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Controllability and Stability

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INTRODUCTION

In this paper we shall study control systems of the form:

$$\frac{dx}{dt} = X(x) + u(t)\, Y(x), \tag{1}$$

where X and Y are analytic vector fields on an n-dimensional real manifold M, and where the control u is a piecewise continuous function defined on $[0, \infty)$. In particular we shall be interested in the relation between controllability and stabilizability of systems described by (1).

The paper will be divided into four sections. The first section consists of notation and the basic definitions. The second section deals with a controllability theorem (Theorem 1). This theorem is a slight generalization of a theorem contained in [3], and its proof is along the lines presented in [1]. The third section deals with global stabilizability (Theorem 2). There we also make a connection between controllability and the existence of stabilizing feedback functions. Finally, Section IV contains examples.

1. NOTATION AND THE BASIC DEFINITIONS

We let $\mathscr{U}$ be the class of all real valued piecewise continuous functions defined on $[0, \infty)$, and we refer to them as the class of admissible controls. If X and Y are a pair of vector fields on M, then $y \in M$ is said to be (X, Y) *accessible from* $x \in M$ if there exist $u \in \mathscr{U}$ and $T > 0$ such that the corresponding solution $x(t)$ of (1) satisfies $x(0) = x$ and $x(T) = y$. In such a case we will also say

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that x *can be* (X, Y) *joined to* y. *The set of all points which are* (X, Y) *accessible from* x *we shall denote by* $A^+(x)$, and *the set of all points which can be* (X, Y) *joined to* x *we denote by* $A^-(x)$. Clearly both $A^+(x)$ and $A^-(x)$ depend on X and Y but for the sake of notational simplicity we omit this dependence in our notation. We will say that a pair of vector fields (X, Y) *is controllable* if $A^+(x) = M$ for all x in M.

It follows that (X, Y) is controllable if and only if there exists a point $x \in M$ such that $A^+(x) = A^-(x) = M$.

If $x_0 \in M$ is such that $X(x_0) = 0$, then (X, Y) *is said to be stabilizable at* x_0 *if there exists an analytic function* f *on* M *with* $f(x_0) = 0$ *such that* x_0 *is a globally asymptotically stable point of the vector field* $X + fY$. This means that

(i) x_0 is a stable point of $X + fY$, and

(ii) $\{x_0\}$ is the positive limit set of each integral curve of $X + fY$.

In the above context, *such an* f *will be termed a stabilizing feedback control.* Every integral curve $x(t)$ of $X + fY$ corresponds to a solution of (1) generated by the control $u(t) = f(x(t))$. Hence, if (X, Y) is stabilizable at x_0, then every point of M can be (X, Y) joined to points arbitrarily close to x_0. If in addition $A^-(x_0)$ contains x_0 in its interior then, necessarily $A^-(x) = M$.

If X and Y are vector fields on M, then $[X, Y]$ denotes their Lie bracket. Recall that $[X, Y]$ is a vector field defined by $[X, Y]f = X(Yf) - Y(Xf)$ for each analytic function f. We will regard the set of all analytic vector fields on M as a Lie algebra under the operations of the pointwise addition and that of the Lie bracket. For any pair of vector fields (X, Y) we will denote by $\mathscr{L}(X, Y)$ the Lie algebra generated by X and Y. For each $x \in M$ we let $\mathscr{L}(X, Y)(x) = \{V(x): V \in \mathscr{L}(X, Y)\}$. Thus, if X and Y are given vector fields, $\mathscr{L}(X, Y)$ is in general an infinite dimensional algebra, but at each point $x \in M$, $\mathscr{L}(X, Y)(x)$ is a linear subspace of M_x the tangent space of M at x.

If Z is any smooth vector field, we let $e^{tZ}(x)$ *be the maximal integral curve of* Z *which passes through* x *at* $t = 0$. Recall that this means that there exists an interval I containing the origin such that

(i) $(d/dt)\, e^{tZ}(x) = Z(e^{tZ}(x))$ for all $t \in I$, and that $e^{tZ}x = x$ for $t = 0$.

