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Lyapunov-type inequality for a class of Dirichlet quasilinear systems involving the $(p_1, p_2, \ldots, p_n)$-Laplacian

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ABSTRACT

We state and prove a generalized Lyapunov-type inequality for one-dimensional Dirichlet quasilinear systems involving the $(p_1, p_2, \ldots, p_n)$-Laplacian. Our result generalize the Lyapunov-type inequality given in Napoli and Pinasco (2006) [12].

1. Introduction

The Lyapunov inequality and many of its generalizations have proven to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations. For authors, who contributed to the Lyapunov-type inequalities, we refer to Brown and Hinton [2], Cheng [3], Çakmak [4], Došlý and Řehák [5], Hartman [7], Kwong [9], Lee et al. [10], Liapunov [11], Pachpatte [13–15], Panigrahi [16], Parhi and Panigrahi [17,18], Pinasco [20], Yang and Lo [24], and the references quoted therein.

Although there is an extensive literature on the Lyapunov-type inequalities for various classes of differential equations, there is not much done for the linear Hamiltonian systems. Recently, Guseinov and Kaymakçalan [6], and Tiryaki, Ünal and Çakmak [21] have obtained the Lyapunov-type inequalities for first order systems. The discrete and time scale analogues of Lyapunov-type inequalities for certain type systems are also given by Ünal, Çakmak and Tiryaki [22], Jiang and Zhou [8], and Ünal and Çakmak [23].

More recently, Napoli and Pinasco [12] have interested in the problem of finding the Lyapunov-type inequality for the following quasilinear systems involving (p, q)-Laplacian operators

$$\left.\begin{aligned} -\left(\left|u'\right|^{p-2}u'\right)' &= f_1(x)|u|^{\alpha-2}u|v|^{\beta} \\ -\left(\left|v'\right|^{q-2}v'\right)' &= f_2(x)|u|^{\alpha}|v|^{\beta-2}v \end{aligned}\right\} \tag{1}$$

where f_1, f_2 are real-valued continuous functions for all $x \in \mathbb{R}$, the exponents satisfy $1 < p, q < \infty$, and the positive parameters α, β satisfy

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1. \tag{2}$$

Their results are as follows:

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Theorem A. *If system* (1) *with $f_i(x) > 0$ for $i = 1, 2$ has a real nontrivial solution $(u(x), v(x))$ such that $u(a) = u(b) = 0 = v(a) = v(b)$ where $a, b \in \mathbb{R}$ with $a < b$ be consecutive zeros, and u and v are not identically zero on $[a, b]$, then the inequality*

$$2^{\alpha+\beta} \leqslant (b-a)^{\alpha+\beta-1}\left(\int_a^b f_1(x)\,dx\right)^{\alpha/p}\left(\int_a^b f_2(x)\,dx\right)^{\beta/q} \tag{3}$$

holds.

This result was used by proving the following theorem which improves the lower bounds on the eigenvalues of the problem

$$\left.\begin{aligned} -\left(\left|u'\right|^{p-2}u'\right)' &= \lambda\alpha r(x)|u|^{\alpha-2}u|v|^{\beta} \\ -\left(\left|v'\right|^{q-2}v'\right)' &= \mu\beta r(x)|u|^{\alpha}|v|^{\beta-2}v \end{aligned}\right\} \tag{4}$$

where $r(x)$ be a positive function for all $x \in \mathbb{R}$.

Theorem B. *There exists a function $h(\lambda)$ such that $\mu \geqslant h(\lambda)$ for every generalized eigenvalue (λ, μ) of problem* (4), *where $h(\lambda)$ is given by*

$$h(\lambda) = \frac{1}{\beta}\left(\frac{C}{\lambda^{\alpha/p}\int_a^b r(x)\,dx}\right)^{q/\beta}, \tag{5}$$

and the constant C is given by

$$C = \frac{2^{\alpha+\beta}}{\alpha^{\alpha/p}(b-a)^{\alpha+\beta-1}}. \tag{6}$$

