with representation $R(t) = \sum_{j=1}^{n} c_j \chi_{(0, t)}(t_j)$, $t \in (0, 1)$, this transfers to

$$\theta_p(F, Q) = \left( \int_0^1 |t - R(t)|^p \alpha(t) \, dt \right)^{1/p}. \quad (4)$$

Equation (4) gives rise to an approximation problem for the uniform distribution on $[0, 1]$ by step functions where the error is measured with weight $\alpha$. We let without ambiguity

$$e_n(\alpha, p) := \inf_{R \in \mathcal{Q}_n} \left( \int_0^1 |t - R(t)|^p \alpha(t) \, dt \right)^{1/p}, \quad (5)$$

where the infimum is taken over all step functions with at most \( n \) jumps. We shall investigate the asymptotic behavior of \( e_n(\alpha, p) \) in this section. Unless in some trivial cases it is not possible to compute the minimal error for a given number of steps exactly. We therefore turn to asymptotic considerations. Such kind of analysis was probably introduced in [9]. Our arguments below are based on investigations similar to [5, 6].

To each design \((t_1, \ldots, t_n)\) of knots in \((0, 1)\) with \( t_j \leq t_{j+1}, j = 0, \ldots, n, \) where \( t_0 := 0 \) and \( t_{n+1} = 1, \) we assign a partition

\[
\Pi := \{ \Delta_j = [t_j, t_{j+1}), \quad j = 0, \ldots, n \}
\]
of \([0, 1]\). A sequence of partitions \((\Pi_n)_{n \in \mathbb{N}}\) is said to be uniformly fine if

\[
\lim_{n \to \infty} \max_{0 \leq j \leq n} |\Delta_j, n| = 0.
\]

We mention that a sequence of partitions \((\Pi_n)_{n \in \mathbb{N}}\) is uniformly fine iff for every \( 0 < a < b < 1 \) we have

\[
\lim_{n \to \infty} \max_{0 \leq j \leq n} |\Delta_j, n \cap [a, b]| = 0.
\]

The following proposition provides a lower bound and indicates a general limitation of approximating distribution functions by step functions.

**Proposition 1.** Let \( \alpha: (0, 1) \to \mathbb{R}^+ \) be continuous and nonnegative. If

\[
\int_0^1 \alpha(t)^{1/(p+1)} \, dt < \infty
\]

then

\[
\liminf_{n \to \infty} n e_n(\alpha, p) \geq \frac{1}{2} \left( \frac{1}{p+1} \right)^{1/p} \left( \int_0^1 \alpha(t)^{1/(p+1)} \, dt \right)^{(p+1)/p}. \tag{6}
\]

**Proof.** Let \((R_n)\) be any sequence of empirical distribution functions. Fix any interval \([a, b] \subset (0, 1). \) For \( n \in \mathbb{N} \) let \( I_n := \{ j, \Delta_j, n \subset [a, b] \}. \) Using representation (4) and applying the mean value theorem we find a sequence \( \xi_j, n \in \Delta_j, n, j \in I_n, n \in \mathbb{N} \) for which

\[
\int_0^1 |t - R_n(t)|^p \alpha(t) \, dt \geq \sum_{j \in I_n} \alpha(\xi_j, n) \min_c \int_{\Delta_j, n} |t - c|^p \, dt
\]

\[
\geq \frac{1}{2^p (p+1)} \sum_{j \in I_n} \alpha(\xi_j, n) |\Delta_j, n|^{p+1}.
\]

From this estimate we see that the error cannot decrease to 0 unless the induced sequence of partitions is uniformly fine. Hölder’s inequality yields

\[
n^p \sum_{j \in I_n} \alpha(\xi_j, n) |\Delta_j, n|^{p+1} \geq \left( \sum_{j \in I_n} \alpha(\xi_j, n)^{1/(p+1)} |\Delta_j, n| \right)^{p+1}.
\]
The right-hand side sum is a Darboux sum for the integral \( \int_a^b \alpha(t)^{1/(p+1)} \, dt \), which is the only possible limit. Thus,

\[
\liminf_{n \to \infty} n^p \int_0^1 |t - R_n(t)|^p \alpha(t) \, dt \geq \frac{1}{2^p (p+1)} \left( \int_a^b \alpha(t)^{1/(p+1)} \, dt \right)^{p+1}.
\]