(ii) if $\sigma(t)$ is any other curve defined on an interval I' which satisfies (i) then $I' \subset I$.

For technical simplicity *we will assume throughout this paper that all vector fields are complete*, i.e., that their maximal integral curves are defined for all $t \in \mathbf{R}$.

If Z is a complete vector field, then for each $t \in \mathbf{R}$, the map $x \to e^{tZ}(x)$ is a diffeomorphism on M. We will denote by e^{tZ} such a map.

As it is well known, the family $\{e^{tZ}\}$ form a one-parameter group of diffeomorphisms on M.

2. Controllability

In studying controllability properties of systems of the form given by (1), it is convenient to recast the problem in certain algebraic terms.

If X and Y are given vectors fields, we denote by $\mathscr{G}(X, Y)$ the group of diffeomorphisms generated by all the elements of the form $e^{t(X+uY)}$ where $(t, u) \in \mathbf{R}^2$.

The group $\mathscr{G}(X, Y)$ acts on M in a natural way; we denote by $\mathscr{G}(X, Y)(x)$ its orbit through the point x in M. It is well known from the integrability theory of families of vector fields, that each orbit $\mathscr{G}(X, Y)(x)$ is a submanifold of M with its dimension equal to the dimension of $\mathscr{L}(X, Y)(x)$. In particular, it then follows that $\mathscr{G}(X, Y)(x) = M$ if and only if the dimensions of $\mathscr{L}(X, Y)(x) = n$ for all $x \in M$.

We shall denote by $\mathscr{S}$ the semi-group of diffeomorphisms generated by the elements of the form $e^{t(X+uY)}$ where $t \geqslant 0$ and $u \in \mathbf{R}$. For each $x \in M$ we let $\mathscr{S}(x)$ *be the semi-orbit of* $\mathscr{S}$ *through* x, i.e., $\mathscr{S}(x) = \{g(x): g \in \mathscr{S}\}$.

For each $x \in M$ and any $g \in \mathscr{S}$, $g(x)$ lies on a trajectory of (1) generated by a piecewise constant control function. Hence, for each $x \in M$, $\mathscr{S}(x) \subset A^+(x)$. Since the class of all piecewise constant control functions are dense in $\mathscr{U}$ (in any reasonable topology), it follows that $\overline{\mathscr{S}(x)} = \overline{A^+(x)}$, where the bar denotes the topological closure in $\mathscr{G}(X, Y)(x)$.

If we let $\mathscr{S}^{-1} = \{g^{-1}: g \in \mathscr{S}\}$, then for each $x \in M$, and each $g \in \mathscr{S}^{-1}$, $x \in \mathscr{S}(g(x))$. Thus, $\mathscr{S}^{-1}(x) \in A^-(x)$ for each $x \in M$. Completely analogously to $\mathscr{S}$, it follows that $\overline{\mathscr{S}^{-1}(x)} + \overline{A^-(x)}$ in the closure relative to the topology of $\mathscr{G}(X, Y)(x)$.

The next lemma shows the relationship between $\mathscr{S}$ and the controllability of (X, Y).

Lemma 1. *Let (X, Y) be a pair of vector fields such that $\mathscr{L}(X, Y)(x) = M_x$ for each $x \in M$. If $\overline{\mathscr{S}(x)} = M$ for x belonging to a dense subset of M, then (X, Y) is controllable.*

Proof. If X and Y are such vector fields that $\mathscr{L}(X, Y)(x) = M_x$ for each $x \in M$, then for any neighborhood U of a point x both $\mathscr{S}^{-1}(x) \cap U$ and $\mathscr{S}(x) \cap U$ contain open sets ([4]).

Let x be a point in M such that $\overline{\mathscr{S}(x)} = M$, and let $y \in M$. Let U be a neighborhood of y. Since $\mathscr{S}^{-1}(y) \cap U$ contains an open set, it follows that there exists $z \in \mathscr{S}(x)$ such that $z \in \mathscr{S}^{-1}(y)$. Hence, $z \in A^-(y)$, and therefore $y \in A(z)$. Thus, $y \in \mathscr{S}(x)$. Therefore, $\mathscr{S}(x) = M$ for all x belonging to a dense set in M. We end this proof by showing that $\mathscr{S}(x) = M$ for all $x \in M$.

