ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 322 (2006) 359–370
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Oscillation properties of nonlinear impulsive delay differential equations and applications to population models ☆

Jurang Yan [a,b,*], Aimin Zhao [a], Quanxin Zhang [b]

[a] *School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, PR China*
[b] *Department of Mathematics, Binzhou College, Binzhou, Shandong 256600, PR China*

Received 5 May 2005

Available online 13 October 2005

Submitted by K. Gopalsamy

Abstract

Comparison theorem and explicit sufficient conditions are obtained for oscillation and nonoscillation of solutions of nonlinear impulsive delay differential equations which can be utilized to population dynamic models. Our results in this paper generalize and improve several known results.

Keywords: Oscillation; Nonoscillation; Impulse; Delay differential equation

1. Introduction

Impulsive delay differential equations may express several real-world simulation processes which depend on their prehistory and are subject to short time disturbances. Such processes occur in theory of optimal control, population dynamics, biotechnologies, economics, etc. In recent years, oscillation theory of solutions of the delay differential equations with impulsive effects or without impulsive effects has been an object of active research; we refer to [1–11,14, 17–19]. For other relative works of study for impulsive delay differential equation we refer to

☆ Supported by the NNSF of China (No. 10071045) and the NSF of Shanxi Province of China.
* Corresponding author.
E-mail address: jryan@sxu.edu.cn (J. Yan).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.09.023

[12,13,15,16]. But, concerning the oscillation properties of impulsive delay differential equation in population dynamics are relatively scarce.

The purpose of this paper is to study oscillation and nonoscillation of nonlinear impulsive delay differential equations and their applications to population models. Our results in this paper generalize and improve several known results in [4,14,19].

Consider the impulsive delay differential equation

$$y'(t) = -\big(1 + y(t)\big) f\big(t, y\big(g(t)\big)\big), \quad t \neq \tau_k,\ t \geqslant 0, \text{ a.e.,} \tag{1.1$_a$}$$

$$y\big(\tau_k^+\big) = (1 + b_k) y(\tau_k), \quad k = 1, 2, \ldots, \tag{1.1$_b$}$$

under the following assumptions:

(A$_1$) $0 \leqslant \tau_1 < \tau_2 < \cdots < \tau_k < \cdots$ are fixed points with $\lim_{k\to\infty} \tau_k = \infty$;
(A$_2$) $f(t, y): [0, \infty) \times R \to R$ satisfies Caratheodory conditions, that is, $f(t, y)$ is locally bounded Lebesgue measurable in t for each fixed y and is continuous in y for each fixed t; $f(t, 0) = 0$ for all $t \geqslant 0$;
(A$_3$) $g(t): [0, \infty) \to R$ is Lebesgue measurable function and $g(t) \leqslant t$ with $\lim_{t\to\infty} g(t) = \infty$;
(A$_4$) $\{b_k\}$ is a sequence of constants and $b_k > -1$, $k = 1, 2, \ldots$.

When $b_k = 0$, $k = 1, 2, \ldots$, (1.1) ((1.1)$_a$ and (1.1)$_b$) reduces to the delay differential equation

$$y'(t) = -\big(1 + y(t)\big) f\big(t, y\big(g(t)\big)\big), \quad t \geqslant 0, \text{ a.e.} \tag{1.2}$$

Oscillation and nonoscillation of some special forms of (1.2) have been extensively investigated in the literature as population dynamics models. For example, the delay logistic equation

$$N'(t) = p(t)N(t)\left[1 - \frac{N(g(t))}{K}\right]$$

is known as Hutchinson's equation. The food-limited equation is

$$N'(t) = p(t)N(t)\frac{K - N(g(t))}{K + s(t)N(g(t))},$$

etc. Their oscillation properties have been studied in [8,11], respectively. They can been reduced to the form of (1.2) by the changes of variable $y(t) = N(t)/K - 1$.

For any $t_0 \geqslant 0$, let $t_0^- = \inf_{t \geqslant t_0} g(t)$. Set $\Phi(t_0)$ denote the set of functions $\phi : [t_0^-, t_0] \to R$ which are bounded Lebesgue measurable on $[t_0^-, t_0]$.

