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Linear Algebra and its Applications 405 (2005) 67–73

LINEAR ALGEBRA AND ITS APPLICATIONS

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On semigroups of matrices with eigenvalue 1 in small dimensions ☆

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Received 15 July 2004; accepted 6 March 2005

Available online 22 April 2005

Submitted by T.J. Laffey

Abstract

Let $S \subset M_4(\mathbb{C})$ be a semigroup such that 1 is an eigenvalue of every $s \in S$. It is shown that S is reducible. A complete list of irreducible semigroups $S \subset M_3(\mathbb{C})$ with this spectral property is given.

Keywords: Matrices with eigenvalue 1; Groups and semigroups of matrices; Reducibility

1. Introduction

Irreducibility and other properties of semigroups of complex $n \times n$ matrices with the property that 1 is an eigenvalue of every matrix have been studied in [2]. There it was shown that for $n = 3$ and $n \geqslant 5$ there exist irreducible semigroups, and

☆ The work of the first author was supported in part by the Ministry of Education, Science and Sport of Slovenia. The work of the second author was supported by a KBN research grant (Poland).

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doi:10.1016/j.laa.2005.03.006

indeed groups, with this spectral property; the most natural examples being the special orthogonal groups in odd dimensions or the image of a connected absolutely simple algebraic group under the adjoint representation. While it is easy to see that for $n = 2$ no such semigroup exists, the case $n = 4$ was left unanswered. The purpose of this paper is to show that the same is true for $n = 4$. In the course of the proof, some information on the structure of such semigroups in the $n = 3$ case is needed. It was shown in [2] that every irreducible semigroup $S \subset M_3(\mathbb{C})$, such that $1 \in \sigma(s)$ for every $s \in S$, is conjugate to a subsemigroup of $SO_3(\mathbb{C})$. We sharpen this result, giving a completely different proof. We refer the reader to [3,7] for the basic facts on linear algebraic groups and their linear representations that are needed in this paper.

2. Semigroups of 3 × 3 matrices

The following result is a slight generalization of [2, Prop. 2.3]. For the sake of completeness we give the whole proof.

Lemma 2.1. *Let $S \subset M_n(\mathbb{C})$ be a semigroup such that $1 \in \sigma(s)$ for every $s \in S$. Assume that the Zariski closure $\overline{S}$ of S in $M_n(\mathbb{C})$ contains an element of rank at most two. Then S is reducible.*

Proof. Observe that the condition that 1 is in the spectrum of every s in S is polynomial, so it is preserved under Zariski closure. Since $\overline{S}$ is a semigroup (see [10, Thm. 1.2]) we may therefore assume that S is Zariski closed. Denote by I the semigroup ideal in S of all matrices with minimal positive rank, say r. By the assumption, $r \in \{1, 2\}$. If $r = 1$, then the trace is constant on I, namely $\mathrm{tr}(s) = 1$ for all $s \in I$. By Kaplansky's theorem [13, Cor. 2.2.3], we conclude that I is reducible. Now [13, Lem. 2.1.10] implies that S is reducible as well.

Consider the case $r = 2$. If for every $s \in I$ the multiplicity of 1 in $\sigma(s)$ is 2, then $\mathrm{tr}(s) = 2$ for all $s \in I$ which completes the proof as before. Otherwise, pick $a \in I$ with $\sigma(a) \supseteq \{1, \alpha\}$, where $\alpha \neq 0, 1$. Without loss of generality we may assume that a is a diagonal matrix $\mathrm{diag}(1, \alpha, 0, \dots, 0)$. Given $s = (s_{ij})_{i,j=1}^n \in S$ and $k \in \{1, 2\}$, the characteristic polynomial of $a^k s$ is equal to

$$\left(\lambda^2 - (s_{11} + \alpha^k s_{22})\lambda + \alpha^k(s_{11}s_{22} - s_{12}s_{21})\right)\lambda^{n-2}.$$

Since $1 \in \sigma(a^k s)$ we have

$$1 - s_{11} + \alpha^k(s_{11}s_{22} - s_{12}s_{21} - s_{22}) = 0$$

for $k = 1, 2$. Because $\alpha \neq 1$ we conclude that $s_{11} = 1$ for every $s \in S$. Then S is reducible by [13, Cor. 2.1.6]. □

Theorem 2.2. *Let $S \subset M_3(\mathbb{C})$ be an irreducible semigroup such that $1 \in \sigma(s)$ for every $s \in S$. Then* $\det(s) = 1$ *for every $s \in S$, S is conjugate to a subsemigroup of*

$SO_3(\mathbb{C})$ and is either a finite group, isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$, or the Zariski closure of S in $M_3(\mathbb{C})$ is conjugate to $SO_3(\mathbb{C})$.