In this paper, by using a similar technique to that of Napoli and Pinasco [12], we state and prove a generalized Lyapunov-type inequality for a Dirichlet problem associated to the following quasilinear systems involving $(p_1, p_2, \ldots, p_n)$-Laplacian operators

$$\left.\begin{aligned} -\left(\left|u_1'\right|^{p_1-2}u_1'\right)' &= f_1(x)|u_1|^{\alpha_1-2}u_1|u_2|^{\alpha_2}\ldots|u_n|^{\alpha_n} \\ -\left(\left|u_2'\right|^{p_2-2}u_2'\right)' &= f_2(x)|u_1|^{\alpha_1}|u_2|^{\alpha_2-2}u_2|u_3|^{\alpha_3}\ldots|u_n|^{\alpha_n} \\ &\cdots \\ -\left(\left|u_n'\right|^{p_n-2}u_n'\right)' &= f_n(x)|u_1|^{\alpha_1}\ldots|u_{n-1}|^{\alpha_{n-1}}|u_n|^{\alpha_n-2}u_n \end{aligned}\right\} \tag{7}$$

where $n \in \mathbb{N}$, f_i are real-valued continuous functions for all $x \in \mathbb{R}$, the exponents satisfy $1 < p_i < \infty$ and the positive parameters α_i satisfy

$$\sum_{i=1}^{n}\frac{\alpha_i}{p_i} = 1 \tag{8}$$

for $i = 1, 2, \ldots, n$. If $n = 1$, then system (7) reduces to the following equation

$$\left(\left|u_1'\right|^{p_1-2}u_1'\right)' + f_1(x)|u_1|^{p_1-2}u_1 = 0 \tag{9}$$

which is known as the half linear equation. Similarly, if $n = 2$, then system (7) reduces to the system

$$\left.\begin{aligned} -\left(\left|u_1'\right|^{p_1-2}u_1'\right)' &= f_1(x)|u_1|^{\alpha_1-2}u_1|u_2|^{\alpha_2} \\ -\left(\left|u_2'\right|^{p_2-2}u_2'\right)' &= f_2(x)|u_1|^{\alpha_1}|u_2|^{\alpha_2-2}u_2 \end{aligned}\right\} \tag{10}$$

which is the same system as (1).

The aim of this paper is to extend and generalize Theorems A and B of Napoli and Pinasco [12] to the general case. Our motivation comes from the recent paper of Afrouzi and Heidarkhani [1].

We derive a Lyapunov-type inequality for quasilinear system (7), where all components of the solution $(u_1(x), u_2(x), \ldots, u_n(x))$ have consecutive zeros at the points $a, b \in \mathbb{R}$ with $a < b$ in $I = [t_0, \infty) \subset \mathbb{R}$. For the special cases of system (7), we also derive some Lyapunov-type inequalities which relates not only points a and b in I at which all components of the solution $(u_1(x), u_2(x), \ldots, u_n(x))$ have consecutive zeros but also any point in (a, b) where all components of the solution $(u_1(x), u_2(x), \ldots, u_n(x))$ are maximized.

Since our attention is restricted to the Lyapunov-type inequality for the quasilinear system of differential equations, we shall assume the existence of the nontrivial solution $(u_1(x), u_2(x), \ldots, u_n(x))$ of system (7).

2. Main results

The main result of this paper is the following theorem:

Theorem 1. *If system* (7) *has a real nontrivial solution* $(u_1(x), u_2(x), \dots, u_n(x))$ *such that* $u_i(a) = 0 = u_i(b)$ *for* $i = 1, 2, \dots, n$ *where* $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* u_i *for* $i = 1, 2, \dots, n$ *are not identically zero on* $[a, b]$, *then the inequality*

$$\prod_{i=1}^{n} \left\{ (c_i - a)^{1-p_i} + (b - c_i)^{1-p_i} \right\}^{\alpha_i/p_i} \leqslant \prod_{i=1}^{n} \left(\int_a^b f_i^+(x)\,dx \right)^{\alpha_i/p_i} \tag{11}$$

holds, where $|u_i(c_i)| = \max_{a<x<b} |u_i(x)|$ *and* $f_i^+(x) = \max\{0, f_i(x)\}$ *for* $i = 1, 2, \dots, n$.

