JOURNAL OF COMPUTER AND SYSTEM SCIENCES: 1, 29-43 (1967)

# Chebyshev Solution of Large Linear Systems*

J. B. ROSEN

*Computer Sciences Department, University of Wisconsin, Madison, Wisconsin*

ABSTRACT

The general problem considered is that of solving a linear system of equations which is singular or almost singular. A method is described which obtains a "solution" to the system which is stable with respect to small changes in the matrix elements. This method will solve an overdetermined system in $m$ variables and $n$ equations ($m < n$) even when the system rank is less than $m$, and should therefore be very useful in many statistical applications. In this case the error of the system is minimized in the Chebyshev norm using a linear programming formulation and solution. A numerical example using the Hilbert matrix is described in detail.

## 1. INTRODUCTION

A wide variety of problems, arising in many areas of the physical and social sciences, lead to a computational problem of the following kind. We are given the elements of an $m \times n$ matrix $A$. The values given will often be subject to small errors. It is desired to get a "best" solution in some sense to the linear system

$$A'y = b, \tag{1.1}$$

where $A'$ is the transpose of $A$, and $b$ is a specified $n$-dimensional vector which may also be subject to error.

One of the most common situations occurs when $m \leqslant n$, so that (1.1) may be an overdetermined system. Provided that $\operatorname{rank}(A) = m$, we can (in principle) get a least-squares solution to (1.1) by solving the normal equations

$$AA'y = Ab \tag{1.2}$$

which gives the least-squares solution

$$y^* = (AA')^{-1} Ab. \tag{1.3}$$

* This research was sponsored in part by NASA Research Grant NGR-50-022-028, and in part by the Mathematics Research Center, United States Army, Madison, Wisconsin under Contract No.: DA-11-022-ORD-2059.

We may consider the standard situation where $A$ is a square ($m \times m$), nonsingular matrix as a special case of this, giving us the usual solution $y = (A')^{-1} b$.

As shown by (1.3) the least-squares solution is unique whenever $\operatorname{rank}(A) = m$, that is, whenever $AA'$ is nonsingular. If we define the error vector

$$\delta = \delta(y) = A'y - b, \tag{1.4}$$

then for $\operatorname{rank}(A) = m$, we have

$$\| \delta(y^*)\| < \| \delta(y)\| \qquad \text{for all} \qquad y \neq y^*, \tag{1.5}$$

where $\| \cdot \|$ denotes the Euclidean norm. When $\operatorname{rank}(A) < m$, there will in general be a linear manifold $Y$ of solutions to (1.4) all of which minimize $\| \delta(y)\|$.

Uniqueness may again be achieved in this situation by imposing an additional requirement on the norm of the solution vector $y$. In particular, the requirement is that the solution $y^\dagger \in Y$ satisfy

$$\| y^\dagger \| \leqslant \| y \| \qquad \text{for all} \qquad y \in Y \tag{1.6}$$

where

$$Y = \{y \mid \| \delta(y)\| = \min_z \| \delta(z)\|\}. \tag{1.7}$$

The unique vector $y^\dagger$ is given in this case by

$$y^\dagger = A^\dagger b, \tag{1.8}$$

where $A^\dagger$ is the pseudoinverse ([*1*], [*2*]) of $A'$.

When the matrix $A$ is square and well-conditioned there are available numerical methods which compute a very accurate inverse. Similarly, if $AA'$ is well-conditioned the value of $y^*$ as given by (1.3) can be obtained to high accuracy by inverting $AA'$. In this latter case, however, it should be noted that the condition number of $AA'$ is the square of that of $A$, so that $AA'$ is never less ill-conditioned than $A$. Recently a numerical method has been proposed [*3*] which does not require the formation or inversion of $AA'$ in order to get $y^*$.

A more fundamental problem also arises when the matrix $A$ or $AA'$ is ill-conditioned. This is the problem of the stability of the solution $y$ with respect to small perturbations in the elements of $A$. Generally speaking, as the condition number of $A$ increases there is an increased sensitivity of the solution $y$ to a small perturbation in an element of $A$. This difficulty is most apparent if $\operatorname{rank}(A) < m$ and we have obtained the solution $y^\dagger$ using the pseudoinverse. An arbitrarily small change in an element of $A$ may now increase the rank and give a completely different solution, so that in a certain sense the solution $y^\dagger$ is completely unstable.

This situation is best illustrated by a very simple numerical example. Suppose we let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1.9}$$

Then it is easy to see that for every scalar $\alpha$, the vector

$$y = y(\alpha) = \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.10}$$

gives a least-squares solution to (1.4) with $\| \delta \| = (0.5)^{1/2}$. The unique pseudoinverse solution is then obtained with $\alpha = 0$, so that

$$y^\dagger = \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}. \tag{1.11}$$

Now suppose that, due to experimental or numerical error, we are actually presented with the matrix

$$A(\epsilon) = \begin{pmatrix} 1 & 1 \\ 1 & 1 + \epsilon \end{pmatrix}. \tag{1.12}$$

For $\epsilon \neq 0$, this matrix is, of course, nonsingular so that (1.1) has the solution

$$y = \begin{pmatrix} 1 + 1/\epsilon \\ -1/\epsilon \end{pmatrix}. \tag{1.13}$$

The situation is illustrated in Fig. 1, with $\epsilon = 0.2$.

