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On quantum corrections to spinning strings and Bethe equations

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Abstract

Recently, it was demonstrated that one-loop energy shifts of spinning superstrings on $AdS_5 \times S^5$ agree with certain Bethe equations for quantum strings at small effective coupling. However, the string result required artificial regularization by zeta-function. Here we show that this matching is indeed correct up to fourth order in effective coupling; beyond, we find new contributions at odd powers. We show that these are reproduced by quantum corrections within the Bethe ansatz. They might also identify the "three-loop discrepancy" between string and gauge theory as an order-of-limits effect.

The investigation of semiclassical spinning superstrings on $AdS_5 \times S^5$ [1–4] [2] and their AdS/CFT duals, local operators of $\mathcal{N} = 4$ SYM in the thermodynamic limit [6],[3] has lead to a number of important insights into both theories. Progress in this subject went hand in hand with the discovery and development of integrable structures in $\mathcal{N} = 4$ SYM [9–12] and string theory on $AdS_5 \times S^5$ [13].[4] The computations of the spinning string correspondence required powerful methods which integrability could provide. Conversely, spinning strings were an ideal testing ground for these methods.

The main tool for obtaining the spectrum of integrable models is the Bethe ansatz. For gauge theory it was developed in [9,11,14–17]. The string counterpart is a set of integral equations for classical strings [18,19] and a proposal for the promotion to Bethe equations for quantum strings was made in [16, 17,20]. The comparison of the classical spectra of both models has shown general agreement at the leading

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[4] See [7,8] for reviews on integrability of gauge theory and strings.

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two orders [6,17,18,21], but also lead to the discovery of a disagreement at third order [14],[5] the so-called "three-loop discrepancy".[6] Note that this mismatch is not necessarily in conflict with the AdS/CFT correspondence [24] though, because order-of-limits effects may spoil the (naive) comparison [14,15].

Recently, the precision tests of the quantum string Bethe equations were performed by comparing their prediction to one-loop effects in quantum string theory. String energies $E(\lambda, J)$ admit an expansion for large string tension $\sqrt{\lambda}$ (or large 't Hooft coupling λ)

$$E(\lambda, J) = \sqrt{\lambda}\mathcal{E}(\mathcal{J}) + \delta E(\mathcal{J}) + \mathcal{O}(1/\sqrt{\lambda}),$$
$$\mathcal{J} = J/\sqrt{\lambda}. \tag{1}$$

Here, $\mathcal{E}$ is the classical string energy and δE is the one-loop energy shift. The effective string tension $\tilde{\lambda}^{1/2} = 1/\mathcal{J}$ (alias the effective spin $\mathcal{J}$) can take any fixed value. The comparison of δE was performed in an expansion in powers of the effective coupling $1/\mathcal{J}$. Agreement at $\mathcal{O}(1/\mathcal{J}^2)$ for the simplest class of spinning string solutions was found in [25,26]. This was later generalized to the full $\mathfrak{su}(2)$ sector [27]. Going to higher orders in $1/\mathcal{J}$, however, is problematic due to the appearance of divergent sums [28]. When these sums are regularized by the first regulator that might come to mind, namely by zeta-function, the result does indeed agree with the Bethe ansatz at $\mathcal{O}(1/\mathcal{J}^6)$ [29]. This is a very good sign of the validity of the Bethe ansatz, given that the computation and the result are rather complex. Merely the need to regularize within this conformal two-dimensional model appears artificial; the unexpanded sums do indeed converge [30].

In this Letter we investigate the divergent sums carefully and find that one can make sense of them. This allows us to compute the coefficients of the expansion of the one-loop energy shift δE. We find that zeta-function regularization actually produces the correct coefficients of $1/\mathcal{J}^4$ and $1/\mathcal{J}^6$. However, we find additional contributions at *odd* powers of $1/\mathcal{J}$ starting at $\mathcal{O}(1/\mathcal{J}^5) = \mathcal{O}(\lambda^{5/2}/J^5)$.[7] This may appear disastrous for the quantum string Bethe ansatz, which does not produce such terms, and for the comparison to gauge theory, due to the unexpected fractional powers of λ. Nevertheless, quite the contrary is true: On the one hand, we will demonstrate that these contributions allow us to determine quantum corrections to the Bethe equations themselves. That this is possible at all is non-trivial and therefore makes us more confident of the Bethe ansatz for quantum strings. On the other hand, they can be interpreted as large-λ effects which might repair the disagreement between string and gauge theory when interpolated down to small λ. Here we even see some quantitative confirmation of this idea.