Proof. It follows from $u_i(a) = 0 = u_i(b)$ for $i = 1, 2, \dots, n$ where $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros and u_i for $i = 1, 2, \dots, n$ are not identically zero on $[a, b]$, one can choose $c_i \in (a, b)$ such that $|u_i(c_i)| = \max_{a<x<b} |u_i(x)| > 0$ for $i = 1, 2, \dots, n$. From Rolle's theorem, clearly $u_i'(c_i) = 0$ for $i = 1, 2, \dots, n$. Therefore, for $c_1 \in (a, b)$ and $u_1(a) = 0$, we have

$$\left|u_1(c_1)\right| = \left| \int_a^{c_1} u_1'(x)\,dx \right| \leqslant \int_a^{c_1} \left|u_1'(x)\right| dx. \tag{12}$$

By using Hölder inequality on the integral of the right-hand side of (12) with indices p_1 and p_1', we obtain

$$\left|u_1(c_1)\right| \leqslant \int_a^{c_1} \left|u_1'(x)\right| dx \leqslant (c_1 - a)^{\frac{1}{p_1'}} \left(\int_a^{c_1} \left|u_1'(x)\right|^{p_1} dx \right)^{\frac{1}{p_1}} \tag{13}$$

where $\frac{1}{p_1} + \frac{1}{p_1'} = 1$. On the other hand, multiplying the first equation of system (7) by u_1 and integrating from a to c_1 and taking into account that $u_1(a) = 0$ and $u_1'(c_1) = 0$, we get

$$\int_a^{c_1} \left|u_1'(x)\right|^{p_1} dx = \int_a^{c_1} f_1(x) \left|u_1(x)\right|^{\alpha_1} \left|u_2(x)\right|^{\alpha_2} \dots \left|u_n(x)\right|^{\alpha_n} dx. \tag{14}$$

Therefore, by using (14) in (13), we have

$$\begin{aligned} \left|u_1(c_1)\right| &\leqslant (c_1 - a)^{\frac{1}{p_1'}} \left(\int_a^{c_1} f_1(x) \left|u_1(x)\right|^{\alpha_1} \left|u_2(x)\right|^{\alpha_2} \dots \left|u_n(x)\right|^{\alpha_n} dx \right)^{\frac{1}{p_1}} \\ &\leqslant (c_1 - a)^{\frac{1}{p_1'}} \left(\int_a^{c_1} f_1^+(x) \left|u_1(x)\right|^{\alpha_1} \left|u_2(x)\right|^{\alpha_2} \dots \left|u_n(x)\right|^{\alpha_n} dx \right)^{\frac{1}{p_1}}. \end{aligned} \tag{15}$$

If we take the p_1-th power of both sides of inequality (15), and $|u_i(x)|$ is maximum at the point c_i for $i = 1, 2, \dots, n$, respectively, we obtain

$$1 \leqslant \left|u_1(c_1)\right|^{\alpha_1 - p_1} \left|u_2(c_2)\right|^{\alpha_2} \dots \left|u_n(c_n)\right|^{\alpha_n} (c_1 - a)^{p_1 - 1} \left(\int_a^{c_1} f_1^+(x)\,dx \right). \tag{16}$$

Now, since $u_1(b) = 0$, we get

$$\left|u_1(c_1)\right| = \left|-u_1(c_1)\right| = \left| \int_{c_1}^{b} u_1'(x)\,dx \right| \leqslant \int_{c_1}^{b} \left|u_1'(x)\right| dx \tag{17}$$

and repeating the above procedure step by step, one can easily obtain

$$1 \leqslant \left|u_1(c_1)\right|^{\alpha_1 - p_1} \left|u_2(c_2)\right|^{\alpha_2} \dots \left|u_n(c_n)\right|^{\alpha_n} (b - c_1)^{p_1 - 1} \left(\int_{c_1}^{b} f_1^+(x)\,dx \right). \tag{18}$$

Thus, summing up inequalities (16) and (18), we have

$$\int_a^b f_1^+(x)\,dx \geqslant \left|u_1(c_1)\right|^{p_1-\alpha_1}\left|u_2(c_2)\right|^{-\alpha_2}\ldots\left|u_n(c_n)\right|^{-\alpha_n}\left\{(c_1-a)^{1-p_1}+(b-c_1)^{1-p_1}\right\}. \tag{19}$$