Since this is valid for every choice of \([a, b]\) the proof of (6) is complete. \( \square \)

It is natural to ask whether this bound can be achieved for a suitable choice of step functions. This is indeed the case if we impose additional assumptions on the weight \( \alpha \). First we recall that the integrability \( \int_0^1 \alpha(t)^{1/(p+1)} \, dt < \infty \) was necessary. Now we let jump locations \( t_{j,n} \in [0, 1] \) be chosen as quantiles (\( t_{0,n} := 0, t_{n+1,n} := 1 \)) by

\[
\int_0^{t_{j,n}} \alpha(t)^{1/(p+1)} \, dt = \frac{j}{n+1} \int_0^1 \alpha(t)^{1/(p+1)} \, dt, \quad j = 1, \ldots, n
\]

and the corresponding weights \( r_{j,n} \) as midpoints

\[
r_{j,n} = \left\{ \begin{array}{ll} 0 & \text{for } j = 0 \\
\frac{t_{j,n} + t_{j+1,n}}{2} & \text{for } j = 1, \ldots, n-1 \\
1 & \text{for } j = n.
\end{array} \right.
\]

This choice of step functions \( R_n(t) = \sum_{j=0}^n r_{j,n} \chi_{\Delta_{j,n}}(t), t \in [0, 1] \), provides asymptotically optimal errors in many cases. We have

**Proposition 2.** Let \( \alpha: (0, 1) \to \mathbb{R}^+ \) be continuous and bounded away from 0. Moreover we assume \( \int_0^1 \alpha(t)^{1/(p+1)} \, dt < \infty \) and that \( t_{j,n} \) and \( r_{j,n} \) are chosen according to (7) and (8), respectively. If

\[
n^p \int_0^{t_{1,n}} t^p \alpha(t) \, dt \longrightarrow 0,
\]

\[
n^p \int_{t_{n,n}}^1 |1-t|^p \alpha(t) \, dt \longrightarrow 0,
\]

and if there is a constant \( C < \infty \) for which

\[
\frac{\alpha(s)}{\alpha(t)} \leq C, \quad \text{whenever } s, t \in \Delta_{j,n}, \ j = 1, \ldots, n-1,
\]

then we have

\[
\lim_{n \to \infty} n \varepsilon_n(\alpha, p) = \frac{1}{2} \left( \frac{1}{p+1} \right)^{1/p} \left( \int_0^1 \alpha(t)^{1/(p+1)} \, dt \right)^{(p+1)/p}.
\]
Remark 1. The integrability assumptions and assumptions (9) and (10) are quite natural within this context and may easily be verified in specific examples. Assumption (11) is technical and can be verified only after having good knowledge of the jump locations. It is not clear whether this may be avoided or replaced by some other more natural condition. Below we shall make use of this assumption while applying the dominated convergence theorem in (14).

Proof of Proposition 2. Let the jump locations \( t_{j,n} \) and weights \( r_{j,n} \) be chosen according to (7) and (8), respectively. We use this representation and the mean value theorem to find \( \xi_{j,n} \in \Delta_{j,n}, j = 1, \ldots, n - 1, \) and to derive

\[
\int_{0}^{1} |t - R_n(t)|^p \alpha(t) \, dt = \int_{0}^{t_{1,n}} t^p \alpha(t) \, dt + \sum_{j=1}^{n-1} \alpha(\xi_{j,n}) \frac{\Delta_{j,n}^{p+1}}{2^p(p+1)} + \int_{t_{n,n}}^{1} |1-t|^p \alpha(t) \, dt. \tag{13}
\]

To each \( \Delta_{j,n} \) we apply Hölder’s inequality with dual indices \( (p+1)/p \) and \( p+1 \) to estimate

\[
|\Delta_{j,n}|^{p+1} = \left( \int_{\Delta_{j,n}} \alpha(t)^{p/(p+1)^2} \times \alpha(t)^{-p/(p+1)^2} \, dt \right)^{p+1} \leq \left( \int_{\Delta_{j,n}} \alpha(t)^{1/(p+1)} \, dt \right)^p \left( \int_{\Delta_{j,n}} \frac{1}{\alpha(t)^{p/(p+1)}} \, dt \right) \cdot \tag{14}
\]