Let us now reinvestigate the one-loop energy shift of a circular spinning string on $AdS_3 \times S^1$ in string theory. The classical solution was found in [32] and quantum corrections to the energy were computed in [28]. We will use the notation of [25,29], i.e., k is the mode number, m is the winding number for S^1 and n is the mode number of the fluctuation. The spin S on AdS_3 and the spin J on S^1 are related by $Sk + Jm = 0$. The energy shift is given by the generic formula

$$\delta E = \sum_{n=-\infty}^{\infty} e(n), \tag{2}$$

where $e(n)$ is the sum of contributions of bosonic and fermionic fluctuations with given mode number n. The expression $e(n)$ can be found in [25,28,29], we recall it in (A.1) in Appendix A.

We first expand for large $\mathcal{J}$ at fixed n and denote the result by $e^{\text{sum}}(n)$. It then turns out that starting from $\mathcal{O}(1/\mathcal{J}^4)$ the sum of $e^{\text{sum}}(n)$ diverges due to contributions with positive powers of n.[8] Let us therefore split the result into a regular part $e^{\text{sum}}_{\text{reg}}(n)$ with contributions of $\mathcal{O}(1/n^2)$ and a singular part $e^{\text{sum}}_{\text{sing}}(n)$ polynomial in n. The expressions are lengthy and we present them in Eqs. (A.8), (A.9) in Appendix A.

[5] See also [22] for a closely related effect in the near plane wave/BMN correspondence.

[6] See [23] for reviews on the comparison between semiclassical spinning strings and gauge theory.

[7] Similar observations are made in [31]. There, systematic analytic methods of handling sums and of computing corrections were developed on several examples. Their methods may be more suitable to understand potential exponential corrections beyond the perturbation series.

[8] We sum order by order in $1/\mathcal{J}$. Technically, the divergencies are caused by an order-of-limits effect.

Clearly, the sum of the regular part converges while the sum of the singular part, an even polynomial, gives identically zero when regularized by zeta-function. For small values of n our answer appears fine, but the large-n behavior is incompatible with the expansion. This problem is not unexpected as we have assumed n to be fixed while taking $\mathcal{J}$ large. This very assumption conflicts with the nature of the sum which goes over *all* modes n.

Let us now attempt to improve the approximation for large values of n. For this we set $n = \mathcal{J}x$ and expand for large $\mathcal{J}$. The resulting expression is given in (A.5) in Appendix A. In this case, the energy shift should be approximated by the integral of $\mathcal{J}\,dx\,e^{\text{int}}(x)$. Once again, we find a problem: The integrand diverges at $x = 0$, as was already noticed for a similar solution on $\mathbb{R} \times S^3$ in [30], and the integral cannot be performed. To see more clearly what happens, we separate the integrand into a regular part $e^{\text{int}}_{\text{reg}}(x)$ which is smooth at $x = 0$ and a singular part $e^{\text{int}}_{\text{sing}}(x)$ with strictly positive powers of $1/x$. The singular part is given in Eq. (A.6) in Appendix A. Despite the singularities at $x = 0$, let us note that $e^{\text{int}}(x)$ has the correct asymptotics at large n, cf. (A.7); its expansion agrees quantitatively with the asymptotics of $e(\mathcal{J}x)$. Apparently, here $e^{\text{int}}(x)$ approximates $e(n)$ well at large values of $n = \mathcal{J}x$, but not at small ones.

The divergencies at large n in the first approach are traded in for divergencies at small n in the second one. We might therefore try to combine the two approaches, use $e^{\text{sum}}(n)$ for small n and $e^{\text{int}}(x)$ for large n. As we will see, this can be done. Moreover, we do not even need a cut-off to separate between the two regimes. Instead we make use of the following observation: The singular part in one regime seems to equal the regular part in the other regime: $e^{\text{int}}_{\text{sing}}(x) = e^{\text{sum}}_{\text{reg}}(\mathcal{J}x)$ and $e^{\text{sum}}_{\text{sing}}(n) = e^{\text{int}}_{\text{reg}}(n/\mathcal{J})$. This property can be confirmed by expanding the regular part after interchanging n and $\mathcal{J}x$.[9] We thus find[10]

$$
\begin{aligned}
e(n) &= e^{\text{sum}}_{\text{reg}}(n) + e^{\text{int}}_{\text{reg}}(n/\mathcal{J}) \\
&= e^{\text{sum}}_{\text{sing}}(n) + e^{\text{int}}_{\text{sing}}(n/\mathcal{J}).
\end{aligned} \tag{3}
$$

