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Thermodynamics of exotic black holes, negative temperature, and Bekenstein–Hawking entropy

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Abstract

Recently, exotic black holes whose masses and angular momenta are interchanged have been found, and it is known that their entropies depend only on the *inner* horizon areas. But a basic problem of these entropies is that the second law of thermodynamics is not guaranteed, in contrast to the Bekenstein–Hawking entropy. Here, I find that there is another entropy formula which recovers the usual Bekenstein–Hawking form, but the characteristic angular velocity and temperature are identified with those of the inner horizon, in order to satisfy the first law of black hole thermodynamics. The temperature has a *negative* value, due to an upper bound of mass as in spin systems, and the angular velocity has a *lower* bound. I show that one can obtain the same entropy formula from a conformal field theory computation, based on classical Virasoro algebras. I also describe several unanswered problems and some proposals for how these might be addressed.

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1. Introduction

Recently, exotic black holes whose masses and angular momenta are interchanged have been found in several different systems. These are (a) asymptotically anti-de Sitter black holes in $(2+1)$-dimensional gravity for the case of a vanishing cosmological constant with minimally coupled topological matter, which is called "BCEA" gravity [1], (b) constant curvature black holes in $(4+1)$-dimensional anti-de Sitter space [2], and (c) BTZ-like black holes in gravitational Chern–Simons theory [3–9]. But, it is known that these black holes do not satisfy the Bekenstein–Hawking entropy formula, but depend only on the area of the *inner* horizons, in order to satisfy the first law of thermodynamics. This looks similar to Larsen's suggestion in another context [10]. But, a basic problem of these approaches is that the second law of thermodynamics is not guaranteed with their entropy formulae, in contrast to the Bekenstein–Hawking form [11]. Actually, without the guarantee of the second law, there is no justification for identifying entropies with the inner horizon areas [12].

In the usual system of black holes, the first law of thermodynamics uniquely determines (up to an arbitrary constant) the black hole entropy with a given Hawking temperature T_H and chemical potential for the event horizon r_+. In this context, there is no choice in the entropy for the exotic black hole, other than proportional to the area of the inner horizon r_-. In this Letter, I show that there is another rearrangement of the first law such as the entropy has the usual Bekenstein–Hawking form, but now the characteristic temperature and chemical potential are those of the inner horizon, in contrast to the previous approaches. And the temperature has a *negative* value, due to an upper bound of mass as in spin systems, and the angular velocity has a *lower* bound. It is not yet clear how to measure these characteristics by a physical observer who is in the outside of the event horizon. But, I show that one can obtain the same entropy from a conformal field theory computation, based on classical Virasoro algebras at the spatial infinity.

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2. Thermodynamics

The three systems [3–9] which I have mentioned in the introduction look different physically. But, they all allow the exotic black hole solution with the following properties.

a) It has the same form of the metric as the BTZ (Banados–Teitelboim–Zanelli) solution [13], or modulus an expanding/contracting 2-sphere for the case of '(b)',

$$ds^2 = -N^2\,dt^2 + N^{-2}\,dr^2 + r^2\big(d\phi + N^\phi\,dt\big)^2 \tag{1}$$

with

$$N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}, \qquad N^\phi = -\frac{r_+ r_-}{l r^2}. \tag{2}$$

Here, r_+ and r_- denote the outer and inner horizons, respectively.

b) But, its mass and angular momentum are interchanged as

$$M = xj/l, \qquad J = xlm \tag{3}$$

with an appropriate coefficient x: $x = 1$ for the BCEA black hole [1], x is a fixed value of $U(1)$ field strength for the case of '(b)' [2], and x is proportional to the coefficient of the gravitational Chern–Simons term for the case of '(c)'. Here, m and j denote the usual mass and angular momentum for the BTZ black hole

$$m = \frac{r_+^2 + r_-^2}{8Gl^2}, \qquad j = \frac{2r_+ r_-}{8Gl} \tag{4}$$

with a negative cosmological constant $\Lambda = -1/l^2$. One remarkable result of (3) is that

$$(lM)^2 - J^2 = x^2\big[j^2 - (lm)^2\big] \leqslant 0 \tag{5}$$

for any non-vanishing x, which shows an upper bound for the mass M, with a saturation by the extremal case of $j^2 = (lm)^2$.