Inserting this into the middle sum in (13) we see that

\[
n^p \sum_{j=1}^{n-1} \alpha(\xi_{j,n}) |\Delta_{j,n}|^{p+1} \leq \sum_{j=1}^{n-1} \left( n \int_{\Delta_{j,n}} \alpha(t)^{1/(p+1)} \, dt \right)^p \left( \int_{\Delta_{j,n}} \frac{\alpha(\xi_{j,n})}{\alpha(t)^{p/(p+1)}} \, dt \right) \leq \left( n \max_{j=1, \ldots, n-1} \int_{\Delta_{j,n}} \alpha(t)^{1/(p+1)} \, dt \right)^p \times \left( \int_{t_{1,n}}^{t_{n,n}} \sum_{j=1}^{n-1} \frac{\alpha(\xi_{j,n})}{\alpha(t)^{p/(p+1)}} \chi_{\Delta_{j,n}}(t) \, dt \right). \]

By assumption (11) we may estimate

\[
\sum_{j=1}^{n-1} \frac{\alpha(\xi_{j,n})}{\alpha(t)^{p/(p+1)}} \chi_{\Delta_{j,n}}(t) \leq C \sum_{j=1}^{n-1} \alpha(t)^{1/(p+1)} \chi_{\Delta_{j,n}}(t) \leq C \alpha(t)^{1/(p+1)}(t), \quad t \in (0, 1),
\]
and apply the dominated convergence theorem to see that

\[
\int_{t_{n, n}}^{t_{n, n-1}} \frac{\alpha(\xi_{j, n})}{\alpha(t)^{p/(p+1)}} \chi_{\Delta_{j, n}}(t) \, dt \to \int_0^1 \alpha(t)^{1/(p+1)} \, dt.
\]

Moreover, by our choice (7) and making use of assumptions (9) and (10) we see that

\[
\limsup_{n \to \infty} n e_n(\alpha, p) \leq \frac{1}{2} \left( \frac{1}{p+1} \right)^{1/p} \left( \int_0^1 \alpha(t)^{1/(p+1)} \, dt \right)^{(p+1)/p}.
\]

Combining this estimate with Proposition 1 the proof is complete.

3. DISTRIBUTION FUNCTION APPROXIMATION

Below we shall provide instances for which the asymptotic behavior of the minimal error for approximating a distribution function by a step function can be characterized. For this purpose we have to find suitable conditions on \( \varphi \) which imply the conditions of Proposition 2. Those conditions consisted of integrability ones and of condition (11), describing the variation of \( \varphi \) on certain sequences of partitions.

To this end, given \( 1 \leq p < \infty \), let \( \varphi : \mathbb{R} \to \mathbb{R}^+ \) be an integrable function which moreover satisfies:

1. \( \varphi \) is a nowhere vanishing continuous function.
2. There is \( 0 < \varepsilon < 1 \) for which \( \lim_{|x| \to \infty} |x|^{(p+1)/p} \varphi(x)^{1-\varepsilon} = 0 \).

Remark 2. The integrability of the weight function is certainly necessary to study integration, since otherwise constant functions would not be integrable. Requirement (i) implies that we concentrate on weights, which are regular on bounded intervals, hence no singularities are allowed there. All we are interested in is the behavior for \( |x| \to \infty \), which is controlled by requirement (ii). Let us mention at this place, that (ii) also implies \( \int_{\mathbb{R}} \varphi(x)^{p/(p+1)} \, dx < \infty \).