Therefore, there is no need to consider the singular parts at all; to obtain the energy shift it suffices to consider the regular parts[11]

$$
\begin{aligned}
\delta E &= \sum_{n=-\infty}^{\infty} e(n) \\
&= \sum_{n=-\infty}^{\infty} e^{\text{sum}}_{\text{reg}}(n) + \int_{-\infty}^{\infty} \mathcal{J}\,dx\,e^{\text{int}}_{\text{reg}}(x).
\end{aligned} \tag{4}
$$

The sum of $e^{\text{sum}}_{\text{reg}}(n)$ is known, it is the zeta-function regularized sum in [29]. The integral however yields a non-trivial contribution

$$
\int_{-\infty}^{\infty} \mathcal{J}\,dx\,e^{\text{int}}_{\text{reg}}(x) = -\frac{(k-m)^3 m^3}{3\mathcal{J}^5} + \mathcal{O}\big(1/\mathcal{J}^7\big). \tag{5}
$$

It is somewhat surprising to see that the integrand $e^{\text{int}}_{\text{reg}}(x)$, cf. (A.5), (A.6), starts at $\mathcal{O}(1/\mathcal{J}^4)$, but its integral vanishes at this order. Nevertheless, this is merely an exception, the integral does not vanish at higher orders. While all the contributions from $e^{\text{sum}}_{\text{reg}}(n)$ are at even powers of $1/\mathcal{J}$, the new contributions are at odd powers. Put differently, the first new term is at order $\lambda^{5/2}/J^5$.

The new term in (5) contradicts the naive expectation that the expansion goes in integer powers of λ and $1/J$ [33] and thus can be directly compared to perturbative gauge theory. It also contradicts the simplest version of the Bethe ansatz for quantum strings [16,17,20] which does not produce such terms [29]. Nevertheless, the appearance of such terms leads to a natural proposal of how to establish the agreement between the gauge and string theory results.

First of all, the one-to-one comparison of perturbative string theory to perturbative gauge theory is suggestive but seemingly plagued by order-of-limits effects. On top of the well-known disagreement of coefficients, the "three-loop discrepancies" [14,22], here we find that also the *structure* of the expansion is different in both limits. This is not in conflict with

[9] In physicist's terms: *resumming* one singular part yields the other regular part.

[10] Note that $e^{\text{int}}_{\text{sing}}$ and $e^{\text{sum}}_{\text{reg}}$ have positive powers of x, n while $e^{\text{sum}}_{\text{sing}}$ and $e^{\text{int}}_{\text{reg}}$ have strictly negative ones. Consequently, we might understand this split as a Laurent expansion in n and a subsequent separation into positive and strictly negative powers.

[11] The integral is merely an approximation to the sum. We however did not find any corrections polynomial in $1/\mathcal{J}$ by improving the integrand using the Euler–Maclaurin formula as in [33].

AdS/CFT; it merely invalidates attempts to compare perturbatively.

Let us assume the AdS/CFT correspondence holds. Then the exact energy E should be some interpolating function between the perturbative string theory expression at large λ and the perturbative gauge theory at small λ. Now we note that the new term at $\mathcal{O}(1/\mathcal{J}^5)$ is accompanied by an old term at the same order in $1/\mathcal{J}$ coming from the expansion of the classical string energy at $\mathcal{O}(\sqrt{\lambda}/\mathcal{J}^5)$. The former should be considered as a quantum correction to the latter. We might combine these two terms with higher-loop corrections into some function $f_5(\lambda)\sqrt{\lambda}/\mathcal{J}^5$ of the coupling. At large λ the function $f_5(\lambda)$ admits an expansion in powers of $1/\sqrt{\lambda}$ starting at $\mathcal{O}(1)$; here we merely see the first two terms. At small λ we expect $f_5(\lambda)$ to have a regular expansion in λ. In between, it should interpolate between $f_5(\infty)$ and $f_5(0)$. Similar effects have been observed in the related context of plane wave string field theory in [34].