The important point to note from this example is that for $\epsilon = 0$ the "correct" answer is $y^\dagger$ as given by (1.11), while for any $\epsilon \neq 0$ the "correct" answer is given by (1.13) and approaches $\infty$ as $\epsilon \to 0$.

The reason for this discontinuous behavior is that we do not impose any requirement on the norm of $y$ except in the special case where $\operatorname{rank}(A) < m$. That is, we are insisting on reducing the error norm to its minimum value regardless of the effect that this has on the norm of the solution vector. This has been pointed out by Levenberg [*4*], who suggested that one should minimize the function

$$\varphi(y) = \| \delta(y) \|^2 + \lambda \| y \|^2 \tag{1.14}$$

for some fixed $\lambda > 0$. This adds the constant $\lambda$ to each diagonal element of the matrix $AA'$ in the normal equations, which improves its conditioning in general, and ensures that it is nonsingular even when $AA'$ is singular without the added constant. Difficulties with this proposal are that it is necessary to solve the normal equations for each selected value of $\lambda$, and—more important—there is no clear relationship between any particular choice of $\lambda$ and the direct effect it has on the solution obtained.

A proposal is made here which appears to eliminate, or at least greatly reduce, the problems arising because $A$ (or $AA'$) is ill-conditioned, and which also avoids the difficulties of the Levenberg scheme. Specifically, it is proposed that the solution vector $y$ minimize the error $\delta(y)$ subject to the additional requirement that each component of $y$ be bounded in absolute value by a parameter $\beta$; that is,

$$|y_i| \leqslant \beta, \qquad i = 1,\dots, m. \tag{1.15}$$

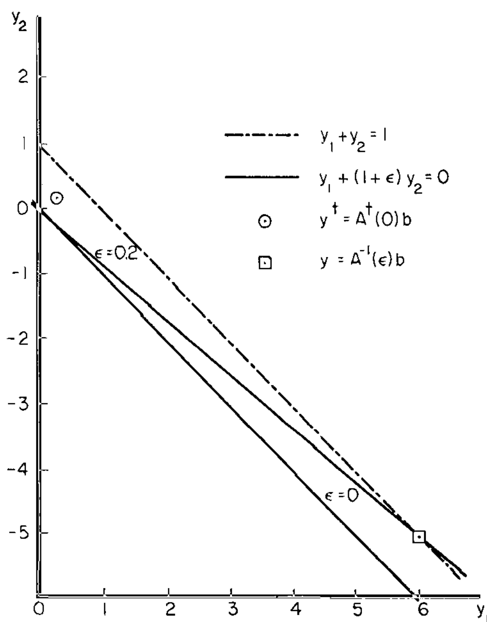


FIG. 1. Graph of solution for $\epsilon = 0.2$ and $\epsilon = 0$.

To illustrate the effect of this additional requirement let us again consider the simple example (1.12). The effect of (1.15) is to require that $y$ lie in, or on the boundary of, a square with sides $2\beta$ and center at the origin. This is shown for several values of $\beta$ in Fig. 2. For each value of $\beta$ the corresponding solution $y(\beta)$ is shown. It is seen that as $\beta$ increases from 0 to $\infty$, the "trajectory" of the solution $y(\beta)$ is traced out. The following points should be noted for $\epsilon > 0$:

1. For $0 \leqslant \beta \leqslant (4 + \epsilon)^{-1}$, the bound (1.15) is active for both components $y_1(\beta)$ and $y_2(\beta)$; that is, $y_1(\beta) = y_2(\beta) = \beta$.

2. At $\beta = \beta_C = (4 + \epsilon)^{-1}$, we obtain the solution $y_1 = y_2 = \beta$, which corresponds to the pseudoinverse solution when $\epsilon = 0$.

3. For $\beta_C < \beta \leqslant 1 + \epsilon^{-1}$, only the first bound is active, that is $y_1(\beta) = \beta$ and $y_2(\beta) < \beta$. However, we have a Chebyshev-type error in this range; that is

$$\delta_2 = -\delta_1 = [1 + (1 - \beta)\,\epsilon]\,(2 + \epsilon)^{-1}.$$

4. For $\beta > 1 + \epsilon^{-1}$, neither bound is active and the solution is given by (1.13), independent of $\beta$, with zero error.

On the basis of these observations we may conclude (at least for this example) that even though the "correct" solution (1.13) depends strongly on $\epsilon$, and in fact is discontinuous at $\epsilon = 0$, the trajectory $y(\beta)$ depends only weakly on $\epsilon$ and may therefore be called stable with respect to the perturbation $\epsilon$. In fact, for any finite positive value $\beta_{max}$, the trajectory $y(\beta)$, $0 \leqslant \beta \leqslant \beta_{max}$, approaches the limiting