Proof. Let G be the Zariski closure of S in $M_3(\mathbb{C})$. By the previous lemma we have $G \subset GL_3(\mathbb{C})$, so G is a linear algebraic group by [12, Thm. 3.18]. Let G^0 be the connected component of the identity in G. We now have to consider two cases according to whether G^0 is trivial or not.

Assume that G^0 is trivial. It follows that $G = S$ is a finite group. Observe that the centre $Z(G)$ of G is trivial. Let $H := \{g \in G; \det(g) = 1\}$. We want to show that $G = H$, so we assume the contrary. Observe that H is not trivial since G is not abelian. Since H is a normal subgroup of G, it is either irreducible or abelian and completely reducible by Clifford's theorem (see [6]). Suppose that the latter is the case. Then the space of column vectors decomposes as a sum of three non-isomorphic one-dimensional weight spaces $N_1 \oplus N_2 \oplus N_3$ under the action of H. Since G permutes these spaces, there is a homomorphism $\varphi : G \to \mathfrak{S}_3$ with $H \subset \ker\varphi$. It follows that $\mathrm{im}\varphi \subset \mathfrak{S}_3$ is an abelian group and therefore $\mathrm{im}\varphi \simeq \mathbb{Z}_3$ since by Ito's theorem 3 divides $[G : \ker\varphi]$ (see [6, Cor. 53.18]). Now, let $g \in G$ be such that $g \notin \ker\varphi$. It is easy to see that in the basis, corresponding to the decomposition described above, g can be written as $g = pd$, where p is a permutation matrix corresponding to a cycle of length three and $d = \mathrm{diag}(\alpha, \beta, \gamma)$ is a diagonal matrix. It follows that $\det(g) = \alpha\beta\gamma$ and $g^3 = \alpha\beta\gamma I$. Recall that the centre of G is trivial which implies $\det(g) = 1$; a contradiction.

It follows that the group H is irreducible. Observe that the conditions $\det(h) = 1$ and $1 \in \sigma(h)$ for every $h \in H$ together imply that $\mathrm{tr}(h) \in \mathbb{R}$ for every $h \in H$ because H is finite. The trace is also real on the (simple) $\mathbb{R}$-subalgebra $A \subset M_3(\mathbb{C})$ spanned by H. Since $H \subset M_3(\mathbb{C})$ is irreducible we have $A \otimes_{\mathbb{R}} \mathbb{C} = M_3(\mathbb{C})$ so A is a central simple $\mathbb{R}$-algebra. For dimensional reasons it follows that A is isomorphic to $M_3(\mathbb{R})$, whence conjugate to $M_3(\mathbb{R})$ by Skolem–Noether theorem. So we may assume, after conjugation, that H is contained in $M_3(\mathbb{R})$. It is well known that every finite (compact) group of invertible real matrices is conjugate to a group of real orthogonal matrices (see [6]). This, combined with the fact that $\det(h) = 1$ for every $h \in H$, allows us to assume, again after conjugation, that H is contained in $SO_3(\mathbb{R})$. It is well known (see [9, Chapter 15]) that H is then isomorphic to one of the following groups: $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$. Observe also, that the natural homomorphism $G \to \mathrm{Aut}(H)$ is injective. We now consider the three possible cases. If H is isomorphic to $\mathfrak{A}_4$, then G is either isomorphic to $\mathfrak{A}_4$ and therefore $G = H$ or G is isomorphic to $\mathfrak{S}_4$ (see [11]). In the second case we observe that of the two non-equivalent representations of $\mathfrak{S}_4$ of degree three only the one, say π, with the property $\det(\pi(g)) = 1$ for every $g \in \mathfrak{S}_4$ has the property that $1 \in \sigma(\pi(g))$ for every $g \in \mathfrak{S}_4$. It then follows $G = H$ by the definition of the group H. If H is isomorphic to $\mathfrak{S}_4$, then $G = H$ since every automorphism of $\mathfrak{S}_4$ is inner. Finally, if H is isomorphic to $\mathfrak{A}_5$, then $G = H$ since there is no irreducible representation of $\mathfrak{S}_5$ of degree three. This completes the proof in the case where G^0 is trivial.