Let x be an arbitrary point of M. If U is any neighborhood of x, then let V be an open set contained in $\mathscr{S}(x) \cap U$. Let $y \in V$ be such that $\mathscr{S}(y) = M$. Since $y \in \mathscr{S}(x)$ we have that $\mathscr{S}(y) \subset \mathscr{S}(x)$. Therefore, $\mathscr{S}(x) = M$. This ends the proof of the lemma.

LEMMA 2. *Let $\hat{\mathcal{S}}$ be the semi-group of diffeomorphisms generated by the elements of the form e^{tX}, $t \geqslant 0$, and e^{sY} where $s \in \mathbf{R}$. Then for each $x \in M$, $\overline{\mathcal{S}(x)} \supseteq \overline{\hat{\mathcal{S}}(x)}$ where the closure is relative to the submanifold topology of $\mathcal{G}(X, Y)(x)$.*

Proof. It suffices to show that for each $x \in M$, and each $t \in R$, $e^{tY}(x) \in \overline{\mathcal{S}(x)}$. Since our vector fields are analytic it follows that for each x there exists a neighborhood U of x such that

$$e^{t(X+uY)}(y) = e^{tX}e^{tuY}e^{Z(t,u)}(y) \tag{2}$$

valid for small t and $y \in U$, where Z

(i) is a vector field belonging to the derived algebra of X and Y, and where

(ii) $Z(t, u) = O(t^2u)$.

This is a consequence of the well known Cambell–Hausdorff formula. Let t be a given number. Let $\{t_n\}$ be any sequence such that $t_n > 0$ and $\lim_{n\to\infty} t_n = 0$. Let $u_n = t/t_n$. For each n, $e^{t_n(X+u_nY)}x \in \mathcal{S}(x)$. Taking the limit of both sides in (2) we get that $e^{tY}(x) \in \overline{\mathcal{S}(x)}$. Therefore, we have proved that $\overline{\hat{\mathcal{S}}(x)} \subset \overline{\mathcal{S}(x)}$.

Using Lemma 1 and Lemma 2 it is very easy to prove the following basic controllability result.

THEOREM 1. *Let (X, X) be a pair of vector fields such that*

(i) *$\mathcal{L}(X, Y)(x) = M_x$ for all $x \in M$,*

(ii) *the set of recurrent points of X is dense in M.*

Then, (X, Y) is controllable.

A proof of this theorem is essentially a paraphrase of a similar theorem done in [3], and therefore we will omit it.

The following is an immediate corollary of this theorem.

COROLLARY 1. *Let $M = \mathbf{R}^n$, and let $X(x) = Ax$ where A is $n \times n$ matrix with real entries such that its eigenvalues are purely imaginary and distinct.*

Let Y be any vector field on M such that $\mathcal{L}(X, Y)(x) = \mathbf{R}^n$ for all $x \in \mathbf{R}^n$. Then (X, Y) is controllable.

3. STABILIZABILITY

We begin this section by reviewing certain local results. If X and Y are given vector fields, then for each integer $n = 0, 1, 2,\ldots$ define a vector field $ad^kX(Y)$ as follows:

$$ad^0X(Y) = Y,$$
$$ad^{k+1}X(Y) = [X, ad^kX(Y)].$$

We will say that a pair of vector fields (X, Y) *satisfies the ad-condition if the linear span of all vectors of the form* $ad^k X(Y)(x)$ is equal to M_x at each $x \in M$.