Definition 1.1. For any $t_0 \geqslant 0$ and $\Phi(t_0)$, a function $y : [t_0^-, \infty) \to R$ is said to be a solution of (1.1) on $[t_0, \infty)$ satisfying the initial value condition

$$y(t) = \phi(t), \quad \phi(t_0) > 0, \quad t \in [t_0^-, t_0], \tag{1.3}$$

if the following conditions are satisfied:

(i) $y(t)$ satisfies (1.3);
(ii) $y(t)$ is absolutely continuous in each interval (t_0, τ_{k_0}), (τ_k, τ_{k+1}), $k \geqslant k_0$, $y(\tau_k^+)$, $y(\tau_k^-)$ exist and $y(\tau_k^-) = y(\tau_k)$;
(iii) $y(t)$ satisfies (1.1)$_a$ a.e. on $[t_0, \infty)\backslash\{\tau_k\}$ and satisfies (1.1)$_b$ for every $\tau_k \geqslant t_0$.

Definition 1.2. A solution of (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

2. Main results

In this section, we first give a fundamental lemma that enables us to reduce the oscillation and nonoscillation of the solutions of (1.1) to the corresponding problems for a nonimpulsive delay differential equation.

For any $t_0 \geqslant 0$, consider the nonlinear delay differential equation

$$x'(t) = -\prod_{t_0 \leqslant \tau_k < t} (1+b_k)^{-1}\Bigl(1+\prod_{t_0 \leqslant \tau_k < t}(1+b_k)x(t)\Bigr) f\Bigl(t, \prod_{t_0 \leqslant \tau_k < g(t)} (1+b_k)x\bigl(g(t)\bigr)\Bigr),$$
$$t \geqslant 0, \text{ a.e.} \tag{2.1}$$

By a solution $x(t)$ of (2.1) and (1.3) on $[t_0, \infty)$, we mean a function which is absolutely continuous on $[t_0, \infty)$, satisfies (2.1) a.e. on $[t_0, \infty)$ and satisfies (1.3) on $[t_0^-, t_0]$. The definition of oscillation is same as Definition 1.2.

The following Lemma 2.1 will be used repeatedly in the proofs of our results below. Its proof is similar to that in [18, Theorem 1] and will be omitted.

Lemma 2.1. *Assume that* (A_1)–(A_4) *hold. For any* $t_0 \geqslant 0$, $y(t)$ *is a solution of* (1.1) *on* $[t_0, \infty)$ *if and only if*

$$x(t) = \prod_{t_0 \leqslant \tau_k < t} (1+b_k)^{-1} y(t) \tag{2.2}$$

is a solution of (2.1) *on* $[t_0, \infty)$.

In the following, we assume from an ecological point of view that

$$1 + y(t) > 0 \quad \text{for } t \geqslant t_0, \tag{2.3}$$

and hence, in view of (2.2),

$$1 + \prod_{t_0 \leqslant \tau_k < t} (1+b_k)x(t) > 0 \quad \text{for } t \geqslant t_0. \tag{2.4}$$

Our first main result below is a comparison theorem for oscillation of solutions of (1.1).

Theorem 2.1. *Assume that* (A_1)–(A_4), (2.3) *hold and there exists a locally bounded Lebesgue measurable function* $p(t) : [0, \infty) \to [0, \infty)$ *satisfying*

$$\frac{f(t, y)}{y} \geqslant p(t) \quad \text{for all } y \neq 0 \text{ and } t \geqslant 0, \tag{2.5}$$

and

$$\prod_{t_0 \leqslant \tau_k < t} (1+b_k) \quad \text{is bounded} \quad \text{and} \quad \liminf_{t\to\infty} \prod_{t_0 \leqslant \tau_k < t} (1+b_k) > 0. \tag{2.6}$$

If there exists a constant $0 < \delta < 1$ *such that all solutions of*

$$z'(t) + (1-\delta) \prod_{g(t) \leqslant \tau_k < t} (1+b_k)^{-1} p(t) z\bigl(g(t)\bigr) = 0 \tag{2.7}$$

are oscillatory, then all solutions of (1.1) *are also oscillatory.*

Proof. First, we show that

$$\int_0^\infty \prod_{g(t)\leqslant\tau_k<t}(1+b_k)^{-1}p(t)\,dt=\infty \tag{2.8}$$

if all solutions of (2.7) are oscillatory. Otherwise, there exists a large $T>0$ such that for all $t\geqslant T$ and any sufficiently small $\delta>0$,

$$(1-\delta)\int_{g(t)}^{t} \prod_{g(s)\leqslant\tau_k<s}(1+b_k)^{-1}p(s)\,ds\leqslant\frac{1}{e}.$$

By a known result (see [8, p. 42]), for any sufficient small $\delta>0$ (2.7) has a nonoscillatory solution. This is a contradiction.