c) On the other hand, since it has the same form of the metric as the BTZ solution, it has the same form of the Hawking temperature and angular velocity of the event horizon r_+ as in the BTZ also

$$T_+ = \frac{\hbar\kappa}{2\pi}\bigg|_{r_+} = \frac{\hbar(r_+^2 - r_-^2)}{2\pi l^2 r_+},$$
$$\Omega_+ = -N^\phi\big|_{r_+} = \frac{r_-}{l r_+} \tag{6}$$

with the surface gravity function $\kappa = \partial N^2/2\partial r$. Now, by considering the first law of thermodynamics as

$$\delta M = \Omega_+ \delta J + T_+ \delta S \tag{7}$$

with T_+ and Ω_+ as the characteristic temperature and angular velocity of the system, one can easily determine the black hole entropy as

$$S = x\frac{2\pi r_-}{4G\hbar}. \tag{8}$$

There is no other choice in the entropy in this usual context [1,2,8,9]. But, a basic problem of this approach is that the second law of thermodynamics is not guaranteed with the entropy formula, which depends only on the inner-horizon area $A_- = 2\pi r_-$: Some of the assumptions for the Hawking's area theorem, i.e., cosmic censorship conjecture might not be valid for the inner horizon in general. Moreover, the usual instability of the inner horizon makes it difficult to apply the Raychaudhuri's equation to get the area theorem, even without worrying about other assumptions for the theorem; actually, this seems to be the situation that really occurs in our exotic black holes also [14,15].

Now, without the guarantee of the second law of thermodynamics, there is no justification for identifying entropy with the inner horizon area, even though its characteristic temperature and angular velocity have the usual identifications [12]. So, in order to avoid this problem, we need another form of the entropy which is *linearly* proportional to the outer horizon area $A_+ = 2\pi r_+$, following the Bekenstein's general argument [12], which should be valid in our case also, but then the first law would be satisfied with some another appropriate temperature and angular velocity. After some manipulation, one finds that the first law can be actually rearranged as

$$\delta M = \Omega_- \delta J + T_- \delta S_{\text{new}} \tag{9}$$

with the black hole entropy

$$S_{\text{new}} = x\frac{2\pi r_+}{4G\hbar} \tag{10}$$

and the characteristic temperature and angular velocity

$$T_- = \frac{\hbar\kappa}{2\pi}\bigg|_{r_-} = \frac{\hbar(r_-^2 - r_+^2)}{2\pi l^2 r_-},$$
$$\Omega_- = -N^\phi\big|_{r_-} = \frac{r_+}{l r_-} \tag{11}$$

for the inner horizon. Here, I note that the entropy (10), for the BCEA gravity [1], gives the exactly the same factor as the usual Bekenstein–Hawking formula, but it depends on other parameters in general [2,8,9].

With this new formulation, we have a dramatic departure from the usual situations. First, the angular velocity has a lower bound $\Omega_- \geqslant 1/l$ due to the fact of $r_+ \geqslant r_-$; it is saturated by the extremal case $r_+ = r_-$ and divergent for the vanishing inner horizon. This implies that this system is always rotating, as far as there is the event horizon r_+. Second, the temperature T_- and the surface gravity κ_- have negative values. [I used the definition of κ as $\nabla^\nu(\chi^\mu \chi_\mu) = -\kappa\chi^\nu$ for the horizon Killing vector χ^μ in order to determine its sign, as well as its magnitude.] The negative-valued temperature looks strange in the usual black hole context, but this is a well-established concept in the spin systems where some *upper bound* of the energy level exists [16]. Actually, this is exactly the same situation as in our case, due to the upper bound of mass in (5), and this provides a physical justification for introducing the negative temperature in our system also.[1] This would be probably the first example in the black hole systems where the negative temperature occurs.

[1] One might consider the positive-valued surface gravity and temperature with $T = |\kappa_-/(2\pi)|$ (as in [15]), but in this case one has an incorrect sign in front of the $T\,dS$ term in (9).