In fact, it is precisely this term, $\mathcal{O}(\sqrt{\lambda}/\mathcal{J}^5)$ in string theory and $\mathcal{O}(\lambda^3/J^5)$ in gauge theory, at which the three-loop discrepancy starts [14]. Put differently, we find $f_p'(\infty) \neq 0$ precisely for that value of p where $f_p(\infty) \neq f_p(0)$. This might be a sign that the mismatch will be resolved by an interpolation between strong and weak coupling.[12] Below, we will present some quantitative evidence for this qualitative statement.

How can the new contribution be interpreted in the quantum string Bethe ansatz [16,17,20]? According to the sophisticated analysis in [29], the expansion of δE is in even powers of $1/\mathcal{J}$, at least up to $\mathcal{O}(1/\mathcal{J}^6)$. Here we go back to the original proposal of the string Bethe equations in [20]. Arutyunov, Frolov and Staudacher's proposal was to modify the gauge Bethe equations by an additional phase shift for the interchange of two excitations[13]

$$\theta(p_k, p_j) = 2\sum_{r=2}^{\infty} c_r(\lambda)\big(\lambda/16\pi^2\big)^r \times \big(q_r(p_k)q_{r+1}(p_j) - q_{r+1}(p_k)q_r(p_j)\big). \tag{6}$$

This dressing phase θ depends on the momenta p of the excitations through their conserved charges q_r. The undetermined functions $c_r(\lambda)$ should approach 1 at $\lambda \to \infty$ to obtain the correct classical limit. If they interpolate to 0 at $\lambda = 0$, the Bethe equations might even agree with the correct gauge result. Apart from these two limits, we know no further constraints for the c_r yet. In [29] it was assumed that the functions $c_r(\lambda) = 1$ are exact, i.e., they do not receive string quantum corrections; that led to an expansion of the string energy in even powers of $1/\mathcal{J}$.

Let us now see whether we can re-establish agreement with one-loop string theory by correcting $c_2 = 1 + \epsilon$. We thus add an overall phase to the Bethe equations[14]

$$2\epsilon\big(\lambda/16\pi^2\big)^{(r+s-1)/2}\big(q_r(p_k)q_s(p_j) - q_s(p_k)q_r(p_j)\big). \tag{7}$$

We solve the Bethe equations for the $\mathfrak{sl}(2)$ sector in the thermodynamic limit with all mode numbers coinciding. This is the one-cut solution studied in [19,25,26,29] corresponding to the above circular spinning string. The equations can be solved by the standard trick of turning them into a quadratic equation for a resolvent. We then find that the classical energy shifts by[15]

$$\delta\mathcal{E} = 4\epsilon\frac{\mathcal{Q}_{r+1}\mathcal{Q}_s - \mathcal{Q}_r\mathcal{Q}_{s+1}}{(4\pi)^{r+s+1}\mathcal{E}} + \mathcal{O}\big(\epsilon^2\big). \tag{8}$$

Here $\mathcal{Q}_r$ are the conserved charges of the solution as defined in [15], here they are normalized to scale as $\mathcal{O}(1/\mathcal{J}^{r-1})$, cf. [37]. We find for the energy shift with $r = 2$ and $s = 3$

$$\delta\mathcal{E} = \epsilon\frac{(k-m)^3m^3}{16\mathcal{J}^5} + \mathcal{O}\big(1/\mathcal{J}^7\big). \tag{9}$$

[12] Similar qualitative statements appeared in [14,15,20,23,35], see also [34].

[13] In the proposal of [20], the interpolating functions could also depend on spin J or length L. This might seem somewhat unnatural from a Bethe ansatz/spin chain/S-matrix point of view. Furthermore, it is not clear how to define L in string theory. Indeed, we will not need dependence on L.

[14] We generalize the form of the corrections to include two uncorrelated charges q_r and q_s. This appears to be the most general form for Bethe equations for certain types of spin chains [36]. We thank T. Klose and M. Staudacher for discussions.

[15] This is the result for the $\mathfrak{su}(2)$ Bethe equations. The result for $\mathfrak{sl}(2)$ is similar.

Remarkably, this is in structural agreement with (5). When we set in (6)

$$c_2(\lambda) = 1 - \frac{16}{3\sqrt{\lambda}} + \mathcal{O}(1/\lambda) \tag{10}$$

the Bethe equations reproduce the correct string result for our class of circular solutions parametrized by k, m.