Now, suppose that (1.1) has an eventually positive solution $y(t)$ which is defined $[t_0,\infty)$ and $y(t)>0$ for $t\geqslant t_1\geqslant t_0$. By Lemma 2.1, $x(t)=\prod_{t_0\leqslant\tau_k<t}(1+b_k)^{-1}y(t)$ is a solution of (2.1) and $x(t)>0$ for $t\geqslant t_1$. From (2.5) we have that for $t\geqslant t_2\geqslant t_1$,

$$f\biggl(t,\prod_{t_0\leqslant\tau_k<g(t)}(1+b_k)x\bigl(g(t)\bigr)\biggr)\geqslant\prod_{t_0\leqslant\tau_k<g(t)}p(t)x\bigl(g(t)\bigr).$$

In view of (2.1) and (2.4), we obtain

$$x'(t)\leqslant-\prod_{g(t)\leqslant\tau_k<t}(1+b_k)^{-1}p(t)x\bigl(g(t)\bigr)\leqslant-(1-\delta)\prod_{g(t)\leqslant\tau_k<t}(1+b_k)^{-1}p(t)x\bigl(g(t)\bigr).$$

By a known result (see [8, p. 50]), (2.7) has an eventually positive solution. This is a contradiction.

Suppose that (1.1) has an eventually negative solution $y(t)>-1$ defined on $[t_0,\infty)$ and $y(t)<0$ for $t\geqslant t_1\geqslant t_0$. By Lemma 2.1 $x(t)=\prod_{t_0\leqslant\tau_k<t}(1+b_k)^{-1}y(t)$ is a negative solution of (2.1) on $[t_1,\infty)$. From (2.5) we have

$$f\biggl(t,\prod_{t_0\leqslant\tau_k<g(t)}(1+b_k)x\bigl(g(t)\bigr)\biggr)\leqslant\prod_{t_0\leqslant\tau_k<g(t)}(1+b_k)p(t)x\bigl(g(t)\bigr),\quad t\geqslant t_2\geqslant t_1.$$

Thus from (2.1) and (2.4), we obtain

$$x'(t)\geqslant-\biggl(1+\prod_{t_0\leqslant\tau_k<t}(1+b_k)x(t)\biggr)\prod_{g(t)\leqslant\tau_k<t}(1+b_k)^{-1}p(t)x\bigl(g(t)\bigr)\geqslant 0,\quad t\geqslant t_2\geqslant t_1.$$

This implies that $x(t)$ is nondecreasing. By using (2.6) and (2.8), it is easy to prove

$$\lim_{t\to\infty}\prod_{t_0\leqslant\tau_k<t}(1+b_k)x(t)=0.$$

Hence there exists a constant $0<\delta_1\leqslant\delta$ such that for all $t\geqslant t_3\geqslant t_2$,

$$x'(t)\geqslant-(1-\delta_1)\prod_{g(t)\leqslant\tau_k<t}(1+b_k)^{-1}p(t)x\bigl(g(t)\bigr).$$

By a known result, the delay differential equation

$$x'(t)+(1-\delta_1)\prod_{g(t)\leqslant\tau_k<t}(1+b_k)^{-1}p(t)x\bigl(g(t)\bigr)=0$$

also has an eventually negative solution and hence (2.7) also has an eventually negative solution. This leads to a contradiction again. The proof of Theorem 2.1 is complete. □

Remark 2.1. Condition (2.6) will be satisfied, for example, if $\lim_{t\to\infty}\prod_{t_0\leqslant\tau_k<t}(1+b_k)$ converges.

Corollary 2.1. *Assume that* (A_1)–(A_4)*,* (2.3)*,* (2.5) *and* (2.6) *hold. If either*

$$\liminf_{t\to\infty}\int_{g(t)}^{t}\prod_{g(s)\leqslant\tau_k<s}(1+b_k)^{-1}p(s)\,ds>\frac{1}{e}\tag{2.9}$$

or $g(t)$ is nondecreasing and

$$\limsup_{t\to\infty}\int_{g(t)}^{t}\prod_{g(s)\leqslant\tau_k<s}(1+b_k)^{-1}p(s)\,ds>1,\tag{2.10}$$

then all solutions of (1.1) *are oscillatory.*

Proof. Suppose that (2.9) is satisfied. Then we can choose a constant $0<\delta<1$ such that

$$(1-\delta)\liminf_{t\to\infty}\int_{g(t)}^{t}\prod_{g(s)\leqslant\tau_k<s}(1+b_k)^{-1}p(s)\,ds>\frac{1}{e}.$$

By a well-known result, all solutions of (2.7) are oscillatory. From Lemma 2.1 and Theorem 2.1, we see that all solutions of (1.1) are also oscillatory.