Next we discuss implications of condition (\( I_p \)) for a possible application of Proposition 2. It is evident from representation (3) that (\( I_p \)) yields with \( u := F(y) \)

\[
\int_0^u \alpha(t)^{1/(p+1)} \, dt = \int_{-\infty}^y \frac{\varphi(x)}{\varphi(x)^{1/(p+1)}} \, dx = \int_{-\infty}^y \varphi(x)^{p/(p+1)} \, dx < \infty, \quad y \in \mathbb{R}.
\]
and also,
\[ \int_0^1 \frac{1}{\alpha(t)^{p/(p+1)}} \, dt = \int_{\mathbb{R}^p} \varphi(x)^{1+p/(p+1)} \, dx \leq \max_{x \in \mathbb{R}^p} \varphi(x)^{p/(p+1)} < \infty. \]

Condition \((I_p)\) also implies the finiteness of \(e_n(F, p)\), which can be seen from

\[ \text{LEMMA 1. For } y \leq 0 \text{ we have} \]
\[ \int_{-\infty}^y F(x)^p \, dx \leq \left( \int_{-\infty}^y \varphi(x)^{p/(p+1)} \, dx \right)^p \left( \sup_{x \leq y} |x|^{(p+1)/p} \varphi(x) \right)^{p/(p+1)} < \infty. \]

A similar inequality holds for the corresponding integral of \(1 - F(x)\) on \((-y, \infty)\).

\[ \text{Proof. The proof is based on a well-known inequality (see, e.g., [7, 22.3.1]) such that below we may reverse the order of integration to conclude} \]
\[ \left( \int_{-\infty}^y F(x)^p \, dx \right)^{1/p} \]
\[ = \left( \int_{-\infty}^y \int_{-\infty}^y \chi(-\infty, x)(u)\varphi(u) \, du \right)^{1/p} \]
\[ \leq \int_{-\infty}^y \left( \int_{-\infty}^0 \chi(-\infty, x)(u)\varphi(u) \, du \right)^{1/p} \]
\[ = \int_{-\infty}^y u^{1/p} \varphi(u) \, du \leq \int_{-\infty}^y \varphi(u)^{p/(p+1)} \, du \left( \sup_{u \leq y} |u|^{(p+1)/p} \varphi(u) \right)^{1/(p+1)}. \]

Both terms above are finite by assumption (ii) of \((I_p)\) and the proof is complete. \[\boxrule 1pt \boxend \]

Furthermore, condition \((I_p)\) also yields assumptions (9) and (10) of Proposition 2. For instance, to verify (9) we observe that \(\int_0^{t_{j,n}} t^p \alpha(t) \, dt = \int_{-\infty}^{x_{j,n}} F(x)^p \, dx\), such that Lemma 1 implies, for \((t_{j,n})\) being quantiles and with (15) in mind:

\[ n^p \int_0^{t_{j,n}} t^p \alpha(t) \, dt \]
\[ \leq \left( n \int_{-\infty}^{x_{j,n}} \varphi(x)^{p/(p+1)} \, dx \right)^p \left( \sup_{x \leq x_{j,n}} |x|^{(p+1)/p} \varphi(x) \right)^{p/(p+1)}. \]

Since \(x_{j,n} \to -\infty\) for \(n \to \infty\) by construction, assumption (ii) of \((I_p)\) completes the argument.
The asymptotically optimal step functions in Proposition 2 were obtained by jump locations \((t_{j,n})\) and weights \((r_{j,n})\) as given in (7) and (8), respectively. Thus, given a weight \(\varphi\) satisfying \((I_p)\) we let \((\overline{x}_{j,n})\) be quantiles
\[
\int_{-\infty}^{\overline{x}_{j,n}} \varphi^{p/(p+1)}(x) \, dx := \frac{j}{n+1} \int_{\mathbb{R}} \varphi^{p/(p+1)}(x) \, dx, \quad j = 1, \ldots, n,
\]
and
\[
\overline{r}_{j,n} := \begin{cases} 
0 & \text{for } j = 0 \\
\frac{F(\overline{x}_{j,n}) + F(\overline{x}_{j+1,n})}{2} & \text{for } j = 1, \ldots, n-1 \\
1 & \text{for } j = n.
\end{cases}
\]
This determines a sequence of step functions \((\overline{Q}_{n+1,n} := \infty)\)
\[
\overline{Q}_n(x) := \sum_{j=1}^{n} \overline{r}_{j,n} \chi_{[\overline{x}_{j,n}, \overline{x}_{j+1,n})}(x), \quad x \in \mathbb{R}.
\] (16)

The remaining condition \((11)\) is fulfilled when assuming

\((Q)\) There is \(C < \infty\) for which \(\varphi(x)/\varphi(y) \leq C, \quad x, y \in [\overline{x}_{j,n}, \overline{x}_{j+1,n}), j = 1, \ldots, n-1.\)

We note that such bound exists always for sequences of intervals within some fixed bounded region, since \(\varphi\) is uniformly bounded and bounded away from 0 there.