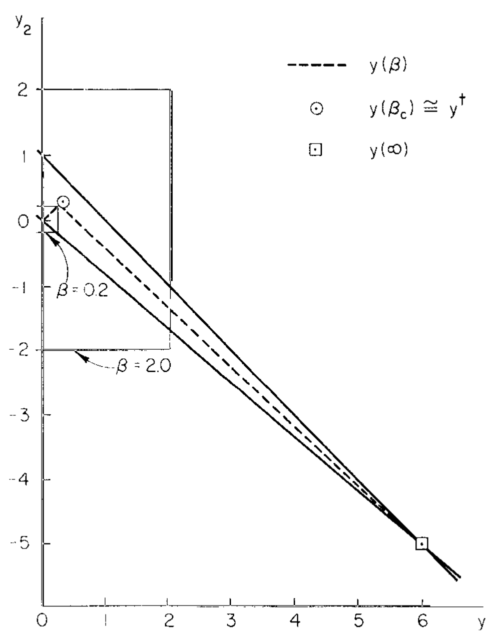


FIG. 2. Solution trajectory with bounds.

trajectory for $\epsilon = 0$, as $\epsilon \to 0$. Furthermore, the first value at which a Chebyshev solution is obtained, $\beta = \beta_C$, gives a vector $y(\beta_C)$ such that $\| y(\beta_C) - y^\dagger \| = O(\epsilon)$. Thus if one were presented with $A(\epsilon)$ for $\epsilon \neq 0$ and $b = \binom{1}{0}$, and asked to solve (1.1), it would be reasonable to choose the solution $y(\beta_C)$.

The example considered above is of course extremely simple. However, as will be shown in the following sections, the essential ideas carry over to the solution of large linear systems of arbitrary rank. In order to carry this out computationally for large systems, it is useful to modify the problem by using the Chebyshev error norm instead of the Euclidean norm. This permits us to use an efficient linear programming (LP) algorithm [5] as the basis of the computational solution method.

Specifically, let

$$\| \delta \|_C = \max_j | \delta_j |. \tag{1.16}$$

Then, in terms of the original system (1.1) and the error vector $\delta(y)$ given by (1.4), the problem we wish to solve may be stated as that of finding $y = y(\beta)$ such that

$$\| \delta(y(\beta)) \|_C = \min_y \left\{ \| \delta \|_C \,\middle|\, \begin{matrix} \delta = A'y - b \\ \| y \|_C \leqslant \beta \end{matrix} \right\}. \tag{1.17}$$

It should also be emphasized that the computational solution of this LP problem requires essentially only that the inverse for a selected $(m + 1) \times (m + 1)$ matrix be computed. This usually requires approximate $2m$ simplex (Gauss–Jordan type) pivot operations. Thus the solution time depends primarily on $m$ and only to a small extent on the value of $n$. The method will therefore be most efficient when $m \ll n$.

In the next section the formulation of (1.17) as a linear programming problem in standard primal form will be derived. This is done by associating (1.17) with the unsymmetric dual problem and actually solving the corresponding primal problem. The parameter $\beta$ then appears in the primal cost row so that the trajectory of solutions $y(\beta)$ as $\beta$ increases from 0 to $\infty$ can be obtained by a single parametric run. If solutions for specific values of $\beta$ are desired, this can easily be done by using the multiple cost row feature of a primal LP code.

An interesting result is obtained which describes the behavior of the error norm $y_0(\beta) = \| \delta(y(\beta)) \|_C$, as $\beta$ increases. It is shown that $y_0(\beta)$ is a nonincreasing, convex, piecewise linear function. The possible decrease in error by increasing the norm of the solution vector is completely described by the function $y_0(\beta)$.

In the final section some numerical results are presented. These results were obtained using the first 5 columns of the $6 \times 6$ Hilbert matrix as the matrix $A'$. The solution trajectory $y(\beta)$ was obtained using the parametric cost row feature of a primal LP code. The results are presented in Table 1. The Chebyshev error norm $y_0(\beta)$ is also given there, and is shown graphically in Fig. 3. It is seen that the error drops rapidly to its value at $\beta_C \cong 1507$, and then decreases only slightly to its minimum value at $\beta_m = 2000$. No basis changes take place for $\beta$ between $\beta_C$ and $\beta_m$ so that $y(\beta)$ and $y_0(\beta)$ are linear functions of $\beta$ over this range. The value $\beta_C$ is the smallest value of $\beta$ for which a Chebyshev-type error is achieved, that is the error $y_0(\beta)$ is attained for at least $m$ of the $n$ Eqs. (1.1). It is also characterized by the fact that only one of the bounds (1.15) is active for $\beta_C \leqslant \beta \leqslant \beta_m$. For $\beta \geqslant \beta_m$, both $y_0(\beta)$ and $y(\beta)$ are constant and none of the bounds (1.15) are active.