Suppose that (2.10) is satisfied. We can choose constant $0<\delta<1$ such that

$$(1-\delta)\limsup_{t\to\infty}\int_{g(t)}^{t}\prod_{g(s)\leqslant\tau_k<s}(1+b_k)^{-1}p(s)\,ds>1.$$

Hence by a known result, all solutions of (1.1) are also oscillatory. □

Remark 2.2. Corollary 2.1 generalizes and improves Theorem 2.2 in [19]. For nonimpulsive delay differential equation $y'(t)=-(1+y(t))p(t)y(g(t))$, $\int_0^\infty p(t)\,dt=\infty$ is assumed in [19, Theorem 2.2] for all solutions of the equation to be oscillatory.

Theorem 2.2. *Assume that* (A_1)–(A_4)*,* (2.3) *hold and there exists a locally bounded Lebesgue measurable function* $p(t):[0,\infty)\to[0,\infty)$ *such that*

$$0<\frac{f(t,y)}{y}\leqslant p(t)\quad\text{for all } y\neq 0 \text{ and } t\geqslant 0,\tag{2.11}$$

and there exists $t_0>0$ satisfying

$$\int_{g(t)}^{t}\prod_{g(s)\leqslant\tau_k<s}(1+b_k)^{-1}p(s)\,ds\leqslant\frac{1}{e},\quad\text{for all } t\geqslant t_0.\tag{2.12}$$

Then (1.1) *has a nonoscillatory solution.*

Proof. By Lemma 2.1, we only prove that (2.1) has a nonoscillatory solution. Let $C[t_0^-,\infty)$ denote a locally convex linear space of all continuous functions on $[t_0^-,\infty)$, where $t_0^- = \inf_{t\geqslant t_0} g(t) < t_0$, with the topology of uniform convergence on compact subsets of $[t_0,\infty)$. Define a set S of continuous functions on $[t_0^-,\infty)$ satisfies the following properties:

(p_1) $x(t)$ is nondecreasing on $[t_0^-,\infty)$;
(p_2) $-(1-\delta) \leqslant x(t) \leqslant -(1-\delta)\exp[-e\int_{t_0}^t \prod_{g(s)\leqslant\tau_k<s}(1+b_k)^{-1}p(s)\,ds]$, $t \geqslant t_0$;
(p_3) $x(t) = -(1-\delta)$ on $[t_0^-, t_0]$;
(p_4) $x(t)e \leqslant x(g(t))$ on $[t_0,\infty)$,

where $0<\delta<1$ is a constant. It is obvious that S is nonempty. For example, the function $\bar{x}(t) = -(1-\delta)\exp[-\int_{t_0}^t \prod_{g(s)\leqslant\tau_k<s}(1+b_k)^{-1}p(s)\,ds] \in S$. Moreover, S is a closed convex subset of $C[t_0^-,\infty)$.

Now, we define a map $F: S \to C[t_0^-,\infty)$ as follows:

$$(Fx)(t) = \begin{cases} -(1-\delta), & t_0^- \leqslant t \leqslant t_0, \\ -(1-\delta)\exp\bigl[-\int_{t_0}^t (\xi x)(s)\,ds\bigr], & t \geqslant t_0, \end{cases} \tag{2.13}$$

where

$$(\xi x)(t) = \biggl[\prod_{t_0\leqslant\tau_k<t}(1+b_k)^{-1}\biggl(1+\prod_{t_0\leqslant\tau_k<t}(1+b_k)x(t)\biggr) \times f\biggl(t, \prod_{t_0\leqslant\tau_k<g(t)}(1+b_k)x\bigl(g(t)\bigr)\biggr)\biggr]\Big/x(t).$$