In fact, one can easily convince oneself that the leading discrepancy between classical string energies E_s and gauge theory energies in the thermodynamic limit E_g is obtained from (9) for $\epsilon = 1$. The general prediction for the $\mathcal{O}(\lambda^{5/2})$ contribution of an arbitrary solution is thus $-\frac{16}{3}(E_s - E_g)/\sqrt{\lambda}$. So our finding is completely consistent with the idea that c_2 interpolates between 1 at strong coupling and 0 at weak coupling. This suggests a natural resolution of the apparent disagreement between the string and gauge theory results at order λ^3 from a string perspective.

Conversely, each effect should have a counterpart on the other side of the duality. How can the disagreement be reduced from a gauge theory point of view? This depends crucially on how the functions $c_r(\lambda)$ approach zero near $\lambda = 0$. For an exponential decline, such as $c_2(\lambda) = \exp(-\frac{16}{3}/\sqrt{\lambda})$, we would see no effects in perturbative gauge theory at all. Another possibility is that $c_r(\lambda) \sim \lambda^L$, where L is the length of the state.[16] This behavior might be associated to "wrapping effects" [15], special types of corrections which start when the range of the Hamiltonian exceeds the length of the state.[17] If, however, $c_r(\lambda) \sim \lambda^a$ with some fixed a, then the scaling behavior in the thermodynamic limit (as well as BMN-scaling [39]) would break down. Proper scaling was a central assumption in the construction of higher-loop gauge theory results (see [7] for a review), but has only been confirmed rigorously up to $\mathcal{O}(\lambda^3)$ [12,40].[18]

Of course, the interpolating functions of the string Bethe ansatz, e.g., (10), must be universal and hold for all other solutions in any sector as well [16,17]. We can thus predict the contributions at odd powers of $1/\mathcal{J}$ from the Bethe equations. To see this, let us repeat the above analysis in a different sector, for a circular string on $\mathbb{R} \times S^3$ [3]. This corresponds to the $\mathfrak{su}(2)$ single-cut solution of [18,42].[19] We restrict to the "half-filling" point ($J_1 = J_2 = J/2$), where most expressions simplify. For the corrections at odd powers of $1/\mathcal{J}$ we appear to find, using the expressions in [30][20,21,22]

$$\int_{-\infty}^{\infty} \mathcal{J}\,dx\, e^{\mathrm{int}}_{\mathrm{reg}}(x) = \frac{m^2}{\sqrt{\mathcal{J}^2+m^2}} + \frac{2\mathcal{J}^2}{\sqrt{\mathcal{J}^2+m^2}} \log\frac{\mathcal{J}^2}{\mathcal{J}^2+m^2} - \frac{\mathcal{J}^2-m^2}{2\sqrt{\mathcal{J}^2+m^2}} \log\frac{\mathcal{J}^2-m^2}{\mathcal{J}^2+m^2}. \tag{11}$$

This agrees with the Bethe equations when c_2 is as in (10).[23] We have also performed a numerical comparison between the exact sum and our expansion of it. We set $m = 1$ and sum up to $|n| = 5000$ for $\mathcal{J}$ between 3 and 10. The coefficients of the $1/\mathcal{J}$ expansion are evaluated numerically up to $\mathcal{O}(1/\mathcal{J}^9)$. The results of both approaches agreed up to about 10^{-7}. If, however, we eliminate the odd powers in $1/\mathcal{J}$ from the expansion, the matching is reduced to about 10^{-4}. This is a clear verification of the presence of the odd powers of $1/\mathcal{J}$ in the expansion.

16 This would, however, be in contradiction with the philosophy of a length-independent S-matrix.

17 Alternatively, the *asymptotic* Bethe ansatz might break down at this order and needs to be replaced by something structurally different from a Bethe ansatz, see e.g., [38].

18 The dispersion relation would still preserve scaling as well as most parts of the S-matrix. Only a global phase would violate proper scaling. A first guess $c_2(\lambda) \sim \lambda$ would imply scaling violations in gauge theory at four loops, just slightly beyond our current horizon. Intriguingly, such scaling violations have been observed in the plane wave matrix model [41]. This latter fact does not necessarily have implications for $\mathcal{N} = 4$ SYM.