By using similar manner, we get the following inequalities

$$\int_a^b f_2^+(x)\,dx \geqslant \left|u_1(c_1)\right|^{-\alpha_1}\left|u_2(c_2)\right|^{p_2-\alpha_2}\left|u_3(c_3)\right|^{-\alpha_3}\ldots\left|u_n(c_n)\right|^{-\alpha_n}\left\{(c_2-a)^{1-p_2}+(b-c_2)^{1-p_2}\right\}, \tag{20}$$

$$\ldots$$

$$\int_a^b f_n^+(x)\,dx \geqslant \left|u_1(c_1)\right|^{-\alpha_1}\ldots\left|u_{n-1}(c_{n-1})\right|^{-\alpha_{n-1}}\left|u_n(c_n)\right|^{p_n-\alpha_n}\left\{(c_n-a)^{1-p_n}+(b-c_n)^{1-p_n}\right\}, \tag{21}$$

respectively. Raising inequality (19) to a power e_1, inequality (20) to a power $e_2, \ldots,$ and inequality (21) to a power e_n, and multiplying the resulting equations, we obtain

$$\left.\begin{aligned}\prod_{i=1}^n\left(\int_a^b f_i^+(x)\,dx\right)^{e_i} \geqslant{}& \left[\prod_{i=1}^n\left\{(c_i-a)^{1-p_i}+(b-c_i)^{1-p_i}\right\}^{e_i}\right]\\ &\times\left|u_1(c_1)\right|^{(p_1-\alpha_1)e_1-\alpha_1e_2-\cdots-\alpha_1e_n}\\ &\times\left|u_2(c_2)\right|^{-\alpha_2e_1+(p_2-\alpha_2)e_2-\alpha_2e_3-\cdots-\alpha_2e_n}\\ &\times\cdots\times\left|u_n(c_n)\right|^{-\alpha_ne_1-\cdots-\alpha_ne_{n-1}+(p_n-\alpha_n)e_n}\end{aligned}\right\} \tag{22}$$

for $i=1,2,\ldots,n$. Now, we choose e_i such that $|u_i(c_i)|$ cancel out for $i=1,2,\ldots,n$ in inequality (22), i.e. solve the homogeneous linear system

$$\left.\begin{aligned}&(p_1-\alpha_1)e_1-\alpha_1e_2-\cdots-\alpha_1e_n=0\\ &-\alpha_2e_1+(p_2-\alpha_2)e_2-\alpha_2e_3-\cdots-\alpha_2e_n=0\\ &\qquad\qquad\ldots\\ &-\alpha_ne_1-\cdots-\alpha_ne_{n-1}+(p_n-\alpha_n)e_n=0.\end{aligned}\right\} \tag{23}$$

We observe that by hypothesis $\sum_{i=1}^n \frac{\alpha_i}{p_i}=1$, this system admits a nontrivial solution, indeed all equations are equivalent to

$$e_i\left(\sum_{\substack{j=1\\ j\neq i}}^n \frac{\alpha_j}{p_j}\right)=\frac{\alpha_i}{p_i}\left(\sum_{\substack{j=1\\ j\neq i}}^n e_j\right) \tag{24}$$

for $i=1,2,\ldots,n$. Hence, we may take $e_i=\frac{\alpha_i}{p_i}$ for $i=1,2,\ldots,n$, and we get inequality (11) which completes the proof. □

Remark 2. The left-hand side of inequality (11) shows that c_i for $i=1,2,\ldots,n$ cannot be too close to a or b, since the exponents satisfy $1<p_i<\infty$ for $i=1,2,\ldots,n$. We have $\int_a^b f_i^+(x)\,dx<\infty$ for $i=1,2,\ldots,n$, but

$$\lim_{\substack{c_i\to a^+\\ c_i\to b^-}}\left\{(c_i-a)^{1-p_i}+(b-c_i)^{1-p_i}\right\}=\infty \tag{25}$$

for $i=1,2,\ldots,n$.