We first verify $FS \subset S$; it is easy to see that $(Fx)(t)$ is nondecreasing and $(Fx)(t) \geqslant -(1-\delta)$ for $t \geqslant t_0^-$. From (2.11), (p_2) and (p_4) we obtain

$$\begin{aligned} &\int_{t_0}^t (\xi x)(s)\,ds \\ &\quad\leqslant \int_{t_0}^t \biggl[\prod_{g(s)\leqslant\tau_k<s}(1+b_k)^{-1}p(s)\biggl(1+\prod_{t_0\leqslant\tau_k<s}(1+b_k)x(s)\biggr)x\bigl(g(s)\bigr)\big/x(s)\biggr]ds \\ &\quad\leqslant e\int_{t_0}^t \prod_{g(s)\leqslant\tau_k<s}(1+b_k)^{-1}p(s) \\ &\qquad\times\biggl[1-(1-\delta)\prod_{t_0\leqslant\tau_k<s}(1+b_k)\exp\biggl(-e\int_{t_0}^s \prod_{t_0\leqslant\tau_k<u}(1+b_k)^{-1}p(u)\,du\biggr)\biggr]ds \\ &\quad\leqslant e\int_{t_0}^t \prod_{g(s)\leqslant\tau_k<s}(1+b_k)^{-1}p(s)\,ds. \end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we find

$$-(1-\delta) \leqslant (Fx)(t) \leqslant -(1-\delta)\exp\left[-e\int_{t_0}^{t} \prod_{g(s)\leqslant\tau_k<s} (1+b_k)^{-1} p(s)\,ds\right].$$

Thus Fx satisfies (p_2). In addition, we also find

$$\frac{(Fx)(t)}{(Fx)(g(t))} = \exp\left[-\int_{g(t)}^{t} (\xi x)(s)\,ds\right] \geqslant \exp\left[-e\int_{g(t)}^{t} \prod_{g(s)\leqslant\tau_k<s} (1+b_k)^{-1} p(s)\,ds\right] \geqslant \frac{1}{e},$$

which implies that Fx satisfies (p_4). Therefore $FS \subset S$.

The continuity of $F : S \to S$ is verified as follows: let $x_n \in S$, $x \in S$ with $\lim_{n\to\infty} x_n = x$. Set $t_2 > t_1$ be a fixed number. By the uniformly convergence of $\lim_{n\to\infty} x_n = x$ on $[t_1, t_2]$, we have that for any $\epsilon > 0$ there exists a positive integer N_ϵ such that

$$\sup_{t_1\leqslant s\leqslant t_2} \left|(\xi x_n)(s) - (\xi x)(s)\right| < \frac{\epsilon}{(1-\delta)(t_2-t_1)} \quad \text{for all } n \geqslant N_\epsilon.$$

Hence from inequality $|e^{-x} - e^{-y}| \leqslant |x-y|$ when $x > 0$ and $y > 0$, we obtain

$$\begin{aligned}\left|(Fx_n)(t) - (Fx)(t)\right| &= (1-\delta)\left|\exp\left(-\int_{t_0}^{t} (\xi x_n)(s)\,ds\right) - \exp\left(-\int_{t_0}^{t} (\xi x)(s)\,ds\right)\right| \\ &\leqslant \left|\int_{t_0}^{t} \left[(\xi x_n)(s) - (\xi x)(s)\right] ds\right| < \epsilon, \quad \text{for all } n \geqslant N_\epsilon \text{ and } t_1 \leqslant t \leqslant t_2.\end{aligned}$$

The continuity of F on S is obtained.

Since

$$\left|\frac{d}{dt}(Fx)(t)\right| = (1-\delta)\exp\left[-\int_{t_0}^{t} (\xi x)(s)\,ds\right]\left|(\xi x)(t)\right|$$

is uniformly bounded in x for t on $[t_1, t_2]$, it follows that the family FS is equibounded, which implies that FS is precompact.

Now, by Schauder–Techonoff fixed point theorem, we conclude that F has a fixed point in S. That is, there is a $\tilde{x} \in S$ and $\tilde{x}(t) = (F\tilde{x})(t)$ on $[t_1, t_2]$. Since $t_2 > t_1$ is arbitrary, we have that for all $t \geqslant t_0$,

$$-(1-\delta) \leqslant \tilde{x}(t) \leqslant -(1-\delta)\exp\left[-\int_{t_0}^{t} \prod_{g(s)\leqslant\tau_k<s} (1+b_k)^{-1} p(s)\,ds\right],$$

and $\tilde{x}(t) = -(1-\delta)$, $t_0^- \leqslant t \leqslant t_0$. $\tilde{x}(t)$ is a nonoscillatory solution of (2.1). The proof of Theorem 2.2 is complete. □