3. Statistical entropy

It is well known that the black hole entropy for the BTZ black hole can be also computed statistically using conformal field theory results [17,18]. So it is natural to expect the similar things in our case also since one has the same form of the metric as in the BTZ. Here I consider, in particular, the case of gravitational Chern–Simons gravity [3–9] which has been interested recently in the context of higher curvature gravities also [7–9] and whose conformal field theory analysis is evident; however, I suspect the similar results for the cases of '(a)' and '(b)' [1,2] also, although the explicit realizations would be different. There are several approaches to compute the statistical entropy from conformal field theory. Here, let me consider the Chern–Simons gauge theory approach in this Letter.

To do this, I first note the equivalence of

$$I_{\mathrm{CS}}\big[A^+\big] - I_{\mathrm{CS}}\big[A^-\big] = I_{\mathrm{GCS}}[e,\omega] \tag{12}$$

for the Chern–Simons gauge action and the gravitational Chern–Simons action [19],

$$I_{\mathrm{CS}}\big[A^\pm\big] = \pm\alpha\frac{k}{4\pi}\int d^3x\left\langle A^\pm\left(dA^\pm + \frac{2}{3}A^\pm A^\pm\right)\right\rangle,$$
$$I_{\mathrm{GCS}}[e,\omega] = -\frac{\alpha}{32\pi G}\int d^3x\left\langle \omega\left(d\omega + \frac{2}{3}\omega\omega\right) + \frac{e}{l^2}T\right\rangle, \tag{13}$$

respectively, with $A^\pm = A^\pm_a J^a = (\omega_a \pm e_a/l)J^a$, $\langle J_a J_b\rangle = (1/2)\eta_{ab}[\eta_{ab} = \mathrm{diag}(-1,1,1)]$, and $T = de + 2\omega e$ is the torsion 2-form. Then, it is easy to see that the BTZ solution (1) satisfies the equations of motion of gravitational Chern–Simons action $C^{\mu\nu} = 0$ with the Cotton tensor $C^{\mu\nu} = \epsilon^{\mu\rho\sigma}\nabla_\rho(R^\nu_\sigma - \delta^\nu_\sigma R/4)/\sqrt{g}$ [3].

Now then, it is straightforward to apply the usual result of Ref. [18], where the Virasoro algebras with *classical* central charges are obtained, since the whole computation is governed by the properties of BTZ solution (1) only. In this way, one can obtain (see Ref. [20] for the details) two sets of Virasoro algebras for the asymptotic isometry group $SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ with the classical central charges

$$c^\pm = \gamma^\pm\frac{3l}{2G} \tag{14}$$

with $\gamma^\pm = \pm\alpha/4l$ and the ground state generators

$$L_0^\pm = \gamma^\pm\frac{1}{2}(lm \pm j) + \frac{c^\pm}{24}. \tag{15}$$

Note that, if one identifies the first term in (15) with $(lM \pm J)/2$ as in the BTZ case [7–9], one finds that M and J are identified with those of (3) with $x = \alpha/(4l)$; however, my computation based on conformal field theory does not depend on the manner of identifications of M and J, but only on r_+ and r_-. With the data of (14) and (15), one can now compute the statistical entropy from the Cardy's formula for the asymptotic states [21] as[2]

$$S_{\mathrm{stat}} = \frac{2\pi}{\hbar}\sqrt{\frac{1}{6}\big(c^+ - L^+_{0(\mathrm{min})}\big)\left(L_0^+ - \frac{c^+}{24}\right)} + \frac{2\pi}{\hbar}\sqrt{\frac{1}{6}\big(c^- - L^-_{0(\mathrm{min})}\big)\left(L_0^- - \frac{c^-}{24}\right)} = \frac{2\pi r_+}{4G\hbar}\left|\frac{\alpha}{4l}\right|, \tag{16}$$

where I have chosen $L^\pm_{0(\mathrm{min})} = 0$ for the minimum value of $L_0^\pm$ as usual [17]; this corresponds to the AdS_3 (three-dimensional anti-de Sitter space) vacuum solution in the usual context, but it has a permanent rotation with the angular momentum $J = -(\alpha/2)(l/16G)$ and the vanishing mass $M = 0$ in our new context [7].