Summarizing the previous discussion, we have thus derived

**THEOREM 1.** Let \(F\) be a distribution function possessing a density \(\varphi\) satisfying \((I_p)\) and \((Q)\). Then we have
\[
\lim_{n \to \infty} ne_n(F, p) = \frac{1}{2} \left( \frac{1}{p+1} \right)^{1/p} \left( \int_{\mathbb{R}} \varphi(x)^{p/(p+1)} \, dx \right)^{(p+1)/p}.
\]
(17)

An asymptotically optimal sequence of step functions is given by \((\overline{Q}_n)_{n \in \mathbb{N}}\) as defined in (16).

Typical instances, for which \((Q)\) is fulfilled will be discussed below, when describing asymptotically optimal quadrature rules, the original theme of the present study.

**Remark 3.** Theorem 1 extends in a natural way to weights which live on bounded or one-sided intervals in \(\mathbb{R}\), which means that they have to satisfy appropriate versions of \((I_p)\).

The situation of weighted integration on a finite interval has (implicitly) been treated in [10]. There the authors indicate a correspondence between the
integration problem and the approximate computation of stochastic integrals, which we also stress below in Section 5.

Also one might include additional weights $g: \mathbb{R} \to \mathbb{R}^+$ and consider

$$\theta_p(F, G, g) := \left( \int_{\mathbb{R}} |F(x) - G(x)|^p g(x) \, dx \right)^{1/p}.$$ 

In this case the functions $\varphi$ in the statements of the results have to be replaced by $\varphi/g$.

4. WEIGHTED NUMERICAL INTEGRATION

We return to the original problem of weighted numerical integration. As emphasized at the end of Section 1 we may translate results on the optimal approximation of distribution functions to ones for numerical integration. We only mention that the weights $c_{j,n}$ of quadrature rules are obtained from the corresponding step functions $Q_n$ as in (16) by $c_{j,n} = \bar{r}_{j,n} - \bar{r}_{j-1,n}, j = 1, \ldots, n$. Thus we may state

**Theorem 2.** Given $1 < q \leq \infty$, let the weight function $\varphi$ satisfy $(I_p)$ for $1/p = 1 - 1/q$ and $(Q)$. Then we have

$$\lim_{n \to \infty} n e_n(\mathcal{F}_q(L), \varphi) = \frac{L}{2} \left( \frac{1}{p + 1} \right)^{1/p} \left( \int_{\mathbb{R}} \varphi(x)^{p/(p+1)} \, dx \right)^{(p+1)/p}.$$ 

A sequence of asymptotically optimal quadrature rules is provided by

$$u_n(f) := \sum_{j=1}^{n} c_{j,n} f(x_{j,n}), \quad f \in \mathcal{F}_q(L),$$

with knots determined by

$$\int_{x_{j,n}}^{x_{j+1,n}} \varphi(x)^{p/(p+1)} \, dx = \frac{1}{n+1} \int_{\mathbb{R}} \varphi(x)^{p/(p+1)} \, dx, \quad j = 0, \ldots, n \quad (18)$$

(where $x_{0,n} := -\infty$ and $x_{n+1,n} = \infty$). Asymptotically optimal weights are given by

$$c_{1,n} := \int_{x_{0,n}}^{x_{2,n}} \varphi(x) \, dx - \frac{1}{2} \int_{x_{1,n}}^{x_{2,n}} \varphi(x) \, dx,$$

$$c_{j,n} := \frac{1}{2} \int_{x_{j-1,n}}^{x_{j+1,n}} \varphi(x) \, dx, \quad j = 2, \ldots, n - 1,$$

$$c_{n,n} := \int_{x_{n-1,n}}^{x_{n+1,n}} \varphi(x) \, dx - \frac{1}{2} \int_{x_{n-1,n}}^{x_{n,n}} \varphi(x) \, dx.$$
The following corollaries make the assumptions, especially (Q), more explicit.