Remark 2.3. By applying linearized oscillation theory in [9], we can improve Theorem 2.1 to obtain necessary and sufficient condition for all solutions of (1.1) to be oscillatory and the improved result can been applied to more general impulsive delay differential equations, for example, the delay food limited equation with impulsive effect

$$N'(t) = p(t)N(t)\frac{K - N(g(t))}{K + a(t)N(g(t))}, \quad t \neq \tau_k,\ t \geqslant 0,\ \text{a.e.,} \tag{2.15$_a$}$$

$$N(\tau_k^+) - N(\tau_k) = b_k K\left(\frac{N(\tau_k)}{K} - 1\right), \quad k = 1, 2, \ldots. \tag{2.15$_b$}$$

By the change of variable $y(t) = (N(t)/K) - 1$, (2.15) becomes

$$y'(t) = -p(t)\big(1 + y(t)\big)\frac{y(g(t))}{1 + a(t)[1 + y(g(t))]}, \quad t \neq \tau_k,\ t \geqslant 0,\ \text{a.e.,} \tag{2.16$_a$}$$

$$y(\tau_k^+) = (1 + b_k)y(\tau_k), \quad k = 1, 2, \ldots, \tag{2.16$_b$}$$

which has the form of (1.1).

Though $(2.16)_a$ does not satisfy (2.5) or (2.11), by same method we can prove the following results. Their proofs will be omitted.

Corollary 2.2. *Assume that* (A$_1$), (A$_3$), (A$_4$) *hold and*

$$a : [0, \infty) \to [0, \infty) \quad \textit{is locally bounded Lebesgue measurable},$$
$$\lim_{t\to\infty} \prod_{0<\tau_k<t} (1 + b_k) \quad \textit{is convergent.}$$

If either

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \prod_{g(s)\leqslant\tau_k<s} (1 + b_k)^{-1} \frac{p(s)}{1 + a(s)}\, ds > \frac{1}{e}$$

or $g(t)$ is nondecreasing and

$$\limsup_{t\to\infty} \int_{g(t)}^{t} \prod_{g(s)\leqslant\tau_k<s} (1 + b_k)^{-1} \frac{p(s)}{1 + a(s)}\, ds > 1,$$

then all solutions of (2.16) *are oscillatory.*

Corollary 2.3. *Assume that* (A$_1$), (A$_3$), (A$_4$), (2.17) *hold. If there exists $t_0 > 0$ such that*

$$\int_{g(t)}^{t} \prod_{g(s)\leqslant\tau_k<s} (1 + b_k)^{-1} \frac{p(s)}{1 + a(s)}\, ds \leqslant \frac{1}{e} \quad \textit{for all } t \geqslant t_0,$$

then (2.16) *has a nonoscillatory solution.*

Remark 2.4. Several related comparison theorems and oscillation or nonoscillation criteria for nonlinear impulsive delay differential equations

$$y'(t)+\sum_{i=1}^{m} r_i(t) f_i\big(y\big(h_i(t)\big)\big)=0, \quad t\neq\tau_k, \tag{2.18$_a$}$$

$$y(\tau_k)=I_k\big(y\big(\tau_k^-\big)\big), \quad \tau_k\geqslant t_0, \tag{2.18$_b$}$$

and

$$y'(t)+a(t)y(t)+f\big(y(t-\sigma_1),\dots,y(t-\sigma_m)\big)=0, \quad t\neq\tau_k, \tag{2.19$_a$}$$

$$y\big(\tau_k^+\big)-y(\tau_k)=b_k y(\tau_k), \quad \tau_k\geqslant t_0, \tag{2.19$_b$}$$

are established respectively by different techniques in [4] and [17], but the results are not able to apply to (1.1), because nonlinear conditions of $(2.18)_a$ and $(2.19)_a$ in [4] and [17] are different from $(1.1)_a$.

3. Generalizations and applications

Consider the impulsive delay differential equation

$$\begin{cases} y'(t)=-(1+y(t))\sum_{i=1}^m f_i(t,y(g_i(t))), & t\neq\tau_k,\ t\geqslant 0,\ \text{a.e.,}\\ y(\tau_k^+)=(1+b_k)y(\tau_k), & k=1,2,\dots \end{cases} \tag{3.1}$$

and the nonimpulsive delay differential equation

$$y'(t)=-\big(1+y(t)\big)\sum_{i=1}^{m} p_i(t)y\big(g_i(t)\big), \quad t\geqslant 0,\ \text{a.e.} \tag{3.2}$$

The following assumptions will be used:

- (A_5) $f_i(t,y):[0,\infty)\times R\to R$, $1\leqslant i\leqslant m$, satisfy Caratheodory conditions and $f_i(t,0)=0$ for all $t\geqslant 0$;
- (A_6) $g_i(t):[0,\infty)\to R$, $1\leqslant i\leqslant m$, are Lebesgue measurable functions and $g_i(t)\leqslant t$ with $\lim_{t\to\infty} g_i(t)=\infty$;
- (A_7) $p_i(t):[0,\infty)\to[0,\infty)$, $1\leqslant i\leqslant m$, are locally bounded Lebesgue measurable functions.