So, one finds an exact agreement for the case of $\alpha > 0$, where M, J, and S_{new} are positive definite, with my new entropy formula (10). Hence, the new entropy formula for the exotic black holes is supported by the statistical computation, based on conformal field theory. Note that, in this case, all $c^\pm$ and $L_0^\pm - c^\pm/24$ are not positive definite, but their self-compensations of the negative signs produce the positive entropy.[3] But for the case of $\alpha < 0$, where S_{new}, as well as M and J, becomes negative, the statistical counterpart does not exist in principle, from its definition $S_{\mathrm{stat}} = \ln\rho \geqslant 0$ for the number of possible states $\rho(\geqslant 1)$. So, it is not so surprising that we have found a disagreement in this latter case.

4. Summary and discussion

I have argued that even the exotic black holes with the interchanged masses and angular momenta have the black hole entropies with the usual Bekenstein–Hawking form, but now their characteristic temperatures and angular momenta are those of the inner horizons. I have found that the new entropy formula agrees with the statistical entropy, based on classical Virasoro algebras at the asymptotic infinity. In the statistical analysis I have considered only the case of gravitational Chern–Simons gravity, and it is believed that similar results would be obtained for the other two cases also. But, there are still several unanswered problems, and I will below describe the problems and some possible proposals for how these might be addressed.

(1) We know that black holes are thermal objects because they emit Hawking radiation with a thermal spectrum. In the

[2] If I consider the system with both the Einstein–Hilbert term as well as the gravitational Chern–Simons term as in Ref. [8], there is the inner-horizon's contribution also, in general. My result can be obtained from the general formula by considering $|\beta|/l \to \infty$ limit, where the inner-horizon's contribution is negligible. However, the resulting formula (5.7) of Ref. [8] does not do the job, and this is basically because it is valid only for $|\beta|/l < 1$ [20].

[3] The application of the Cardy's formula to the case of negative c and L_0 might be questioned due to the existence of negatives-norm states with the usual condition $L_n|h\rangle = 0 (n > 0)$ for the highest-weight state $|h\rangle$. However, this problem can be easily cured by considering another representation of the Virasoro algebra with $\hat{L}_n \equiv -L_{-n}$, $\hat{c} \equiv -c$, and $\hat{L}_n|\hat{h}\rangle = 0$ $(n > 0)$ for the new highest-weight state $|\hat{h}\rangle$ [22]. So, the formula (16), which is invariant under this substitution, can be understood in this more precise context also.

standard analysis initiated by Hawking [23], this spectrum is determined by the metric alone. However, this work implies that two black holes with identical BTZ metrics will emit radiation with different spectra, one a black body spectrum corresponding to a positive temperature T_+ for the ordinary black hole and one a very non-black-body spectrum corresponding to a negative temperature T_- for the exotic black holes. Then: "Can we give a plausible explanation of why the standard computation of black hole temperature should fail in the exotic cases?" and "How can we compute the Hawking radiation if the standard computation fails?".