The stability of this (unperturbed) solution trajectory was investigated by making similar runs on eight perturbed matrices obtained by making random changes in the

TABLE 1

UNPERTURBED SOLUTION

| $\beta$ | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|---|
| 0 | 791.6765 | 0 | 0 | 0 | 0 | 0 |
| 100.000 | 563.3432 | 100.000 | 100.000 | 100.000 | 100.000 | 100.000 |
| 1000.000 | 3.2286 | 1.146 | 7.937 | 1000.000 | 1000.000 | 1000.000 |
| 1100.000 | 0.8744 | 81.355 | −304.439 | 1100.000 | 1100.000 | 1100.000 |
| 1200.000 | 0.2268 | 101.321 | −307.982 | 913.719 | 1200.000 | 1200.000 |
| 1300.000 | 0.1504 | 82.012 | −125.339 | 562.454 | 1300.000 | 1300.000 |
| 1400.000 | 0.0778 | 62.735 | 57.272 | 211.151 | 1400.000 | 1400.000 |
| 1500.000 | 0.0133 | 43.435 | 240.127 | −140.425 | 1500.000 | 1500.000 |
| 1507.232 | 0.0121 | 41.696 | 254.790 | −167.042 | 1507.232 | 1507.232 |
| 2000.000 | 0.0100 | 50.000 | 100.000 | 500.000 | 500.000 | 2000.000 |
| $\infty$ | 0.0100 | 50.000 | 100.000 | 500.000 | 500.000 | 2000.000 |

| | $w_0$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ |
|---|---|---|---|---|---|---|
| $\beta_C \leqslant \beta \leqslant \beta_m$ | $-4.27 \times 10^{-6}$ | 0.01685 | −0.31412 | 1.35366 | −2.04403 | 1.00000 |

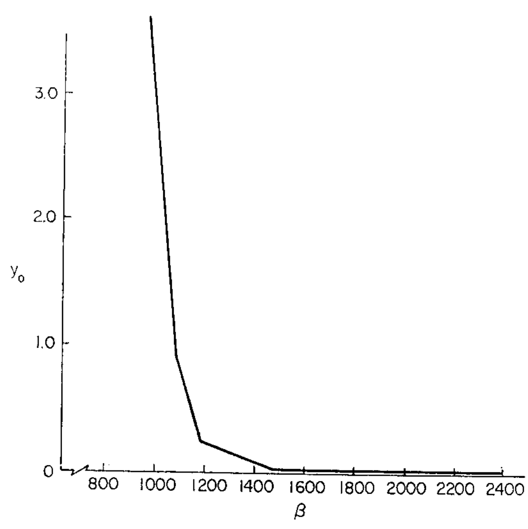


FIG. 3. Chebyshev error norm.

6th decimal digit of the elements of $A$. It was found, as shown in Table 2, that the "correct" solutions $y(\infty)$ to the perturbed problems differed greatly from the unperturbed solution. However, the trajectory $y(\beta)$ and error norm $y_0(\beta)$ for $\beta \leqslant \beta_C$, were found to be almost independent of the perturbations and therefore $y(\beta)$, for $\beta \leqslant \beta_C$, represents a stable solution to the linear system. This is clearly shown in Fig. 4, where the maximum difference between the unperturbed and perturbed solutions are plotted as a function of $\beta$. The stable solution with minimum error is given by the vector $y(\beta_C)$, which may therefore be taken as the "best" solution to the original problem.

TABLE 2

COMPARISON OF SOLUTIONS

| Matrix | $\beta_C$ | $\beta_m$ | $\lVert y_p(\infty) - y_u(\infty)\rVert_C$ | $y_0(\infty)$ |
|---|---|---|---|---|
| Unperturbed | 1507.232 | 2000.000 | — | 0.0100 |
| Perturbed I | 1512.995 | 2308.638 | 1808 | 0.0087 |
| Perturbed II | 1514.952 | 1724.730 | 583 | 0.0162 |
| Perturbed III | 1510.452 | 1748.373 | 524 | 0.0120 |
| Perturbed IV | 1496.171 | 1933.700 | 119 | 0.0093 |
| Perturbed V | 1504.621 | 1598.359 | 1098 | 0.0099 |
| Perturbed VI | 1502.963 | 2768.087 | 768 | 0.0008 |
| Perturbed VII | 1502.788 | 2035.204 | 83 | 0.0104 |
| Perturbed VIII | 1512.958 | 1889.224 | 122 | 0.0072 |

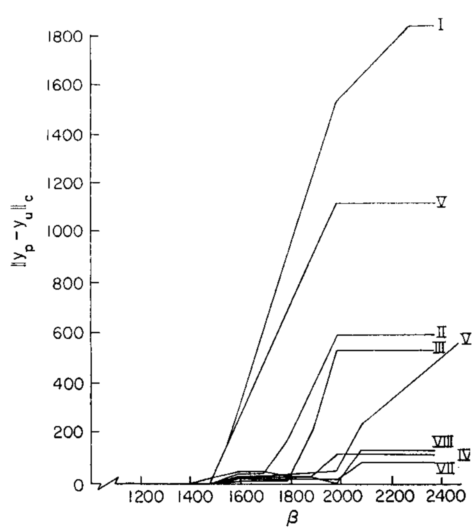


FIG. 4. Difference between perturbed and unperturbed solutions.

In many actual problems a "reasonable" value of the solution vector may be known. To be specific, suppose we are presented with the system

$$A'z = d \tag{1.18}$$

and an *a priori* estimate $\bar{z}$ for a reasonable value of the solution. In such a case we would want to proceed as discussed above except that we would want to replace (1.15) by

$$|z_i - \bar{z}_i| \leqslant \beta, \qquad i = 1,..., m. \tag{1.19}$$

This is easily accomplished by letting

$$y = z - \bar{z}, \qquad b = d - A'\bar{z} \tag{1.20}$$

and proceeding as before.