Let $g(t)=\max_{1\leqslant i\leqslant m} g_i(t)$, $g^*(t)=\min_{1\leqslant i\leqslant m} g_i(t)$. By similar to the arguments in Section 2 we can establish the following results.

Theorem 3.1. *Assume that* (A_1), (A_4)–(A_7), (2.3) *and* (2.6) *hold and there exists a sufficiently large T such that for each i,*

$$\frac{f_i(t,y)}{y}\geqslant p_i(t) \quad \textit{for all } y\neq 0 \textit{ and } t\geqslant T. \tag{3.3}$$

If there exists a positive constant $\delta<1$ such that all solutions of

$$u'(t)+(1-\delta)\sum_{i=1}^{m}\prod_{g_i(t)\leqslant\tau_k<t}(1+b_k)^{-1}p_i(t)u\big(g(t)\big)=0$$

are oscillatory, then all solutions of (3.1) *are also oscillatory.*

By applying Theorems 3.4.1 and 3.4.3 in [8] and Theorem 3.1, we have the following results.

Corollary 3.1. *Assume that* (A_1), (A_4)–(A_7), (2.3) *and* (2.6) *hold and either*

$$\liminf_{t\to\infty}\int_{g(t)}^{t}\sum_{i=1}^{m}\prod_{g_i(s)\leqslant\tau_k<s}(1+b_k)^{-1}p_i(s)\,ds>\frac{1}{e}\tag{3.4}$$

or $g(t)$ is nondecreasing and

$$\limsup_{t\to\infty}\int_{g(t)}^{t}\sum_{i=1}^{m}\prod_{g_i(s)\leqslant\tau_k<s}(1+b_k)^{-1}p_i(s)\,ds>1,\tag{3.5}$$

then all solutions of (3.1) *are oscillatory.*

Corollary 3.2. *Assume that* (A_6) *and* (A_7) *hold and either* (3.4) *or* (3.5) *is satisfied. Then all solutions of* (3.2) *are oscillatory.*

Remark 3.1. Corollary 3.2 improves respectively [3, Corollary 6] and [14, Theorem 2]. There assume conditions

$$\int_{0}^{\infty}\sum_{i=1}^{m}p_i(t)\,dt=\infty\quad\text{and}\quad\liminf_{t\to\infty}\int_{g(t)}^{t}\sum_{i=1}^{m}p_i(s)\,ds>0\tag{3.6}$$

for all solutions of (3.2) to be oscillatory.

Theorem 3.2. *Assume that* (A_1), (A_4)–(A_7), (2.3) *hold and there exists a sufficiently large T such that for each i, $1\leqslant i\leqslant m$,*

$$0<\frac{f_i(t,y)}{y}\leqslant p_i(t)\quad\textit{for all } y\neq 0 \textit{ and } t\geqslant T$$

and

$$\int_{g^*(t)}^{t}\sum_{i=1}^{m}\prod_{g^*(s)\leqslant\tau_k<s}(1+b_k)^{-1}p_i(s)\,ds\leqslant\frac{1}{e}\quad\textit{for all } t\geqslant T.\tag{3.7}$$

Then (3.1) *has a nonoscillatory solution.*

By Theorem 3.2 we obtain the following result.

Corollary 3.3. *Assume that* (A_6) *and* (A_7) *hold. Then* (3.2) *has a nonoscillatory solution.*

Remark 3.2. Corollary 3.3 improves respectively [3, Corollary 8] and [14, Theorem 5]. The condition (3.6) is assumed in [3,14] for (3.2) to have a nonoscillatory solution.