19 This solution is unstable due to tachyonic modes at small $n < 2m$ (IR). Here we consider corrections which are associated to large mode numbers (UV) and thus unaffected by the instability.

20 This result is independent of way periodicity is handled for fermions, cf. [3,30] vs. [25,32].

21 This result can also be obtained from string theory with an infinite world sheet confirming that the origin of the contribution is a local quantum effect rather than a finite-size effect.

22 Comparing [32] and [42] we expect $-m^3(k-m)^3/3\mathcal{J}^5 + \mathcal{O}(1/\mathcal{J}^7)$ for the generic case. Here $k = 2m$.

23 A preliminary analysis using (8) yields the leading corrections for the higher $c_r(\lambda)$. The coefficients for $c_2 \ldots c_6$ seem to be: $-16/3$, $-16/3$, $-184/15$, $-182/15$, $-3268/175, \ldots$ without an apparent pattern.

There are other cases for which one might compute these odd contributions. For instance, there are further one-cut solutions which should be easily accessible, such as solutions on $\mathbb{R} \times S^5$ [2,3,43]. These are interesting because they add "flavor" to the Bethe equations. One could also try to generalize to two-cut solutions, such as the folded string [4,6], but these are more involved due to their elliptic nature. An expansion around an algebraic solution along the lines of [44] might simplify the analysis.

The universality of the Bethe ansatz also predicts the existence of these types of corrections in the near plane wave limit of $AdS_5 \times S^5$. There, the first terms of fractional order in λ would occur at $\mathcal{O}(\lambda^{5/2}/J^7)$ representing a $1/J^2$ effect. At second order in $1/J$ a sum over intermediate channels appears and this may become divergent when first expanding in $\lambda' = \lambda/J^2$.[24] Partial results were obtained in [45]. Once the exact expressions for finite λ' are known, the regularization of the sum might proceed in a similar way as above and the result should be compared to the Bethe ansatz. For instance, in the $\mathfrak{su}(2)$ sector, the leading difference between gauge and string theory in the near plane wave limit is given by the general formula derived from the results in [20]

$$E_\mathrm{s} - E_\mathrm{g} = -\frac{\lambda^3}{16J^7} \sum_{k,j=1}^{M} n_k^2 n_j^2 (n_k - n_j)^2 + \mathcal{O}\big(\lambda^4/J^9\big). \quad (12)$$

Here, M is the number of excitations and n_k are their mode numbers (which are allowed to coincide). The $\mathcal{O}(\lambda^{5/2}/J^7)$ contribution is predicted to be $-\frac{16}{3}(E_\mathrm{s} - E_\mathrm{g})/\sqrt{\lambda}$.

One might also wonder how to obtain the odd powers in $1/\mathcal{J}$ in the fast string expansion of [46].[25] Here, one expands in $1/\mathcal{J}$ at the level of the classical action. Therefore, one can possibly obtain only the summands $e^{\mathrm{sum}}(n)$ expanded at finite mode number n. As we have demonstrated, the integrand $e^{\mathrm{int}}(x)$ may be recovered from $e^{\mathrm{sum}}(n)$. However, this requires resumming of all orders and thus the odd powers in $1/\mathcal{J}$ are non-perturbative contributions in this effective field theory.

There are many aspects which deserve further investigation. For instance, it would be important to understand how to disentangle finite-size $(1/J)$ and finite-tension $(1/\sqrt{\lambda})$ effects: We have interpreted the odd powers in $1/\mathcal{J}$ as quantum corrections to classical contributions. They correspond to $1/\sqrt{\lambda}$ corrections to the Bethe equations. Also, when extrapolating to perturbative gauge theory, these terms should go away. On the other hand, the corrections at even powers in $1/\mathcal{J}$ remain and can be compared to gauge theory. There, they correspond to finite-size $(1/J)$ corrections to the thermodynamic limit. If we knew how to disentangle them, we could focus only on finite-tension effects and find higher loop corrections to the Bethe equations.[26]

Another direction to proceed would be to generalize the findings of [27] to finite $1/\mathcal{J}$. At $\mathcal{O}(1/\mathcal{J}^2)$ it was shown in generality that the one-loop energy shift equals a regularized sum over fluctuation energies. As above, the regularization should be equivalent to adding quantum corrections to the Bethe equations. Now, the fluctuation energies and the energy shift can both be computed from the Bethe equations. By comparing the two, one should thus be able to derive the complete one-loop quantum corrections as a consistency requirement of the Bethe ansatz framework with quantum mechanics.