## 2. Equivalent Linear Programming Problem

We now want to convert (1.17) into a linear programming (LP) format. We observe that, in component form, (1.4) becomes

$$\delta_i = a_i'y - b_i\,, \qquad i = 1,..., n, \tag{2.1}$$

where $a_i$ is the $i$th column of $A$, and $b_i$ is the $i$th component of the vector $b$. Introducing a new nonnegative scalar variable $y_0 \geqslant 0$, we consider the system of inequalities

$$y_0 \geqslant |a_i'y - b_i|, \qquad i = 1,..., n \tag{2.2}$$

or the equivalent system of $2n$ linear inequalities,

$$y_0 \geqslant a_i'y - b_i \geqslant -y_0\,, \qquad i = 1,..., n. \tag{2.3}$$

It is clear that for any fixed $y$, the minimum value of $y_0$ satisfying (2.2) or (2.3) gives the value of $\| \delta(y)\|_C$ as defined by (1.16). We may therefore state the problem (1.17) as that of finding $y_0(\beta)$ and $y(\beta)$ such that $y_0$ attains its minimum value over all $y_0 \geqslant 0$ and vectors $y$ satisfying (2.3) and (1.15). Putting this in standard format we obtain an LP problem in $m + 1$ variables and $[2(m + n) + 1]$ linear inequality constraints as follows:

$$\min_{y_0, y} \left\{ y_0 \;\middle|\; \begin{array}{ll} y_0 + a_i'y \geqslant b_i\,, & i = 1,..., n \\ y_0 - a_i'y \geqslant -b_i\,, & i = 1,..., n \\ -y_i \geqslant -\beta, & i = 1,..., m \\ y_0 \geqslant 0, & \\ y_i \geqslant -\beta, & i = 1,..., m \end{array} \right\}. \tag{2.4}$$

We observe that in this form we have a problem with more inequality constraints than variables, and in which the variables are not necessarily nonnegative. We therefore associate (2.4) with the dual form [6] of an LP problem. We define the $(m + 1) \times (2n + 2m + 1)$ matrix $\hat{A}$ and corresponding vector $\hat{b}$:

$$\hat{A} = \left[\begin{array}{c|c|c|c} 1 \quad 1 \ldots 1 & 1 \ldots 1 & 0 \quad 0 \ldots 0 & \\ A & -A & -I_m & I_{m+1} \end{array}\right]; \tag{2.5}$$

$$\hat{b}' = (b_1 b_2 \ldots b_n \mid -b_1 \ldots -b_n \mid -\beta \ldots -\beta \mid 0 - \beta \ldots -\beta).$$

The vertical lines are used merely to help clarify the structure of $\hat{A}$ and $\hat{b}$. We also define two $(m + 1)$-vectors

$$\hat{y} = \begin{pmatrix} y_0 \\ y \end{pmatrix}, \qquad \hat{c} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.6}$$

The problem (2.4) may now be written in the concise form

$$\min_{\hat{y}}\{\hat{c}'\hat{y} \mid \hat{A}'\hat{y} \geqslant \hat{b}\}. \tag{2.7}$$

Considering this as a problem in dual form we can immediately write down the corresponding primal problem in terms of a nonnegative $(2n + 2m + 1)$-vector $x$,

$$\max_{x} \left\{\hat{b}'x \,\middle|\, \begin{array}{l} \hat{A}x = \hat{c} \\ \quad x \geqslant 0 \end{array}\right\}. \tag{2.8}$$

This is the standard primal LP format with a cost row $\hat{b}'$ and right-hand side $\hat{c}$. An initial feasible basis for this primal problem is given by the identity matrix $I_{m+1}$. By the duality theory of linear programming, if (2.8) has an optimal solution then so does (2.7), and the optimal function values are equal. Furthermore, the computational solution of (2.8) determines not only the optimal value of the vector $x$, but also the corresponding optimal dual (shadow-price) vector $\hat{y}$, so that a standard LP solution of (2.8) also gives us the desired solution to (2.7).

Another well known result of duality theory is that any dual feasible solution (a solution satisfying the dual constraints) gives an upper bound to the primal function value for any primal feasible vector $x$. Now $y = 0$, $y_0 = \max_i \mid b_i \mid$, is a feasible solution to (2.7) for any $\beta \geqslant 0$. Therefore we know that the solution to (2.8) always exists (i.e., it has a finite solution) and in fact we have $\hat{b}'x \leqslant \max_i \mid b_i \mid$.

The parameter $\beta$ appears in the cost row of the primal problem. Therefore we can use the parametric cost row option (available on most LP codes) to obtain the optimal solution to (2.8) as a function of $\beta$, as $\beta$ goes from 0 to $\infty$. Starting with the initial basis $I_{m+1}$, the optimal basis for $\beta = 0$ will be obtained by replacing one or more columns of $I_{m+1}$ by columns selected from the first $(2n + m)$ columns of $\hat{A}$. The optimal value of $\hat{y}$ for $\beta = 0$ is given by $y_0(0) = \max_i |b_i|$ and $y(0) = 0$. As $\beta$ is increased the optimal $y_0(\beta)$ will (as shown below) decrease linearly with $\beta$ until a basis change is required to maintain optimality. Following the basis change, the optimal $y_0(\beta)$ will again be linear in $\beta$ until the next required basis change takes place.