If X and Y satisfy the *ad*-condition then for any $\epsilon > 0$, any $x \in M$ and any $t > 0$, $e^{tX}(x)$ lies in the interior of the set of attainability generated by controls n with $|u| < \epsilon$. This follows from the fact that there exist n-smooth functions $v_1, \ldots, v_n$ defined on $[0, t]$ such that the rank of the differential of the map

$$(\epsilon_1, \ldots, \epsilon_n) \to x(x, \epsilon_1 v_1 + \cdots + \epsilon_n v_n, t) \qquad \text{at} \quad \epsilon_1 = \epsilon_2 = \cdots = \epsilon_n = 0$$

is equal to the dimension of M. Hence, this map covers a neighborhood of $e^{tX}(x)$. Here, $x(x, \epsilon_1 v_1 + \cdots + \epsilon_n v_n, t)$ denotes the trajectory of (1) through x generated by the control $\epsilon_1 v_1 + \cdots + \epsilon_n v_n$.

A slight modification of the preceding argument shows that x is contained in the interior of the set of points which can be steered to $e^{tX}x$ in positive time along the trajectories of (1) generated by controls u with $|u| < \epsilon$.

In particular, if $x \in M$ is such that $X(x) = 0$, then as is well known from the linear theory, the *ad*-condition is equivalent to $b, Ab, \ldots, A^{n-1}b$ being linearly independent where $b = Y(x)$ and where A is the matrix with entries $(\partial X^i / \partial x_j)(x)$.

We now address the question of existence of stabilizing controls. If f is any smooth function on M, let $Z = X + fY$. If $x \in M$ is such that $X(x) = 0$ and $f(x) = 0$ then a necessary condition for asymptotic stability of Z at x is that its differential dZ at x have eigenvalues with negative real parts. If in addition (X, Y) satisfies the *ad*-condition then it follows from the linear theory that the spectrum of Z at x can be completely controlled for different choices of df, and hence we have local stabilizability.

Easy examples show that in general local stabilizability does not extend globally. The following theorem however gives conditions under which (X, Y) is globally stabilizable.

THEOREM 2. *Let $M = \mathbf{R}^n$ and let $X(x) = Ax$ where A is $n \times n$ matrix whose spectrum consists of n distinct imaginary eigenvalues. If Y is any vector field on M such that (X, Y) satisfies the ad-condition for all $x \neq 0$ then (X, Y) is globally stabilizable at $x = 0$.*

Proof. Since A has purely imaginary spectrum, then there exists a basis in respect to which A is skew-symmetric. Therefore, there is no loss of generality in assuming that A is skew-symmetric. Let $f(x) = -\langle x, Y(x) \rangle$ where $\langle\ ,\ \rangle$ denotes the inner product in $\mathbf{R}^n$, and let $Z = X + fY$.

If $x \in \mathbf{R}^n$, then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \| e^{tZ}x \| &= \left\langle e^{tZ}(x), \frac{d}{dt} e^{tZ}x \right\rangle = \langle e^{tZ}x, Z(e^{tZ}x) \rangle \\ &= \langle e^{tZ}x, Ae^{tZ}x \rangle + f(e^{tZ}x) \langle e^{tZ}x, Y(e^{tZ}x) \rangle \\ &= f(e^{tZ}x) \langle e^{tZ}x, Y(e^{tZ}x) \rangle \\ &= -f(e^{tZ}x)^2 \leqslant 0 \qquad \text{for all} \quad t \geqslant 0. \end{aligned}$$

If we let $\Omega = \{x: f(x) = 0\}$ then by LaSalle's invariance theorem [2, p. 66] all integral curves of Z will tend to the largest subset of Ω which is invariant under Z.

If we denote by E this invariant subset of Ω, the proof will be finished if we show that $E = \{0\}$.

Let $x \in E$. Since $e^{tZ}x \in E$ for all t we have that $f(e^{tZ}x) \equiv 0$. Hence, $e^{tZ}x = e^{At}x$. Thus, it follows that $\langle e^{tA}x, Y(e^{tA}x)\rangle \equiv 0$ for all t. The successive differentiation of the above curve at $t = 0$ shows that x is orthogonal to all vectors of the form $ad^k X(Y)x$, $k = 0, 1,\dots$. Since (X, Y) satisfies the *ad*-condition, it follows that $x = 0$. Thus, $E = \{0\}$ and hence f is the global stabilizing control at $x = 0$. The proof is now complete.

