Realization of the Dynamical Group for the Generalized Laguerre Functions

SHI-HAI DONG
Instituto de Ciencias Nucleares, UNAM
Apdo. Postal. 70-543, Circuito Exterior, C. U. 04510 Mexico, D. F., Mexico
and
Programa de Ingeniería Molecular, Instituto Mexicano del Petróleo
Lázaro Cardenas 152, 07730 Mexico, D.F., Mexico
dongsh2@yahoo.com

(Received May 2002; accepted February 2003)

Abstract—The ladder operators for the generalized Laguerre functions are constructed by a factorization method. It is shown that these ladder operators satisfy the commutation relations of an SU(1,1) algebra. The matrix elements of some operators $x$ and $2x \frac{d}{dx}$ are analytically evaluated from these ladder operators. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Factorization method, Ladder operators, SU(1,1) algebra, Generalized Laguerre functions, Matrix element.

1. INTRODUCTION

The factorization method plays an important role in physics [1-3]. The advantage of this method is that we can construct the ladder operators for some given wave functions and then constitute a suitable dynamical group. Recently, we have carried out the ladder operators for the Morse potential, the modified Pöschl-Teller potential, and the pseudoharmonic potential [4-7]. It is found that those of the former two molecular potentials satisfy an SU(2) algebra, but those of the latter satisfy an SU(1,1) one. It is stressed that our approach is different from the traditional one, where an auxiliary nonphysical variable was introduced [8]. Simultaneously, the matrix elements of some related operators, which are of significance in physics, can be analytically calculated from the ladder operators. Recently, such a factorization method has been generalized to some special functions [9]. The coherent states for generalized Laguerre functions has been worked out by Jellal, where the coherent states such as the Klauder-Perelomov coherent states, the Gazeau-Klauder coherent states, and Barut-Girardello coherent states have been studied by SU(1,1) algebra [10]. Following our previous works [4-7], the purpose of this paper is to study the dynamical group for the generalized Laguerre functions with our approach, which are of great
importance to the solutions of the Schrödinger equation with the central symmetric potentials, whose dynamical group can be carried out in the similar way as done in this work.

This paper is organized as follows. In Section 2, we study the properties of the generalized Laguerre functions. In Section 3, we construct the ladder operators by the factorization method and then constitute a suitable dynamical group. The matrix elements of some related operators \( x \) and \( 2x \frac{d}{dx} \) are calculated from the ladder operators. The concluding remarks are given in Section 4.

2. THE GENERALIZED LAGUERRE FUNCTIONS

We begin with the definition of the generalized Laguerre functions [10]

\[
|n\rangle \equiv \Psi_n(x) = N_n^\beta e^{-x/2} x^{\beta/2} L_n^\beta(x),
\]

with

\[
N_n^\beta = \sqrt{\frac{n!}{(n + \beta)!}},
\]

where \( \beta \) is integer.

The generalized Laguerre functions obey the orthonormal condition

\[
\int_0^\infty \Psi_n^\beta(x) \Psi_{n'}^\beta(x) \, dx = \delta_{nn'}.
\]

Since the associated Laguerre functions \( L_n^\alpha(x) \) satisfies the following differential equation (see [11, p. 1064])

\[
x \frac{d^2}{dx^2} L_n^\alpha(x) + (\alpha + 1 - x) \frac{d}{dx} L_n^\alpha(x) + \alpha L_n^\alpha(x) = 0,
\]

we obtain the differential equation for the generalized Laguerre functions as

\[
x \frac{d^2}{dx^2} \Psi_n^\beta(x) + \frac{d}{dx} \Psi_n^\beta(x) + \frac{1}{4} \left( 2 + 2\beta - x - \frac{\beta^2}{x} \right) \Psi_n^\beta(x) + n\Psi_n^\beta(x) = 0.
\]