This would be the most important but the most difficult question whose complete answer is still missing. But here, I would like to only mention the possible limitation of the standard approach in the exotic black hole case and how this *might* be circumvented. To this end, I first note that, in the standard computation of Hawking, the background metric is considered fixed such as the back-reaction effects are neglected. Now, the question is how much we can trust this approximation to get the *leading* Hawking radiation effects for the real dynamical geometry? In order to clarify this, let me consider a black hole with "rotation". Then, I note that we need to choose an appropriate coordinate, called *co-rotating* coordinate, with the condition $\tilde{N}^\phi \equiv N^\phi + \Omega_+ \equiv 0$ at the "outer" horizon r_+ in order to have a well-defined analysis, i.e., analyticity, *near* the outer horizon [24,25], where the Hawking radiation occurs. And also this makes the s-wave or WKB approximation to be justified [26] since the radial wave number approaches infinity near the horizon due to the coincidence of the infinite redshift surface and outer horizon, even for a rotating black hole. Now, let me turn to the "dynamical" geometry where the back-reaction effects during the emission process are considered. Then, it is easy to see that, for the emitted particles *without* carrying the angular momentum, the standard computation with a fixed background is perfectly well defined "at the initially fixed horizon $r_{+(i)}$", though the actual outer horizon shrinks dynamically at the loss of the emitted positive energy: With the initial choice of the co-rotating angular velocity Ω_+, one has still $\tilde{N}^\phi = 0$ at the initially fixed horizon $r_{+(i)}$ such as the infinite redshift surface agrees with the initial horizon in the co-rotating coordinate system. However, when there is a change of angular momentum, the situation is quite different. Actually, in this case there is a *finite* separation of the infinite redshift surface and the initial horizon if we take into account the loss of the angular momentum, i.e., $\tilde{N}^\phi|_{r_{+(i)}} = s/2r^2_{+(i)}$, due to angular momentum s of the emitted particles, with the initially chosen co-rotating angular velocity Ω_+. So, in the standard computation one does not know whether to use the angular velocity Ω_+ before emission, the angular velocity after emission, or something in between when consider the co-rotating coordinate system. This problem looks similar to the situation in the near extremal black holes when determine a thermal temperature [27], but it would be qualitatively different.

Now, let me explain why this might be relevant to the possible failure of the standard computation for the exotic black holes. Here, the important point is that, for the exotic black holes, the emission of energy ω with an initially chosen co-rotating coordinate system would reduce the black holes's mass M from the conservation of energy, but this corresponds to the change of the angular momentum j of (4) in the ordinary BTZ black hole context, due to the interchange of the roles of the mass and angular momentum as in (3). This is in sharp contrast to the case of ordinary black hole. This seems to be a key point to understand the peculiar Hawking radiation for the exotic black holes, and in this argument the conservations of energy and angular momentum, which are not well enforced in the standard computation, have a crucial role. So in this respect, the Parikh and Wilczek's approach [28], which provides a direct derivation of Hawking radiation as a quantum tunneling by considering the global conservation law naturally, would be an appropriate framework to study the problem. But before this, we first need to study the self-gravitating shells with rotation in Hamiltonian gravity for our exotic black hole system, as a generalization of Kraus and Wilczek's [29]. Currently this is under study.

(2) The Green's function methods for determining the temperature of a black hole require an equilibrium with matter at the corresponding temperature [25]. This work now implies that the analysis assumes such an equilibrium with "some exotic surrounding matter" which has a negative temperature, with an upper bound of energy levels as in spin systems: Otherwise, i.e., with the ordinary surrounding matter, the negative temperature black hole cannot be at equilibrium with positive temperature surroundings since an object with a negative temperature is hotter than one with any positive temperature. Then: "How one could build a container with walls held at a negative temperature in order that such an equilibrium can exist—the Universe might have to be filled with such "exotic matter"?".

This would be a physically interesting question which might be relevant to understand our Universe with a dark side. But I suspect that the resolution would be rather simple in our case from the fact that in the anti-de Sitter space the artificial container is not needed in order to study the canonical (or grand-canonical) ensemble [30,31]. But, in the context without the explicit container, there is a critical angular velocity [31] at which the action of the black hole or the partition function of its corresponding conformal field theory diverges. However, I note that the critical value is the same as the lower bound of Ω_- such as we are beyond the critical point with our angular velocity Ω_-. So, from this fact, it seems clear that the ensemble, if there is, in this strong coupling regime would be quite different from that of the usual BTZ black hole such as one cannot simply apply the usual result to the strong coupling case. It seems that we need an independent analysis for this case. But presumably, the ensemble may be still defined even in the strong coupling case, due to the symmetry of the BTZ metric under the $r_+ \leftrightarrow r_-$ exchange.

Finally, I would like to remark that in the standard Green's function approach the determination of the equilibrium temperature from the "fundamental period", known as the KMS (Kugo–Martin–Schwinger) condition [32–34], can be defined without the implicit assumption of a positive temperature, though not quite well known in the gravity community (see Ref. [35], for example). Physicswise, this should also be the case since the negative temperature is perfectly well defined in the ordinary statistical mechanics of spin systems and its Green's function formulation similarly will reflect the same temperature, if there is.

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