In view of Corollary 1, the preceding theorem shows a connection between controllability and stabilizability. It also suggests the extent of such a connection because the results of the preceding theorem also hold when A has eigenvalues with non-positive real parts. However, in such a case there exist many pairs (X, Y) which are not controllable.

4. Examples

In the first example we characterize controllability in $\mathbf{R}^2$ of pairs of vector fields given by matrices with real and distinct eigenvalues.

Example 1 (controllability of matrices with real and distinct eigenvalues). Let $M = \mathbf{R}^2/\{0\}$, and let $X(x) = Ax$ and $Y(x) = Bx$ where A and B are real matrices of dimension 2. Without loss of generality we assume that B is diagonal.

For a continuous control $u(t)$, let $v(t) = \int_0^t u(s)\,ds$. If we denote by $y(t) = e^{-v(t)B}x(t)$ where $x(t)$ is a trajectory of (1), then $y(t)$ satisfies the following differential equation:

$$\frac{dy}{dt} = (e^{-v(t)B} A e^{v(t)B})\, y(t).$$

If

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad \text{and} \quad B = \begin{vmatrix} b_1 & 0 \\ 0 & b_2 \end{vmatrix},$$

then the components y_1 and y_2 of y satisfy:

$$\frac{dy_1}{dt} = a_{11}y_1(t) + a_{12}e^{\Delta v}y_2(t)$$

and

$$\frac{dy_2}{dt} = a_{21}e^{-\Delta v}y_1(t) + a_{22}y_2(t),$$

where $\Delta = b_1 - b_2$.

If we multiply dy_1/dt by y_2, dy_2/dt by y_1, and then add the results, we get:

$$\frac{d}{dt}(y_1 y_2) = (a_{11} + a_{22})\, y_1 y_2 + a_{12} e^{\Delta v} y_2^2 + a_{21} e^{-\Delta v} y_1^2.$$

The variation of parameters method yields:

$$y_1(t)\, y_2(t) = e^{at}\left[\left(y_1(0)\, y_2(0) + a_{12}\int_0^t e^{-as} e^{\Delta v(s)} y_2^2(s)\, ds + a_{21}\int_0^t e^{-\Delta v(s)} y_1^2(s)\, ds\right)\right],$$

where $a = a_{11} + a_{22}$.

There are two cases to consider.

(i) If both a_{21} and a_{12} are non-negative, then $y_1(t)\, y_2(t) \geqslant 0$ whenever $y_1(0)\, y_2(0) \geqslant 0$ for all $t \geqslant 0$ no matter what v is chosen. Analogously, when a_{21} and a_{12} are both non-positive $y_1(t)\, y_2(t) \geqslant 0$ whenever $y_1(0)\, y_2(0) \geqslant 0$ for all $t \leqslant 0$.

In this case, the set of points which can be reached via the trajectories of (1) starting from

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{with} \quad x_1 x_2 > 0$$

is contained in the quadrant which contains x, while in the second case the set of points which can be steered to x is contained in the quadrant which contains x.

Therefore in either case the control system described by (1) is not controllable. In particular, from the above, it follows that when A and B are symmetric 2×2 matrices the resulting linear system is never controllable. The condition that $a_{12}a_{21} > 0$ is stable under small perturbations of the entries of A, hence the preceding argument shows that there is an open subset of 2×2 linear systems which are not controllable.

(ii) Here we assume that $a_{12}a_{21} < 0$. It then follows by direct computation there exists a value u such that $A + uB$ has complex roots. This implies that there exists an admissible trajectory of (1), namely $e^{(A+uB)t}(x)$, which cuts all the trajectories of B, and this in turn implies *controllability* of (1).

For higher dimensions not only that this method breaks down, but there seems not to be any simple characterization of controllability. An account of this theory will appear in a joint paper of V. Jurdjevic and I. Kupka.