Since the function $h(x)=x^{1-p_i}$ is convex for $x>0$ and $p_i>1$ for $i=1,2,\ldots,n$, Jensen's inequality $h(\frac{y+z}{2})\leqslant\frac{1}{2}[h(y)+h(z)]$ with $y=c_i-a$ and $z=b-c_i$ for $i=1,2,\ldots,n$ implies

$$(c_i-a)^{1-p_i}+(b-c_i)^{1-p_i}\geqslant 2^{p_i}(b-a)^{1-p_i} \tag{26}$$

for $i=1,2,\ldots,n$. Thus, by using inequality (26), Theorem 1 reduces to the following result:

Corollary 3. *If system* (7) *has a real nontrivial solution* $(u_1(x), u_2(x), \dots, u_n(x))$ *such that* $u_i(a) = 0 = u_i(b)$ *for* $i = 1, 2, \dots, n$ *where* $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros and* u_i *for* $i = 1, 2, \dots, n$ *are not identically zero on* $[a, b]$, *then the inequality*

$$2^{(\sum_{i=1}^{n}\alpha_i)}(b-a)^{1-(\sum_{i=1}^{n}\alpha_i)} \leqslant \prod_{i=1}^{n}\left(\int_a^b f_i^+(x)\,dx\right)^{\alpha_i/p_i} \tag{27}$$

holds, where $|u_i(c_i)| = \max_{a<x<b}|u_i(x)|$ *and* $f_i^+(x) = \max\{0, f_i(x)\}$ *for* $i = 1, 2, \dots, n$.

Remark 4. Let $n = 1$. If we compare inequalities (16) and (18) with the inequalities of (2.3) in Lemma 2.1 of Pinasco [19] respectively, it is easy to see that the restricted condition, i.e. a bounded positive function, on the function r in Lemma 2.1 can be dropped. Thus, Theorem 1 (or Corollary 3) generalizes and extends Theorem 2.3 of Pinasco [19].

Remark 5. Let $n = 2$. If we compare Theorem 1 with Theorem A of Napoli and Pinasco [12], since (26) holds, we conclude that Theorem 1 is more general than Theorem A.

Remark 6. Corollary 3 with $n = 2$ and $f_i(x) > 0$ for $i = 1, 2$ reduces to Theorem A.

Remark 7. When $\alpha_i = p_i$ for $i = 1, 2, \dots, n$, and for $j \neq i$, $\alpha_j = 0$ for $j = 1, 2, \dots, n$, we obtain the result for the case of a single equation from Theorem 1 or Corollary 3.

Remark 8. Since

$$f^+(x) \leqslant \left|f(x)\right|, \tag{28}$$

the integrals of $\int_a^b f_i^+(x)\,dx$ for $i = 1, 2, \dots, n$ in the above results can also be replaced by $\int_a^b |f_i(x)|\,dx$ for $i = 1, 2, \dots, n$, respectively.

Now, we present an application of the obtained Lyapunov-type inequality for system (7).

Let λ_i for $i = 1, 2, \dots, n$ be generalized eigenvalues of system (7), and $r(x)$ be a positive function for all $x \in \mathbb{R}$. Therefore, system (7) with $f_i(x) = \lambda_i\alpha_i r(x) > 0$ for $i = 1, 2, \dots, n$ and all $x \in \mathbb{R}$ reduces to the following system:

$$\left.\begin{aligned}
-\left(\left|u_1'\right|^{p_1-2}u_1'\right)' &= \lambda_1\alpha_1 r(x)|u_1|^{\alpha_1-2}u_1|u_2|^{\alpha_2}\dots|u_n|^{\alpha_n}\\
-\left(\left|u_2'\right|^{p_2-2}u_2'\right)' &= \lambda_2\alpha_2 r(x)|u_1|^{\alpha_1}|u_2|^{\alpha_2-2}u_2|u_3|^{\alpha_3}\dots|u_n|^{\alpha_n}\\
&\dots\\
-\left(\left|u_n'\right|^{p_n-2}u_n'\right)' &= \lambda_n\alpha_n r(x)|u_1|^{\alpha_1}\dots|u_{n-1}|^{\alpha_{n-1}}|u_n|^{\alpha_n-2}u_n.
\end{aligned}\right\} \tag{29}$$

By using similar techniques to the technique in Napoli and Pinasco [12], we obtain the following result which gives lower bounds for the n-th eigenvalue of λ_n. The proof of the following theorem is based on above generalization of the Lyapunov-type inequality, as in that of Theorem 1.4 of Napoli and Pinasco [12].