If G^0 is not trivial, then it is clearly a reductive group by Clifford's theorem and can be written as an almost direct product of its central torus T and its derived subgroup $H = (G^0, G^0)$, where both T and H are normal in G. We want to show that T is trivial, so we assume the contrary. Since T is not central in G, the space of column vectors N decomposes as the sum of three one-dimensional weight spaces $N = N_1 \oplus N_2 \oplus N_3$, with the corresponding characters $\chi_i : T \to C^*$ non-trivial and mutually distinct. But there exists $t \in T$, such that $\chi(t) \neq 1$ for every non-trivial character χ of T (see [3, Prop. 8.8]). It follows that $1 \notin \sigma(t)$; a contradiction. Therefore, G^0 is a semisimple group. For dimensional reasons, the only two possibilities are that G^0 is of type A_2 and therefore equal to $SL_3(\mathbb{C})$ or that G^0 is of type A_1 and therefore isomorphic, and even conjugate, to $SO_3(\mathbb{C})$. The first possibility is ruled out by the spectral condition on G^0, therefore G^0 is isomorphic to $SO_3(\mathbb{C})$. Again we consider the natural morphism $G \to \mathrm{Aut}(G^0)$ of algebraic groups, which is clearly injective. Since every (algebraic) automorphism of $SO_3(\mathbb{C})$ is inner (see [3]), it follows that $G = G^0$ is conjugate to $SO_3(\mathbb{C})$ as claimed. □

We record the following immediate corollary of this theorem.

Corollary 2.3. *Let $S \subset M_3(\mathbb{C})$ be an irreducible semigroup, such that $1 \in \sigma(s)$ for all $s \in S$. Then every $s - I$, $s \in S$, has either rank two or zero.*

3. Semigroups of 4×4 matrices

The purpose of this section is to show that every semigroup $S \subset M_4(\mathbb{C})$ such that $1 \in \sigma(s)$ for every $s \in S$ is reducible. We first consider the possibility that S is a finite simple group of invertible matrices.

Lemma 3.1. *Let $G \subset GL_4(\mathbb{C})$ be an irreducible finite group such that $1 \in \sigma(g)$ for every $g \in G$. Then G is not simple.*

Proof. After checking the list of all possible finite simple groups with an irreducible character of degree 4 (see [1,8]) and using the isomorphisms of finite simple groups of small order (see [5]) it follows that, up to isomorphism, the only finite simple group with an irreducible character of degree 4 is the alternating group $\mathfrak{A}_5$ and this character is unique. Let π denote the corresponding representation of $\mathfrak{A}_5$ and let $a \in \mathfrak{A}_5$ be any element of order 5. It is easy to see that $1 \notin \sigma(\pi(a))$. □

Next we consider the possibility that S is a finite group of invertible matrices.

Proposition 3.2. *Let $G \subset GL_4(\mathbb{C})$ be a finite group such that $1 \in \sigma(g)$ for every $g \in G$. Then G is reducible.*

Proof. Assume that there exist irreducible finite groups in $GL_4(\mathbb{C})$ such that 1 is an eigenvalue of every element. Let G be such a group with the minimal possible order. By the above lemma G is not simple, so it contains a non-trivial maximal normal subgroup H. By our assumptions H cannot act irreducibly.

Suppose that H is not abelian. By Clifford's theorem the space of column vectors decomposes as $V_1 \oplus V_2$ for some subspaces V_1, V_2 of dimension 2, on which H acts irreducibly. Let π_1, π_2 be the corresponding representations. If H' is the derived subgroup of H then for $h \in H'$ we have $1 \in \sigma(\pi_i(h))$, $i = 1, 2$, if and only if $\pi_i(h) = I_2$. So every element $h \in H'$ is either of the form $\pi_1(h) \oplus I_2$ or $I_2 \oplus \pi_2(h)$. Now, by our assumptions there exist $h_i \in H'$, $i = 1, 2$, such that $\pi_i(h_i) \neq I_2$. But then $h = h_1 h_2 \in H'$ does not have eigenvalue 1; a contradiction. This shows that H must be abelian.

It is well known [14, Lemma 1.12] that the simple group G/H embeds into $\mathfrak{S}_r$ for some $r \leqslant 4$ in this case. Hence G/H is isomorphic either to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. On the other hand 4 divides $[G : H]$ by Ito's theorem, see [6, Cor. 53.18]; a contradiction. $\square$

We now consider the possibility that the semigroup S contains an element of rank at most three.