**Corollary 1.** Assume that the weight φ satisfies (I_p) and that there are constants K, M > 0 for which log(φ(x)) is Lipschitz with constant M for |x| ≥ K. Then

\[
\lim_{n \to \infty} n e_n(\mathcal{F}_q(L), \varphi) = \frac{L}{2} \left( \frac{1}{p + 1} \right)^{1/p} \left( \int_{\mathbb{R}} \varphi(x)^{p/(p+1)} \, dx \right)^{(p+1)/p}.
\]

All that remains to be checked is condition (Q). As mentioned before, (Q) is satisfied for intervals within [−K, K]. But for |x| ≥ K we conclude

\[
\frac{\varphi(x)}{\varphi(y)} = e^{\log \varphi(x) - \log \varphi(y)} \leq e^{M|y-x|}, \quad x, y \in \mathbb{R},
\]

by the Lipschitz assumption. Since the lengths of the intervals \([x_{j,n}, x_{j+1,n})\) tend to 0, condition (Q) is seen to hold.

**Example.** The weight \(\varphi(x) := 1/(1 + |x|^r)\), \(x \in \mathbb{R}\), satisfies (I_p) for \(r > (p + 1)/p\) and (Q).

**Example.** Weights \(\varphi(x) := e^{-a|x|}, x \in \mathbb{R}, a > 0\), satisfy (I_p) for \(r > (p + 1)/p\) and (Q).

Next we turn to Gaussian weights, the original setup in [3].

**Corollary 2.** Gaussian weights

\[
\varphi_\sigma(x) := \frac{1}{\sqrt{2\pi \sigma^2}} e^{-x^2/(2\sigma^2)}, \quad x \in \mathbb{R},
\]

satisfy (I_p) for all \(p \geq 1\) and (Q). Consequently we have (with \(1/p + 1/q = 1\))

\[
\lim_{n \to \infty} n e_n(\mathcal{F}_q(L), \varphi_\sigma) = L \left( \frac{\pi \sigma^2}{2p} \left( \frac{p + 1}{4} \right)^{-1/p} \right)^{1/(2p)}.
\]

It is easily verified that \(\varphi_\sigma\) does not obey the assumptions of Corollary 1. The function \((\log \varphi_\sigma)'\) is linear instead of uniformly bounded. But a different argument works for these cases.

**Proof of Corollary 2.** The weights \(\varphi_\sigma\) plainly obey conditions (I_p). It remains to check the validity of (Q). But

\[
\frac{\varphi(x)}{\varphi(y)} = e^{-(1/(2\sigma^2))(x^2 - y^2)} \leq e^{(1/(2\sigma^2))(x^2 - y^2)} \leq e^{(1/\sigma^2) \max(|x|, |y|)|x-y|}.
\]
We are done once we can bound the right-hand side for values $x_{j,n}$ and $x_{j+1,n}$ determined by (18). This is essentially proven in Lemma 3 in [3]. For the convenience of the reader we sketch the argument briefly. We do calculations for indices $(j, n)$ with $1 \leq n - j \leq k_0$; hence $x_{j+1,n} \to \infty$. We need to show that $x_{j+1,n} (x_{j+1,n} - x_{j,n}) \leq C$ for some constant $C < \infty$. To simplify expressions we introduce

$$
\psi_{\sigma}(x) := \frac{\varphi_{\sigma}^{p/(p+1)}(x)}{\int_{\mathbb{R}} \varphi_{\sigma}^{p/(p+1)}(x) \, dx}, \quad x \in \mathbb{R},
$$

and let $G(x) := \int_{-\infty}^{x} \psi_{\sigma}(u) \, du$. By our choice of $(x_{j,n})$ we conclude

$$
x_{j+1,n} - x_{j,n} = \int_{x_{j,n}}^{x_{j+1,n}} \psi_{\sigma}(u) \, \frac{1}{\psi_{\sigma}(u)} \, du \leq \left( \max_{x_{j,n} \leq u \leq x_{j+1,n}} 1 \right) \frac{1}{\psi_{\sigma}(x_{j+1,n})} \frac{1}{n+1} (n + 1) - (j + 1) \leq \psi_{\sigma}(x_{j+1,n}) \frac{1}{n+1}.
$$

Hence, we arrived at

$$
x_{j+1,n} (x_{j+1,n} - x_{j,n}) \leq (1 - G(x_{j+1,n})) \frac{x_{j+1,n}}{\psi_{\sigma}(x_{j+1,n})}, \quad 1 \leq n - j \leq k_0,
$$

which is bounded; it even tends to some finite value for $x_{j+1,n} \to \infty$ (see the proof of Lemma 3 in [3] for such assertion or [1, Chap. 7]).