Theorem 9. *There exists a function* $h_1(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$ *such that* $\lambda_n \geqslant h_1(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$ *for every generalized eigenvalue* $(\lambda_1, \lambda_2, \dots, \lambda_n)$ *of the quasilinear system* (29), *where* $|u_i(c_i)| = \max_{a<x<b}|u_i(x)|$ *for* $i = 1, 2, \dots, n$ *and*

$$h_1(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) = \frac{1}{\alpha_n}\left(\frac{\prod_{i=1}^{n}\{(c_i-a)^{1-p_i} + (b-c_i)^{1-p_i}\}^{\alpha_i/p_i}}{(\prod_{i=1}^{n-1}(\lambda_i\alpha_i)^{\alpha_i/p_i})(\int_a^b r(x)\,dx)}\right)^{p_n/\alpha_n}. \tag{30}$$

Proof. Let $(\lambda_1, \lambda_2, \dots, \lambda_n)$ be a generalized eigenpair, and $u_1, u_2, \dots, u_n$ be the corresponding nontrivial solution of system (29). For $i = 1, 2, \dots, n$ and all $x \in \mathbb{R}$, by substituting $\lambda_i\alpha_i r(x) > 0$ for $f_i(x)$ in the Lyapunov inequality (11), we obtain

$$\prod_{i=1}^{n}\left\{(c_i-a)^{1-p_i} + (b-c_i)^{1-p_i}\right\}^{\alpha_i/p_i} \leqslant \prod_{i=1}^{n}\left(\int_a^b \lambda_i\alpha_i r(x)\,dx\right)^{\alpha_i/p_i}. \tag{31}$$

Rearranging the terms, and by using condition $\sum_{i=1}^{n}\frac{\alpha_i}{p_i} = 1$, we obtain

$$\prod_{i=1}^{n}\left\{(c_i-a)^{1-p_i} + (b-c_i)^{1-p_i}\right\}^{\alpha_i/p_i} \leqslant \left(\prod_{i=1}^{n}(\lambda_i\alpha_i)^{\alpha_i/p_i}\right)\left(\int_a^b r(x)\,dx\right), \tag{32}$$

which gives

$$\left(\frac{\prod_{i=1}^{n}\{(c_i-a)^{1-p_i}+(b-c_i)^{1-p_i}\}^{\alpha_i/p_i}}{(\prod_{i=1}^{n-1}(\lambda_i\alpha_i)^{\alpha_i/p_i})(\int_a^b r(x)\,dx)}\right)^{p_n/\alpha_n} \leqslant \lambda_n\alpha_n, \tag{33}$$

and the proof is completed. □

Remark 10. Since h_1 is a continuous function, then $h_1(\lambda_1,\lambda_2,\dots,\lambda_{n-1}) \to +\infty$ as any eigenvalue of $\lambda_i \to 0^+$ for $i = 1,2,\dots,n-1$. Therefore, there exists a ball centered in the origin such that the generalized spectrum is contained in its exterior. Also, by rearranging terms in (33) we obtain

$$\prod_{i=1}^{n}\lambda_i^{\alpha_i/p_i} \geqslant \left(\frac{\prod_{i=1}^{n}\{(c_i-a)^{1-p_i}+(b-c_i)^{1-p_i}\}^{\alpha_i/p_i}}{(\prod_{i=1}^{n}\alpha_i^{\alpha_i/p_i})(\int_a^b r(x)\,dx)}\right). \tag{34}$$

It is clear that when the interval collapses, right-hand side of (34) goes to infinity. Hence, we obtain the desired generalizations of Napoli and Pinasco [12]'s result for one-dimensional nonlinear systems.

Remark 11. Let $n=1$. If we compare inequality (34) with inequality (1.2) with $n=1$ in Theorem 1.1 of Pinasco [19], it is easy to see that inequality (34) gives better lower bound than inequality (1.2) with $n=1$. Thus, Theorem 9 generalizes and extends Theorem 1.1 with $n=1$ of Pinasco [19].

Remark 12. Let $n=2$. If we compare Theorem 9 with Theorem B of Napoli and Pinasco [12], we obtain $h_1(\lambda_1) \geqslant h(\lambda)$ since (26) holds. Therefore, Theorem 9 gives better lower bound than Theorem B.