In conclusion, we have found new effects in the small effective coupling expansion of the one-loop energy shift (5); these might be interpreted as a resolution of the three-loop puzzle. We have also derived parts of the first quantum correction to the string scattering phase. This is given by the interpolating function (10) for the dressing phase θ within the string Bethe ansatz. It would be important to understand how this phase behaves for finite values of λ, not just for small or strong coupling. In view of many exact results for scattering phases in sigma models, e.g., [47] (see also [35] in the present context), this is not a hopeless goal. Also, the above argument of self-consistent quantum corrections seems suggestive in this direction.

[24] A similar problem has been observed in the context of plane wave string field theory in [34] when the expansion for small $\lambda' = \lambda/J^2$ is done prior to summing.

[25] See [23] for a review of the fast string expansion.

[26] Related issues and ideas have been discussed in [35] which might be useful in this respect.

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Appendix A

Here we present some lengthy expressions which arise in the sum over frequencies $e(n)$. The exact expression for $e(n)$ is given in [25,28,29]

$$\begin{aligned} e(n) = {} & \frac{1}{4\kappa}(\omega_1+\omega_2-\omega_3-\omega_4) + \frac{1}{\kappa}\sqrt{n^2+\kappa^2} \\ & + \frac{2}{\kappa}\sqrt{n^2+\mathcal{J}^2-m^2} \\ & - \frac{2}{\kappa}\sqrt{(n-\gamma)^2+\tfrac{1}{2}\left(\kappa^2+\mathcal{J}^2-m^2\right)} \\ & - \frac{2}{\kappa}\sqrt{(n+\gamma)^2+\tfrac{1}{2}\left(\kappa^2+\mathcal{J}^2-m^2\right)}. \end{aligned} \tag{A.1}$$

Here the first two terms correspond to bosonic fluctuations along AdS_5, the third to bosonic fluctuations along S^5 and the remaining two to fermionic fluctuations. The frequencies $\omega_1, \dots, \omega_4$ are the solutions to the quartic equation

$$\begin{aligned} & \left(\omega^2-n^2\right)^2 - \frac{4\mathcal{J}m\kappa^2\omega^2}{k\sqrt{\kappa^2+k^2}} \\ & \quad - 4\left(1-\frac{\mathcal{J}m}{k\sqrt{\kappa^2+k^2}}\right)\left(\omega\sqrt{\kappa^2+k^2}-kn\right)^2 = 0 \end{aligned} \tag{A.2}$$

ordered in magnitude from largest to smallest. The shift γ is given by

$$\gamma = \frac{\kappa m}{\sqrt{\kappa^2+k^2}}\,\frac{\kappa^2-\mathcal{J}^2+k^2}{\kappa^2-\mathcal{J}^2+m^2} \tag{A.3}$$

and, finally, κ is determined by the equation

$$\left(\kappa^2-\mathcal{J}^2-m^2\right)\sqrt{\kappa^2+k^2} + 2\mathcal{J}km = 0. \tag{A.4}$$

When we expand for large $\mathcal{J}$ assuming $n = \mathcal{J}x$ to be of the order $\mathcal{J}$, we obtain

$$\begin{aligned} e^{\text{int}}(x) = {} & \frac{(k-m)^2}{32\mathcal{J}^4x^2(1+x^2)^{7/2}} \\ & \times \left[-16m^2 + \left(-3k^2+14km-75m^2\right)x^2\right. \\ & + \left(12k^2-32km+60m^2\right)x^4 \\ & \left. + \left(-16km-16m^2\right)x^6\right] \\ & + \frac{(k-m)^2}{256\mathcal{J}^6x^4(1+x^2)^{11/2}} \\ & \times \left[\left(-256km^3+256m^4\right)\right. \\ & + \left(64k^2m^2-1536km^3+1344m^4\right)x^2 \\ & + \left(15k^4-248k^3m+1118k^2m^2\right. \\ & \left. - 4624km^3+2907m^4\right)x^4 \\ & + \left(-180k^4+1420k^3m-2168k^2m^2\right. \\ & \left. - 3204km^3+2276m^4\right)x^6 \\ & + \left(120k^4+568k^3m-1892k^2m^2\right. \\ & \left. - 1728km^3+1076m^4\right)x^8 \\ & + \left(224k^3m-688k^2m^2-736km^3\right. \\ & \left. + 368m^4\right)x^{10} \\ & + \left(64k^3m-128k^2m^2-128km^3\right. \\ & \left. \left. + 64m^4\right)x^{12}\right] + \mathcal{O}\left(1/\mathcal{J}^8\right). \end{aligned} \tag{A.5}$$