A basis consists of $m + 1$ linearly independent columns selected from $\hat{A}$. Let us denote by $B$ the current basis matrix. Then $\hat{y} = B^{-1}\bar{b}(\beta)$ where $\bar{b}(\beta)$ is an $(m + 1)$-vector consisting of the elements of $\hat{b}$ which correspond to the current basis. The linearity of $\hat{y}(\beta)$, and $y_0(\beta)$ in particular, for a fixed basis follows immediately. More generally, the important properties of $y_0(\beta)$ are given by

THEOREM 1. *$y_0(\beta)$ is a piecewise linear, nonincreasing, convex function of $\beta$.*

*Proof.* The piecewise linearity of $y_0(\beta)$ has been shown above. Let us denote by $D(\beta)$ the set of feasible points satisfying the constraints of (2.4) or (2.7), that is $\hat{y} \in D(\beta) \Leftrightarrow \hat{A}'\hat{y} \geqslant \hat{b}(\beta)$. Then $D(\beta_2) \supset D(\beta_1)$ for $\beta_2 > \beta_1$. Now let $\hat{y}_1$, with first component $y_0(\beta_1)$ be the optimal solution to (2.7) with $\beta = \beta_1$, and $\hat{y}_2$ with first component $y_0(\beta_2)$ be the optimal solution to (2.7) with $\beta = \beta_2 > \beta_1$. Then $\hat{y}_1 \in D(\beta_1) \subset D(\beta_2)$ so that $\hat{y}_1 \in D(\beta_2)$ and therefore $y_0(\beta_2) \leqslant y_0(\beta_1)$. Thus $y_0(\beta)$ is nonincreasing. Finally, the convexity of $y_0(\beta)$ follows from the known result that the optimal function value for (2.7) is a convex function of the vector $\hat{b}$.

Note that if $b$ lies entirely in the null space of $A$ (i.e., $Ab = 0$), then we have $y_0(\beta) = \| b \|$, and we can choose $y(\beta) = 0$ for all $\beta \geqslant 0$. Therefore we assume that $Ab \neq 0$.

The duality between (2.7) and (2.8) is also exhibited by the fact that, if a specific column of $\hat{A}$ is in the optimal basis $B$, then the corresponding inequality constraint in (2.7) is satisfied as an equality. In particular, a bound of the type (1.15) can only be active if the corresponding column from the last $(2m + 1)$ columns of $\hat{A}$ is in the basis. Now as $\beta$ increases, it seems reasonable that those $x_i > 0$ with coefficients $-\beta$ would be driven to zero in order to maximize the primal function value. This corresponds to one or more of the last $(2m + 1)$ columns of $\hat{A}$ going out of the basis, which also means that the corresponding bounds (1.15) become strict inequalities. We therefore have

THEOREM 2. *The total number of active bounds, $y_i = \pm\beta$, goes to zero for sufficiently large $\beta$.*

*Proof.* An active bound $y_i = \pm\beta$, in the dual problem requires that the corresponding primal activity be positive. That is, we have $x_i > 0$, corresponding to $-\beta$ in $\hat{b}$. Now if such a basis were optimal for arbitrarily large $\beta$, we would have

$\hat{b}'x < 0$ for sufficiently large $\beta$. But this contradicts the fact that for each fixed $\beta$ the optimal function values of (2.7) and (2.8) are equal, and $\hat{c}'\hat{y} = y_0 \geqslant 0$.

Suppose that we have an optimal solution $\hat{y}(\beta)$ for some fixed $\beta$, such that only one of the last $(2m + 1)$ columns of $\hat{A}$ is in the optimal basis. Then either $y_0(\beta) = 0$ or $y_i = \pm\beta$ for one value of $i$. Furthermore, since we must have $m$ out of the first $2n$ columns of $\hat{A}$ in the basis we will have

$$a_i'y(\beta) - b_i = \pm y_0 \tag{2.9}$$

for at least $m$ values of $i$. That is, the maximum error is *attained* for at least $m$ equations. When this occurs we will say that the corresponding optimal solution $y(\beta)$ gives a Chebyshev error. We will denote by $\beta_C$ the smallest value of $\beta$ for which there is at most one active bound. For some value of $\beta \geqslant \beta_C$, the value of $y_0(\beta)$ will reach its minimum value (which may be either zero or positive). We will denote by $\beta_m \geqslant \beta_C$, this value of $\beta$. For $\beta \geqslant \beta_m$, the vector $y(\beta)$ remains constant; that is,

$$\begin{aligned} y_0(\beta) &= y_0(\beta_m) \\ y(\beta) &= y(\beta_m) \end{aligned}, \qquad \beta \geqslant \beta_m . \tag{2.10}$$

It follows that for $\beta > \beta_m$, there are no active bounds.

As we saw in the simple example discussed in the Introduction, the solution $y(\beta_C)$ is closely related to the pseudoinverse solution $y^\dagger$. The numerical results of the next section show that $y(\beta_C)$ is also stable with regard to small perturbations in the elements of $A$, even when $A$ is very ill-conditioned and the "correct" solution is highly unstable.