**Remark 4.** As mentioned we exhibit the result proven in [3]. Corollary 2 with $p = 1$ yields

$$
\lim_{n \to \infty} n e_n(\mathcal{F}_\infty(L), \varphi_{\sigma}) = L \sigma \sqrt{\pi/2},
$$

which corresponds to [3, p. 16] by noting that $n$ there corresponds to $2n + 1$ here.

We conclude this section with a discussion of the asymptotically optimal quadrature rules as given in Theorem 2. The above asymptotically optimal weights can hardly be calculated in most cases. But we have the following approximation for $j = 2, \ldots, n - 1$:

$$
c_{j,n} = \frac{1}{2} \left\{ \int_{\Delta_{j-1,n}} \varphi(x) \, dx + \int_{\Delta_{j,n}} \varphi(x) \, dx \right\}
$$

$$
= \frac{1}{2} \left\{ \varphi(x_{j,n} - \tau_{j,n})^{1/(p+1)} \int_{\Delta_{j-1,n}} \varphi(x)^{p/(p+1)} \, dx
$$

$$
+ \varphi(x_{j,n} + \tau'_{j,n})^{1/(p+1)} \int_{\Delta_{j,n}} \varphi(x)^{p/(p+1)} \, dx \right\}
$$

$$
= \frac{\varphi(x_{j,n} - \tau_{j,n})^{1/(p+1)} + \varphi(x_{j,n} + \tau'_{j,n})^{1/(p+1)}}{2 \varphi(x_{j,n})^{1/(p+1)}}\frac{\varphi(x_{j,n})^{1/(p+1)}}{n+1}
$$

$$
\times \int_{\mathbb{R}} \varphi(x)^{p/(p+1)} \, dx,
$$
where the $\tau_{j,n}$ and $\tau'_{j,n}$ were obtained using the mean value theorem. Letting

$$\tilde{c}_{j,n} := \frac{\varphi(x_{j,n})^{1/(p+1)}}{n+1} \int_{\mathbb{R}} \varphi(x)^{p/(p+1)} \, dx, \quad j = 2, \ldots, n-1,$$

we derive

$$c_{j,n} = \frac{\varphi(x_{j,n} - \tau_{j,n})^{1/(p+1)} + \varphi(x_{j,n} + \tau'_{j,n})^{1/(p+1)}}{2\varphi(x_{j,n})^{1/(p+1)}} \tilde{c}_{j,n}.$$  

By the choice of $x_{j,n}$ it is evident that for any $k \in \mathbb{N}$ we have $x_{j,kj-1} = x_{1,k-1}$, $j \in \mathbb{N}$, determined as $(1/k)$-quantiles, and

$$\lim_{j \to \infty} \frac{\varphi(x_{j,kj-1} - \tau_{j,kj-1})^{1/(p+1)} + \varphi(x_{j,kj-1} + \tau'_{j,kj-1})^{1/(p+1)}}{2\varphi(x_{j,kj-1})^{1/(p+1)}} = 1$$

(uniformly for $k \leq K_0$). Thus in practice, uniformly for weights belonging to knots on some fixed bounded interval, the weights $c_{j,n}$ may be replaced by $\tilde{c}_{j,n}$. Although formally these weights do not obey the necessary condition $\sum_{j=1}^{n} \tilde{c}_{j,n} = 1$, they work well in many cases as reported in [3].

**Remark 5.** As a special instance we recover the asymptotic quadrature rule provided in [3, Theorem 3]. However, we do not pay attention to results concerning additional properties, although the regular sequence of knots described above in (18) will be distributed symmetrically for odd $n$.

For a recent publication concerning rigorous results on existence and uniqueness of optimal knots we refer to [2].

### 5. APPLICATION TO WEIGHTED INTEGRATION OF BROWNIAN PATHS

Below we are going to exploit a general principle relating the worst case error of integration to an average case one, which probably goes back to [10]. We will not give many details and refer the reader to [4], where further information as well as references are given.