Now, let us consider the impulsive delay logistic equation

$$\begin{cases} N'(t)=N(t)\sum_{i=1}^{m}p_i(t)\left(1-\frac{N(g_i(t))}{K}\right), & t\neq\tau_k,\ t\geqslant 0,\ \text{a.e.},\\ N(\tau_k^+)-N(\tau_k)=b_kK\left(\frac{N(\tau_k)}{K}-1\right), & k=1,2,\ldots,\end{cases}\tag{3.8}$$

where K is a positive constant. By the change of variable $y(t) = (N(t)/K - 1)$, (3.8) is transformed into

$$\begin{cases} y'(t) = -(1 + y(t)) \sum_{i=1}^{m} p_i(t) y(g_i(t)), & t \neq \tau_k,\ t \geqslant 0, \text{ a.e.,} \\ y(\tau_k^+) = (1 + b_k) y(\tau_k), & k = 1, 2, \ldots. \end{cases} \tag{3.9}$$

From an ecological point of view we restrict attention to the positive solutions of (3.8). A positive solution of (3.8) is said to be oscillatory about K if function $N(t) - K$ is oscillatory. A solution of (3.8) is said to be nonoscillation about K if $N(t) - K$ is either eventually positive or eventually negative. Since $1 + y(t) > 0$, oscillation (or nonoscillation) of $N(t)$ about K is equivalent to oscillation (nonoscillation) of solution of (3.9).

Thus by using Corollary 3.1 and Theorem 3.2, we obtain the following results.

Corollary 3.4. *Assume that* (A_1), (A_4), (A_6), (A_7), (2.3) *and* (2.6) *hold, and either* (3.4) *or* (3.5) *is satisfied. Then all solutions of* (3.8) *is oscillatory about* K.

Corollary 3.5. *Assume that* (A_1), (A_4), (A_6), (A_7), (2.3) *hold and* (3.7) *is satisfied. Then* (3.8) *has a nonoscillatory solution about* K.

Acknowledgment

The authors thank the referee for careful reading of the manuscript and useful suggestions that helped to improve the presentation.

References

[1] L. Berezansky, E. Braverman, Oscillation of a linear delay impulsive differential equation, Comm. Appl. Nonlinear Anal. 3 (1996) 61–77.
[2] L. Berezansky, E. Braverman, Oscillation properties of a logistic equation with several delays, J. Math. Anal. Appl. 247 (2000) 110–125.
[3] L. Berezansky, E. Braverman, On oscillation of a logistic equation with several delays, J. Comput. Appl. Math. 113 (2000) 255–265.
[4] L. Berezansky, E. Braverman, Linearized oscillation theory for a nonlinear delay impulsive equation, J. Comput. Appl. Math. 161 (2003) 477–495.
[5] L.E. Erbe, Q. Kong, B.G. Zhang, Oscillation Theory for Functional Differential Equations, Dekker, New York, 1995.
[6] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic, Dordrecht, 1992.
[7] S.R. Grace, I. Györi, B.S. Lalli, Necessary and sufficient conditions for the oscillations of multiplicative logistic equation, Quart. Appl. Math. 53 (1995) 69–79.
[8] I. Györi, G. Ladas, Oscillation Theory of Delay Differential Equations, Clarendon, Oxford, 1991.
[9] V.L. Kocic, G. Ladas, C. Qian, Linearized oscillations in nonautonomous delay differential equations, Differential Integral Equations 6 (1993) 671–683.
[10] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, New York, 1993.
[11] G. Ladas, C. Qian, Oscillation and global stability in a delay logistic equation, Dynam. Stability Syst. 9 (1994) 153–162.
[12] W.T. Li, H.F. Huo, Existence and global attractivity of positive periodic solutions of functional differential equations with impulses, Nonlinear Anal. 59 (2004) 857–877.
[13] Y. Liu, W. Ge, Global attractivity in delay "food-limited" models with exponential impulses, J. Math. Anal. Appl. 287 (2003) 200–216.
[14] J. Luo, Oscillation and linearized oscillation of a logistic equation with several delays, Appl. Math. Comput. 131 (2002) 469–476.

[15] S. Tang, L. Chen, Global attractivity in a "food-limited" population model with impulsive effects, J. Math. Anal. Appl. 292 (2004) 211–221.
[16] X. Tang, J. Yu, Global attractivity of zero solution of logistic type functional differential equations with impulses, Acta Math. Sinica 45 (2002) 941–952 (in Chinese).
[17] J. Yan, Oscillation of nonlinear delay impulsive differential equation and inequalities, J. Math. Anal. Appl. 265 (2002) 332–342.
[18] J. Yan, A. Zhao, Oscillation and stability of linear impulsive delay differential equations, J. Math. Anal. Appl. 227 (1998) 187–194.
[19] B.G. Zhang, K. Gopalsamy, Oscillation ans non-oscillation in a nonautonomous delay-logistic equation, Quart. Appl. Math. 46 (1988) 267–273.