Remark 13. Let $n=2$. Theorem 9 with $c_1 = \frac{a+b}{2} = c_2$ or with inequality (26) reduces to Theorem B.

References

[1] G.A. Afrouzi, S. Heidarkhani, Existence of three solutions for a class of Dirichlet quasilinear elliptic systems involving the $(p_1,p_2,\dots,p_n)$-Laplacian, Nonlinear Anal. 70 (2009) 135–143.
[2] R.C. Brown, D.B. Hinton, Opial's inequality and oscillation of 2nd order equations, Proc. Amer. Math. Soc. 125 (1997) 1123–1129.
[3] S.S. Cheng, Lyapunov inequalities for differential and difference equations, Fasc. Math. 23 (1991) 25–41.
[4] D. Çakmak, Lyapunov-type integral inequalities for certain higher order differential equations, Appl. Math. Comput. 216 (2010) 368–373.
[5] O. Došlý, P. Řehák, Half-Linear Differential Equations, Math. Stud., vol. 202, North-Holland, 2005.
[6] G. Guseinov, B. Kaymakçalan, Lyapunov inequalities for discrete linear Hamiltonian system, Comput. Math. Appl. 45 (2003) 1399–1416.
[7] P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964, Birkhäuser, Boston, 1982.
[8] L. Jiang, Z. Zhou, Lyapunov inequality for linear Hamiltonian systems on time scales, J. Math. Anal. Appl. 310 (2005) 579–593.
[9] M.K. Kwong, On Lyapunov's inequality for disfocality, J. Math. Anal. Appl. 83 (1981) 486–494.
[10] C. Lee, C. Yeh, C. Hong, R.P. Agarwal, Lyapunov and Wirtinger inequalities, Appl. Math. Lett. 17 (2004) 847–853.
[11] A.M. Liapunov, Probleme général de la stabilit é du mouvement (French translation of a Russian paper dated 1893), Ann. Fac. Sci. Univ. Toulouse 2 (1907) 27–247, reprinted in Ann. of Math. Stud. 17 (1947).
[12] P.L. Napoli, J.P. Pinasco, Estimates for eigenvalues of quasilinear elliptic systems, J. Differential Equations 227 (2006) 102–115.
[13] B.G. Pachpatte, On Lyapunov-type inequalities for certain higher order differential equations, J. Math. Anal. Appl. 195 (1995) 527–536.
[14] B.G. Pachpatte, Lyapunov type integral inequalities for certain differential equations, Georgian Math. J. 4 (2) (1997) 139–148.
[15] B.G. Pachpatte, Inequalities related to the zeros of solutions of certain second order differential equations, Facta Univ. Ser. Math. Inform. 16 (2001) 35–44.
[16] S. Panigrahi, Lyapunov-type integral inequalities for certain higher order differential equations, Electron. J. Differential Equations 2009 (28) (2009) 1–14.
[17] N. Parhi, S. Panigrahi, On Liapunov-type inequality for third-order differential equations, J. Math. Anal. Appl. 233 (2) (1999) 445–460.
[18] N. Parhi, S. Panigrahi, Liapunov-type inequality for higher order differential equations, Math. Slovaca 52 (1) (2002) 31–46.
[19] J.P. Pinasco, Lower bounds for eigenvalues of the one-dimensional p-Laplacian, Abstr. Appl. Anal. 2004 (2) (2004) 147–153.
[20] J.P. Pinasco, Comparison of eigenvalues for the p-Laplacian with integral inequalities, Appl. Math. Comput. 182 (2006) 1399–1404.
[21] A. Tiryaki, M. Ünal, D. Çakmak, Lyapunov-type inequalities for nonlinear systems, J. Math. Anal. Appl. 332 (2007) 497–511.
[22] M. Ünal, D. Çakmak, A. Tiryaki, A discrete analogue of Lyapunov-type inequalities for nonlinear systems, Comput. Math. Appl. 55 (2008) 2631–2642.
[23] M. Ünal, D. Çakmak, Lyapunov-type inequalities for certain nonlinear systems on time scales, Turkish J. Math. 32 (2008) 255–275.
[24] X. Yang, K. Lo, Lyapunov-type inequality for a class of even-order differential equations, Appl. Math. Comput. 215 (2010) 3884–3890.