Furthermore, there are some recursion relations for the associated Laguerre functions (see [11, p. 1062]), which will be used to construct the ladder operators below

\[
(n + 1)L_{n+1}^\beta(x) + (n + \beta)L_{n-1}^\beta(x) + (x - 2n - \beta - 1)L_n^\beta(x) = 0,
\]

\[
x \frac{d}{dx} L_n^\beta(x) = nL_n^\beta(x) - (n + \beta)L_{n-1}^\beta(x),
\]

\[
x \frac{d}{dx} L_n^\beta(x) = (n + 1)L_{n+1}^\beta(x) - (n + \beta + 1 - x)L_n^\beta(x).
\]

Calculating the derivative of the generalized Laguerre functions \( \Psi_n^\beta(x) \), we have

\[
\frac{d}{dx} \Psi_n^\beta(x) = -\frac{1}{2} \Psi_n^\beta(x) + \frac{\beta + 2n}{2x} \Psi_n^\beta(x) + N_n^\beta e^{-x/2} x^{\beta/2} \frac{d}{dx} L_n^\beta(x),
\]

which, together with equation (6), allows us to obtain the recursion relations for the generalized Laguerre functions

\[
\sqrt{(n + 1)(n + \beta + 1)}|n+1\rangle + \sqrt{n(n + \beta)|n-1\rangle} = (2n + \beta + 1 - x)|n\rangle,
\]

\[
\left[ \frac{d}{dx} - \frac{1}{2} + \frac{\beta + 2n}{2x} \right] \Psi_n^\beta(x) = \frac{n + \beta}{x} N_n^\beta \Psi_{n-1}^\beta(x),
\]

\[
\left[ \frac{d}{dx} - \frac{1}{2} + \frac{\beta + 2n + 2}{2x} \right] \Psi_n^\beta(x) = \frac{n + 1}{x} N_n^\beta \Psi_{n+1}^\beta(x).
\]

In the next section, we will realize the dynamical group for the generalized Laguerre functions, which is the main purpose of this paper.
3. THE REALIZATION OF THE DYNAMICAL GROUP

We now address how to construct the ladder operators for the generalized Laguerre functions by the factorization method and then constitute their dynamical group. We intend to find the differential operators \( \hat{\mathcal{L}}_{\pm} \) with the property

\[
\hat{\mathcal{L}}_{\pm} |n> = \ell_{\pm} |n \pm 1>.
\]

For convenience, we first define the “number” operator \( \hat{n} \)

\[
\hat{n} |n> = n |n>.
\]

The creation and annihilation operators \( \hat{\mathcal{L}}_{\pm} \) can be obtained from equations (9) and (10)

\[
\frac{d\hat{\mathcal{L}}_+}{dx} = \frac{x}{2} + \frac{\beta}{2} + \hat{n} + 1, \quad \frac{d\hat{\mathcal{L}}_-}{dx} = -\frac{x}{2} + \frac{\beta}{2} + \hat{n},
\]

which satisfy

\[
\hat{\mathcal{L}}_+ |n> = \ell_+ |n + 1>, \quad \ell_+ = \sqrt{(n + 1)(\beta + n + 1)},
\]

\[
\hat{\mathcal{L}}_- |n> = \ell_- |n - 1>, \quad \ell_- = \sqrt{n(\beta + n)}.
\]

Obviously, the operator \( \hat{\mathcal{L}}_- \) annihilates the ground state

\[
|0> = \frac{1}{\sqrt{\beta!}} e^{\beta/2} e^{-x/2}.
\]

The commutator \([\hat{\mathcal{L}}_-, \hat{\mathcal{L}}_+]\) can be calculated in the bases \(|n>\)

\[
[\hat{\mathcal{L}}_-, \hat{\mathcal{L}}_+] |n> = (2n + \beta + 1) |n>.
\]

We can thus define the operator \( \hat{\mathcal{L}}_0 \)

\[
\hat{\mathcal{L}}_0 = \hat{n} + \frac{\beta + 1}{2},
\]

\[
\hat{\mathcal{L}}_0 |n> = \ell_0 |n>, \quad \ell_0 = n + \frac{\beta + 1}{2}.
\]