Its singular part is given by

$$\begin{aligned} e^{\text{int}}_{\text{sing}}(x) = {} & -\frac{(k-m)^2m^2}{2\mathcal{J}^4x^2} - \frac{(k-m)^3m^3}{\mathcal{J}^6x^4} \\ & + \frac{(k-m)^2m^2}{4\mathcal{J}^6x^2}\left(k^2-2km-m^2\right) \\ & + \mathcal{O}\left(1/\mathcal{J}^8\right). \end{aligned} \tag{A.6}$$

We also state the large-n asymptotics of $e^{\text{int}}(x)$ which is agreement with $e(\mathcal{J}x)$

$$e^{\text{int}}(x) = -\frac{(k-m)^2(k+m)m}{x^3} \times \left(-\frac{1}{2\mathcal{J}^4} + \frac{k^2-3km+m^2}{4\mathcal{J}^6} + \mathcal{O}(1/\mathcal{J}^8)\right) + \mathcal{O}(1/x^4). \tag{A.7}$$

Now we assume n to be fixed and expand. The regular part in this case was found in [29]

$$\begin{aligned} e^{\text{sum}}_{\text{reg}}(n) = {} & \frac{(n^4-4(k-m)mn^2)^{1/2}}{4\mathcal{J}^2} \\ & - \frac{1}{4\mathcal{J}^2}\big[n^2 + (-2km+2m^2)\big] \\ & - \frac{(n^4-4(k-m)mn^2)^{-1/2}}{16\mathcal{J}^4} \\ & \times \big[n^6 + (6k^2-22km+12m^2)n^4 \\ & + (-20k^3m+80k^2m^2 \\ & - 84km^3+24m^4)n^2\big] \\ & + \frac{1}{16\mathcal{J}^4}\big[n^4 + (6k^2-20km+10m^2)n^2 \\ & + (-8k^3m+30k^2m^2-28km^3+6m^4)\big] \\ & + \frac{(n^4-4(k-m)mn^2)^{-3/2}}{32\mathcal{J}^6} \\ & \times \big[n^{12} + (15k^2-44km+25m^2)n^{10} \\ & + (15k^4-218k^3m+603k^2m^2 \\ & - 556km^3+164m^4)n^8 \\ & + (-106k^5m+1068k^4m^2-3128k^3m^3 \\ & + 3796k^2m^4-2014km^5+384m^6)n^6 \\ & + (180k^6m^2-1656k^5m^3 \\ & + 5256k^4m^4-7744k^3m^5 \\ & + 5684k^2m^6-1960km^7+240m^8)n^4\big] \\ & - \frac{1}{32\mathcal{J}^6}\big[n^6 + (15k^2-38km+19m^2)\,n^4 \\ & + (15k^4-128k^3m+279k^2m^2 \\ & - 202km^3+44m^4)n^2 \\ & + (-16k^5m+120k^4m^2-282k^3m^3 \\ & + 262k^2m^4-94km^5+10m^6)\big] \\ & + \mathcal{O}(1/\mathcal{J}^8) \end{aligned} \tag{A.8}$$

and the singular part reads

$$\begin{aligned} e^{\text{sum}}_{\text{sing}}(n) = {} & -\frac{(k-m)^2}{32\mathcal{J}^4}(3k^2-16km+19m^2) \\ & + \frac{(k-m)^2n^2}{64\mathcal{J}^6}(45k^2-162km+153m^2) \\ & + \frac{(k-m)^2}{256\mathcal{J}^6}(15k^4-248k^3m+766k^2m^2 \\ & - 752km^3+91m^4) + \mathcal{O}(1/\mathcal{J}^8). \end{aligned} \tag{A.9}$$

It can be verified that $e^{\text{int}}_{\text{sing}}(x) = e^{\text{sum}}_{\text{reg}}(\mathcal{J}x)$ and $e^{\text{sum}}_{\text{sing}}(n) = e^{\text{int}}_{\text{reg}}(n/\mathcal{J})$, at least as far as the expansion goes.

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