Proposition 3.3. *Let $S \subset M_4(\mathbb{C})$ be a semigroup such that $1 \in \sigma(s)$ for every $s \in S$. Assume that S contains an element of rank at most three. Then S is reducible.*

Proof. Again, we can assume that S is Zariski closed in $M_4(\mathbb{C})$. We want to show that S is reducible, so we assume the contrary. Then the ideal $I \subset S$ of elements of rank at most three is irreducible as well, so we may assume $S = I$. It then follows from Lemma 2.1 that every $s \in S$ has rank three. Let $J \subset S$ be the kernel of S, which exists by [12, 3.28] and is completely simple. In fact we get $S = J$, see [10, Thm. 3.5]. We identify S with its Rees matrix presentation $\mathscr{M}(G, X, Y; P)$, where X, Y are non-empty (possibly infinite) sets, G is a maximal subgroup of S and $P = (p_{ij})$ is a $Y \times X$ sandwich matrix with entries in G. So $G \subseteq eM_4(\mathbb{C})e$ for an idempotent e of rank 3 which is the identity of G, whence G is a subgroup of $GL_3(\mathbb{C})$. Let $\overline{P}$ be the matrix over $\mathbb{C}$ obtained by erasing the matrix brackets in every entry of the matrix P. It is known that irreducibility of S is equivalent to the condition $\mathrm{rk}(\overline{P}) = 4$ [10, Thm. 4.26]. Therefore, there exists a 2×2 submatrix Q of P such that $\mathrm{rk}(\overline{Q}) = 4$. Say

$$Q = \begin{pmatrix} p_{im} & p_{in} \\ p_{jm} & p_{jn} \end{pmatrix}.$$

Let $T \cong \mathscr{M}(G, 2, 2; Q)$ be the corresponding subsemigroup of S. In other words, T consists of all elements $(g, \alpha, \beta) \in \mathscr{M}(G, X, Y; P)$ such that $\alpha \in \{m, n\}$, $\beta \in \{i, j\}$. By the criterion used above, T also is irreducible. Hence, replacing S by T we may assume that $S = T$, whence $P = Q$ and $|X| = |Y| = 2$. It is well known that P may be normalized in such a way that every entry in the first row and every entry in the first column of P is the identity matrix, see [4, Cor. 3.17]. Notice that $G = eSe$ is an irreducible subgroup of $GL_3(\mathbb{C})$. As shown in Corollary 2.3, the rank of $p_{22} - I$ is either 0 or 2. Therefore, $\overline{P}$ has rank 3 or 5. This contradiction completes the proof of the proposition. □

We are now in a position to prove the main result of this paper.

Theorem 3.4. *Let $S \subset M_4(\mathbb{C})$ be a semigroup such that $1 \in \sigma(s)$ for every $s \in S$. Then S is reducible.*

Proof. Assume that S is irreducible and let G be the Zariski closure of S in $M_4(\mathbb{C})$. By the previous proposition we have $G \subset GL_4(\mathbb{C})$, so G is a linear algebraic group by [12, Thm. 3.18]. Clearly, G is irreducible and $1 \in \sigma(g)$ for every $g \in G$. Let G^0 denote the connected component of the identity in G. If it were trivial, then G would be an irreducible finite group, but this is impossible by Proposition 3.2. So we may assume that G^0 is not trivial. Since it is a normal subgroup of G, Clifford's theorem implies that G^0 is a reductive group. As such, it is an almost direct product of its central torus T and its derived group $H = (G^0, G^0)$ which is semisimple, where both T and H are normal in G. Assume that T is not trivial. Since T is not central in G, the space N of column vectors decomposes under the action of T either as the sum of two weight spaces $N = N_1 \oplus N_2$ or as the sum of four weight spaces $N = N_1 \oplus \cdots \oplus N_4$, where in both cases the corresponding characters $\chi_i : T \to \mathbb{C}^*$ are non-trivial and mutually distinct. But, again, there exists $t \in T$, such that $\chi(t) \neq 1$ for every non-trivial character χ of T. It follows that $1 \notin \sigma(t)$; a contradiction. Therefore, $G^0 = H$ is a semisimple group. We now consider different possibilities for G^0. If it acts completely reducibly on N, then for dimensional reasons G^0 is either of type A_1 or $A_1 + A_1$. In either case there exists a $g \in G^0$ such that $1 \notin \sigma(g)$; a contradiction. If it acts irreducibly on N, then the possible types for G^0 are A_1, $A_1 + A_1$, A_3 and B_2. Again, it is easy to see that in each case there exists an element $g \in G^0$ such that $1 \notin \sigma(g)$; a contradiction. □

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