At least, in the spaces spanned by the generalized Laguerre functions \(|n>\), the operators \( \hat{\mathcal{L}}_{\pm} \) and \( \hat{\mathcal{L}}_0 \) satisfy the commutation relations of the SU(1,1) algebra, which is isomorphic to an SO(2,1) algebra

\[
[\hat{\mathcal{L}}_-, \hat{\mathcal{L}}_+] = 2 \hat{\mathcal{L}}_0, \quad [\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_{\pm}] = \pm \hat{\mathcal{L}}_{\pm}.
\]

As is well known [12], there are four series of irreducible unitary representations for the SU(1,1) algebra except for the identity representation. They are the representation \( D^+(j) \) with a spectrum bounded below, the representation \( D^-(j) \) with a spectrum bounded above, the supplementary series \( D_s(Q,q_0) \) and the principle series \( D_P(Q,q_0) \). Since there is the ground state \(|0>\) in the generalized Laguerre functions \(|n>\), the representation of the dynamical group SU(1,1) belongs to \( D^+(j) \) [12]

\[
I_0|j, m> = m|j, m>,
\]

\[
I_-|j, m> = [(m + j)(m - j - 1)]^{1/2} |j, m - 1>,
\]

\[
I_+|j, m - 1> = [(m + j)(m - j - 1)]^{1/2} |j, m>,
\]
In comparison with equations (14), (15), and (19), we have \( j = -(\beta + 1)/2, \ m = n + (\beta + 1)/2, \) and \( |n\rangle = |j, m\rangle. \)

The Casimir operator can be written as

\[
\hat{C} = \hat{L}_0 \left( \hat{L}_0 - 1 \right) - \hat{L}_+ \hat{L}_-, \quad \hat{C}|n\rangle = \frac{\beta^2 - 1}{4}|n\rangle.
\]

Finally, we define the Hamiltonian as

\[
\hat{H}|n\rangle = \hat{L}_+ \hat{L}_- |n\rangle = n(n + \beta)|n\rangle,
\]

The generalized Laguerre functions can be expressed as

\[
|n\rangle = N_n^\beta \mathcal{L}_n^\alpha(0),
\]

with

\[
N_n^\beta = \sqrt{\frac{\beta!}{n!(n + \beta)!}}.
\]

The following expressions can be easily obtained from the operators \( \hat{L}_\pm \) and \( \hat{L}_0 \)

\[
x = 2\hat{L}_0 - \hat{L}_+ - \hat{L}_-, \quad 2x \frac{d}{dx} = \hat{L}_+ - \hat{L}_- - 1,
\]

from which we have

\[
\langle m|x|n\rangle = (2n + \beta + 1)\delta_{mn} - \sqrt{(n + 1)(n + \beta + 1)}\delta_{m(n+1)} - \sqrt{n(n + \beta)}\delta_{m(n-1)}
\]

and

\[
\langle m|2x \frac{d}{dx}|n\rangle = \sqrt{(n + 1)(n + \beta + 1)}\delta_{m(n+1)} - \sqrt{n(n + \beta)}\delta_{m(n-1)} - \delta_{mn}.
\]

Before ending this section, it is worthwhile to emphasize that equation (8) can be reexpressed as

\[
x|n\rangle = \left(2\hat{L}_0 - \hat{L}_+ - \hat{L}_-\right)|n\rangle.
\]

4. CONCLUDING REMARKS

In this paper, we have established the creation and annihilation operators \( \hat{L}_\pm \) for the generalized Laguerre functions by the factorization method. It is shown that these operators \( \hat{L}_\pm \) and \( \hat{L}_0 \) satisfy the commutation relations of an SU(1,1) algebra. The matrix elements of some operators \( x \) and \( 2x \frac{d}{dx} \) are analytically calculated from these operators. This method can be generalized to other wave functions and presents a simple and elegant approach to calculate some matrix elements.

REFERENCES