EXAMPLE 2 (case of a controllable but not stabilizable system). Here

$$M = \mathbf{R}^2, \qquad X(x) = \begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix} x \qquad \text{and} \qquad Y(x) = \begin{vmatrix} x_2^2 \\ 1 \end{vmatrix}.$$

The components x_1 and x_2 of trajectories of (1) satisfy:

$$\frac{dx_1}{dt} = 4x_1 + u(t)\, x_2^2, \qquad \frac{dx_2}{dt} = x_2 + u(t).$$

This system can be solved explicitly to yield:

$$x_1(t) = \frac{x_2^3(t)}{3} + \frac{e^{4t}}{3} \int_0^t e^{-s} v^3(s)\, ds, \qquad x_2(t) = e^t v(t),$$

where $v(t) = \int_0^t e^{-s} u(s)\, ds$.

As u varies over all piecewise continuous functions on $[0, t]$, $t > 0$ the range of

$$\begin{vmatrix} \frac{e^{4t}}{3} \int_0^t e^{-s} v^3(s)\, ds \\ e^t v(t) \end{vmatrix}$$

covers all of $\mathbf{R}^2$.

Thus, any point of $\mathbf{R}^2$ can be reached via the trajectories of (1) which originate from the origin.

A completely similar argument applied to functions on $[-t, 0]$ shows that all points of $\mathbf{R}^2$ can be steered to the origin in time t. Thus, (X, Y) is controllable in $\mathbf{R}^2$.

However, $Z = X + fY$ is not asymptotically stable at $x = 0$ no matter what scalar function f is chosen. For if f is any smooth function on $\mathbf{R}^2$ such that $f(0) = 0$, then the differential of Z at $x = 0$ is given by:

$$\begin{vmatrix} 4 & 0 \\ \frac{\partial f}{\partial x_1}(0) & 1 + \frac{\partial f}{\partial x_2}(0) \end{vmatrix}.$$

The above matrix has 4 as its eigenvalue independently of f, and therefore no stability is possible.

EXAMPLE 3 (a Liénards equation example). If we let X and Y be matrix vector fields in $\mathbf{R}^2$ given by

$$X(x) = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} x \qquad \text{and} \qquad Y(x) = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} x$$

then it is easy to check that X and Y satisfy the *ad*-condition at all points of $M = \mathbf{R}^2/\{0\}$. Hence, the pair (X, Y) is controllable on M, and the origin is globally stabilizable. The choice of feedback function $f(x) = -\langle x, Y(x)\rangle = -x_1 x_2$ results in:

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = -x_1 - x_2 x_1^2.$$

This system is equivalent to $(d^2x_1/dt^2) + x_1(t)^2(dx_1/dt) + x_1(t) = 0$, which is a specific case of Liénards equation. It is well known that such equation is asymptotically stable.

It might be interesting to note that the above pair (X, Y) cannot be stabilized by means of a constant control function because the spectrum of $A + uB$ contains at least one eigenvalue with non-negative real part, no matter what u is chosen.

EXAMPLE 4 (Van Der Pol's equation and practical stability). Let X and Y be matrix vector fields in $\mathbf{R}^2$ given by

$$X(x) = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} x \qquad \text{and} \qquad Y(x) = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} x.$$

It is easy to check that (X, Y) satisfies the *ad*-condition at all points of $M = \mathbf{R}^2/\{0\}$, and that it is controllable in M. However, (X, Y) is not stabilizable at the origin because the differential of $Z = X + fY$ at the origin always contains an eigenvalue with a positive real part. If we set $f(x) = \alpha - \beta\langle x \cdot Y(x)\rangle = \alpha - \beta x_2 x_1$ where α and β are positive constants, then the system $dx/dt = X(x) + f(x)\, Y(x)$ is equivalent to:

$$\frac{d^2x_1}{dt^2}(t) + \frac{dx_1}{dt}(\beta x_1{}^2 - 1) + (1 - \alpha)\, x_1(t) \equiv 0.$$

This equation is a specific case of Van Der Pol's equation. It is well known that such an equation has a limit cycle to which all solutions tend (see, for instance, [2]). Moreover, by choosing α sufficiently close to 1 and by choosing β sufficiently large, this cycle can be deformed to any neighborhood of the origin. Thus, in this case, following LaSalle and Lefschetz [2], the pair (X, Y) might be called practically globally stabilizable. Perhaps in this context controllability is more related to such a notion than to asymptotic stability.

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