We now consider the interesting case where the basis does not change as $\beta$ increases from $\beta_C$ to $\beta_m$. Let us denote by $B_C$ the basis obtained when $\beta = \beta_C$. $B_C$ will consist of $m$ columns from the first $2n$ columns of $\hat{A}$ and one column from the last $(2m + 1)$ columns of $\hat{A}$. If this one column corresponds to the zero element in $\hat{b}$, then $y_0(\beta_C) = 0$ and $\beta_m = \beta_C$. Otherwise, we have a single column in $B_C$, say the $l$th column, corresponding to one of the elements $-\beta$ in $\hat{b}$. Since the basis does not change we have

$$\hat{y}(\beta) = (B_C^{-1})'\,\bar{b}(\beta), \qquad \beta_C \leqslant \beta \leqslant \beta_m, \tag{2.11}$$

where $\bar{b}(\beta)$ consists of the elements of $\hat{b}$ corresponding to the columns of $B_C$. Now every element of $\bar{b}(\beta)$ is constant except for the $l$th element which is $-\beta$. Therefore

$$\hat{y}(\beta) = \hat{y}(\beta_C) + (\beta - \beta_C)\,\hat{w}, \qquad \beta_C \leqslant \beta \leqslant \beta_m, \tag{2.12}$$

where the constant vector $\hat{w}$ is the $l$th column of $(B_C^{-1})'$ or the $l$th row of $B_C^{-1}$. At $\beta = \beta_m$ a new basis is obtained. This new basis has no columns corresponding to the elements $-\beta$ in $\hat{b}$. Therefore $\bar{b}(\beta)$ is a constant vector for $\beta \geqslant \beta_m$. The error $y_0(\beta)$ and the trajectory $y(\beta)$ are then completely described for $\beta \geqslant \beta_C$, in terms of their values at $\beta_C$ and the vector $\hat{w}$:

$$\hat{y}(\beta) = \begin{cases} \hat{y}(\beta_C) + (\beta - \beta_C)\,\hat{w}, & \beta_C \leqslant \beta \leqslant \beta_m, \\ \hat{y}(\beta_m) = \hat{y}(\beta_C) + (\beta_m - \beta_C)\,\hat{w}, & \beta_m \leqslant \beta. \end{cases} \tag{2.13}$$

It follows from the relations (2.2) and (2.12) that

$$\| A'w \|_C \leqslant w_0 \tag{2.14}$$

where $w_0$ is the first component of the vector $\hat{w}$ and $w$ represents the last $m$ components of $\hat{w}$. Thus the ratio $w_0/\| w \|_C$ is a good measure of the linear independence of the columns of $A'$, and will be small if the columns are almost linearly dependent (in the numerical example, this ratio is approximately $2.1 \times 10^{-6}$). Furthermore, if this ratio is small we can say that the vector $w$ lies in an "approximate null space" of $A'$.

## 3. Numerical Results

This concluding section is concerned with the most important aspect of this work: the computational test of the theory. Since inversion of the Hilbert matrix is a test which every aspiring linear equation solver must pass, an interesting test problem was constructed using this matrix. The matrix $A$ was chosen to be the first 5 rows of the $6 \times 6$ Hilbert matrix; that is,

$$(A)_{ij} = \frac{1}{i + j - 1}, \qquad \begin{matrix} i = 1,\dots, 5 \\ j = 1,\dots, 6. \end{matrix} \tag{3.1}$$

Each element of $A$ was rounded to 6 decimal digits and input as a 6-digit number; i.e., $\frac{1}{6} = 0.166667$. A numerical solution was then obtained to the problem (1.17) with the vector $b$ given by

$$b = A'\bar{y} + \delta, \bar{y} = \begin{pmatrix} 50 \\ 100 \\ 500 \\ 500 \\ 2000 \end{pmatrix}, \qquad \delta = \begin{pmatrix} .01 \\ -.01 \\ .01 \\ -.01 \\ .01 \\ -.01 \end{pmatrix}. \tag{3.2}$$

For this choice of $b$, the solution to (1.17) is $y(\beta) = \bar{y}$ and $\| \delta(y(\beta))\|_C = 0.01$, for $\beta \geqslant 2000$. The equivalent primal LP problem (2.8) was formulated as described in Section 2, and solved using the Control Data Corporation CDM 4 LP code [7] on the CDC 3600. The solution trajectory $y(\beta)$ was obtained using the parametric and multiple cost row features of CDM 4. The Chebyshev error $y_0(\beta)$ and the solution vector $y(\beta)$ at selected values of $\beta$, are shown in Table 1. The value $\beta_C = 1507.232$ was obtained, and the corresponding error $y_0(\beta_C)$ and solution $y(\beta_C)$ are given. The computed solution gave $\beta_m = 2000$, $y_0(\beta_m) = 0.01$ and $y(\beta_m) = \bar{y}$, to a larger number of figures than shown in the table. There were no basis changes between $\beta_C$ and $\beta_m$, so that the solution in this range is given by (2.13). The required coefficients $w_i$, $i = 0,..., 5$, are also given in Table 1. We will call the solution vector $y(\beta)$ obtained in this way the unperturbed solution, and we will denote it by $y_u(\beta)$. A graph of the Chebyshev error $y_0(\beta)$ is given in Fig. 3. This illustrates the piecewise linear, non-increasing convex property of $y_0(\beta)$ as it approaches its minimum value $y_0(\beta_m) = 0.01$.