Suppose we are given a Brownian motion $X := (X_t)_{t \geq 0}$, $X_0 = 0$, on a probability space $(\Omega, \mathcal{F}, P)$, which has almost surely continuous paths and has covariance kernel

$$\mathbb{E} P X_s X_t = \min\{s, t\}, \quad s, t \geq 0.$$  

Given, as above, a weight $\varphi$, now satisfying $(I_2)$, with integral extending from 0 to $\infty$, and $(Q)$, we aim at approximating

$$I_\varphi(X(\omega)) := \int_0^\infty X_t(\omega)\varphi(t) \, dt, \quad \omega \in \Omega,$$
by a quadrature formula

$$u(X(\omega)) := \sum_{j=1}^{n} c_j X_t(\omega), \quad \omega \in \Omega.$$ 

Note that both $I_\varphi(X)$, as well as $u(X)$, are real random variables. The following observation is important. For any Borel measure, say $\mu$, on $[0, \infty)$ we let $\langle X, \mu \rangle := \int_{\omega}^\infty X_t(\omega) \, d\mu(t)$. Then we obtain the following equalities (in case the distribution function of $\mu$ was square integrable):

$$\begin{align*}
E|\langle X, \mu \rangle|^2 &= E \int_{0}^{\infty} \int_{0}^{\infty} X_s(\omega) X_t(\omega) \, d\mu(t) \, d\mu(s) \\
&= \int_{0}^{\infty} \int_{0}^{\infty} \min\{s, t\} \, d\mu(t) \, d\mu(s) \\
&= \int_{0}^{\infty} |\mu([s, \infty)))|^2 \, ds = \int_{0}^{\infty} |\mu([0, s)) - \mu([0, \infty)))|^2 \, ds. \quad (19)
\end{align*}$$

Especially, $E|I_\varphi(X)|^2 < \infty$, and the corresponding error may be measured in mean square sense. Hence,

$$e^{\text{avg}}(I_\varphi, u) := \left( E|I_\varphi(X) - u(X)|^2 \right)^{1/2},$$

and we let

$$e_n^{\text{avg}}(I_\varphi) := \inf_{u \in \mathcal{U}_n} e^{\text{avg}}(I_\varphi, u)$$

denote the $n$th minimal error on the average (with respect to the Wiener measure).

As before we denote by $F$ and $Q$ the distribution functions corresponding to the weight $\varphi$ and the quadrature formula $u$, respectively. Applying the above reasoning to $\mu([0, s)) = F(s) - Q(s)$ and noting that $1 = F(\infty) = Q(\infty)$ is required to make (19) finite, this amounts to

$$\begin{align*}
E|I_\varphi(X) - u(X)|^2 &= \int_{0}^{\infty} |F(s) - Q(s)|^2 \, ds.
\end{align*}$$

Thus Theorem 1 immediately implies

**Corollary 3.** If the weight $\varphi$ obeys (I$_2$) (with integral extending from 0 to $\infty$) and (Q) then

$$\lim_{n \to \infty} ne_n^{\text{avg}}(I_\varphi) = \frac{1}{\sqrt{12}} \left( \int_{0}^{\infty} \varphi(x)^{2/3} \, dx \right)^{3/2}.$$
Corresponding sequences of asymptotically optimal quadrature rules are given as described in Section 4 (for $p = 2$).

Remark 6. As mentioned above such a result (on a bounded interval) is discussed in the running example in [10] (see e.g., Eq. (3.16) there). As indicated there the condition on the weight function $\varphi$ can be relaxed. The authors also establish the relation between average case integration error for a measure with given covariance and the worst case integration error over functions from the unit ball in the reproducing kernel Hilbert space. Here, this relation of worst and average case errors is provided by relating Theorems 1 and 2 and Corollary 3 and reads

$$e_n(\mathcal{F}_2(1), \varphi) = e_{\text{avg}}^n(I_\varphi), \quad n \in \mathbb{N},$$

after mentioning that $\mathcal{F}_2(1)$ is the unit ball of the reproducing kernel Hilbert space $W^1_2$ of the Brownian motion $X$.

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