In order to test the stability of this solution, eight similar parametric runs were made using a different perturbed matrix $A$ for each run. The perturbed matrices were obtained by adding to each element of $A$, a random number in the range $[-5 \times 10^{-6}, 5 \times 10^{-6}]$. Eight different perturbed matrices were generated in this way, and each such matrix was used in a parametric run.

The solution obtained for large $\beta$ (no active bounds) is highly unstable with respect to these perturbations. Let us denote by $y_p(\beta)$ the solution obtained with one of these perturbed matrices. We measure the difference between $y_p(\beta)$ and $y_u(\beta)$ in the Chebyshev norm; that is

$$\| y_p(\beta) - y_u(\beta) \|_C = \max_{i=1,...,5} | y_{pi}(\beta) - y_{ui}(\beta) |. \tag{3.3}$$

Each solution vector $y_p(\beta)$ obtained for $\beta \geqslant \beta_m$ is independent of $\beta$ and in fact is what would be called the "correct" solution to (1.1) for the corresponding perturbed matrix $A$. The difference between these eight solutions and the unperturbed solution $y_u(\beta_m) = y_u(\infty) = \bar{y}$, is shown in Table 2. The values of $\beta_C$, $\beta_m$ and $y_0(\beta_m) = y_0(\infty)$ are also given there. We see that the change in the solution may be as large as 1808, so that a perturbation in the sixth decimal digit of $A$ may cause a relative change in the solution of almost 100 %. This large change is due to the fact that we are dealing with an almost singular system, so that any perturbation in the matrix $A$ is greatly amplified. It should be emphasized that the results presented in Table 2 are not due to computational difficulties. Essentially the same results would be obtained with arbitrarily high precision arithmetic. Thus the instability of the "solution" to perturbations is not essentially a numerical problem but is inherent in the kind of solution being sought.

In contrast, let us look at the solution vectors $y_p(\beta)$ for $\beta \leqslant \beta_C$. The results are shown in Fig. 4, where the Chebyshev norm of the difference, as defined by (3.3), is plotted for each of the eight perturbed matrices. It is seen that, relative to the differences for larger $\beta$, the differences are very small for $\beta \leqslant \beta_C$. That is, the solution trajectory $y(\beta)$ for $\beta \leqslant \beta_C$ is essentially independent of the perturbations in the matrix $A$. As $\beta$ increases, with values $\beta \geqslant \beta_C$, the differences increase to their maximum values given in Table 2. Thus if we were presented with the matrix $A$ given by (3.1), and subject to small errors in the elements, we would say that $y(\beta)$ for $\beta \leqslant \beta_C$, represents the stable solution trajectory to the minimum-error problem. If we also take into account the error $y_0(\beta)$ as given by Table 1 and shown in Fig. 3, we see that the most reasonable choice for a stable solution is the vector $y(\beta_C)$, given in Table 1.

The single parametric run on the unperturbed matrix has given us the stable solution trajectory $y(\beta)$ and the corresponding error $y_0(\beta)$ for $\beta \leqslant \beta_C$. It also gives the special vector on this trajectory, $y(\beta_C)$, which is the stable solution with the smallest error. Finally, it gives (in terms of the vector $w$) the unstable solution trajectory $y(\beta)$ for $\beta \geqslant \beta_C$.

As an indication of computation time, a typical parametric LP run as described above, required about 30 seconds. The computation time for a large primal LP problem depends primarily on the number of rows in the problem, and only to a smaller extent on the number of columns. The number of rows in the primal problem (2.8) is $m + 1$, where $m$ is the number of variables in the original problem (1.1). The CDM 4 code will handle up to 400 rows, and other codes are available which handle even larger problems. On the basis of LP computational experience it should be possible to obtain the solution trajectory to a system (1.1) with several hundred variables and possibly several thousand equations in 30–60 minutes.

## Acknowledgment

The paper by Levenberg was called to my attention by Dr. Martin Beale during a very helpful discussion of some preliminary results I had obtained. The computational results presented here were obtained by Dennis Kuba. I also want to thank Dr. Klaus Ritter for several suggested improvements.

## References

1. R. Penrose. *Proc. Cambridge Phil. Soc.* **51**, 406-413 (1955).
2. T. N. E. Greville. *Soc. Ind. Appl. Math. Rev.* **2**, 15-22 (1960).
3. G. Golub. *Numerische Math.* **7**, 206-216 (1965).
4. K. Levenberg. *Quart. Appl. Math.* **2**, 164-168 (1944).
5. G. B. Dantzig. "Linear Programming and Extensions." Princeton University Press, Princeton, New Jersey (1963).
6. G. Hadley. "Linear Programming." Addison-Wesley, Reading, Mass. (1962), p. 237.
7. "CDM 4 Reference Manual." Control Data Corp., Minneapolis, Minnesota, June, 1965.