On the Structure at Infinity of a Structured System

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ABSTRACT

We develop a graph-theoretic characterization of the generic structure at infinity of the transfer matrix of a structured system. We show that the generic structure at infinity can be determined by means of algorithms from combinatorial optimization based on the max-flow min-cut theorem, and on results concerning minimal-cost flows. As an application of the obtained characterization, we propose a structural version of two well-known disturbance decoupling problems, and we derive graph-theoretic necessary and sufficient conditions for the solvability of each of the two problems.

1. INTRODUCTION

In the present paper we introduce the generic rank and the generic orders of the zeros at infinity, together forming the generic structure at infinity, of transfer matrices of a general class of structured linear systems. We represent structured systems by means of directed graphs, and we develop graph-theoretic characterizations of the generic rank and the generic orders of the zeros at infinity of the corresponding transfer matrix. We show that the obtained characterizations can be checked by means of well-known and efficient algorithms from combinatorial optimization. As an application of

*The results in this paper were obtained while the author was affiliated with the Centre for Mathematics and Computer Science in Amsterdam, The Netherlands.

LINEAR ALGEBRA AND ITS APPLICATIONS 148:145–169 (1991) 145

655 Avenue of the Americas, New York, NY 10010 0024-3795/91/$3.50

the obtained characterization, we propose a structural version of two well-known disturbance-decoupling problems, and we derive graph-theoretic necessary and sufficient conditions for the solvability of each of the two problems.

Having briefly sketched the contents of the paper, we now want to make clear why the study of structured systems is useful. To do this we may consider any well-established control problem, formulated for an appropriate linear system. For instance, we may think of the pole assignment problem, the disturbance-decoupling problem, or the problem of noninteracting control (cf. Wonham [24]).

One of the main ideas behind the present paper is that, before applying algorithms that check the solvability of the control problem and that compute the corresponding feedback control laws, it may be worthwhile to investigate if the system has any structure. If so, it may then be useful to try to determine, from this structure, whether or not in some structural sense the control problem is solvable. Of course, it is therefore required that we have a characterization of the structural solvability of the control problem in terms of the structure of the system. Furthermore, it is clear that it might be useful to have an algorithm by which we can verify the characterization in an efficient way. Finally, it should be clear that when we can derive such a characterization and algorithm, we obtain a powerful tool which exploits the structure present in the system and which, in addition to the existing algorithms, helps us to decide about the solvability of the control problem.

In the present paper we are motivated by the problem of disturbance decoupling by state feedback, well known from the geometric approach to control theory (cf. Wonham [24]). We recall that the solvability of the disturbance-decoupling problem is equivalent to the fact that certain elementary transfer matrices have the same rank and have zeros at infinity of the same orders. Since we represent structured systems by means of graphs, it is therefore clear that our first interest lies in the development of a graph-theoretic characterization of the rank and the orders of the zeros at infinity of the transfer matrix of a structured system.

The outline of the present paper is as follows. In Section 2 we introduce structured systems and describe a way in which they can be parametrized. In Section 3 we introduce the rank and the orders of the zeros at infinity of proper rational matrices. Furthermore, we state a result on the solvability of proper rational matrix equations over the proper rational matrices. In Section 4 we introduce the generic rank and the generic orders of the zeros at infinity of the transfer matrix of a structured system. In Section 5 we introduce the graphs corresponding to the structured systems, and we recall some important notions and results from graph theory.

In Section 6 we state our main results. We first prove that the generic rank of the transfer matrix of a structured system is equal to the largest number of disjoint paths from the set of input vertices to the set of output vertices in the graph corresponding to the structured system. If this rank equals r, we next prove that the generic orders of the zeros at infinity can be determined by computing, for i from 1 to r, the smallest number of state vertices appearing in any i-tuple of disjoint paths from the set of input vertices to the set of output vertices.

In Section 7 we discuss some of the computational aspects of the main results. We indicate that for a given structured system the generic rank and the generic orders of the zeros at infinity can be computed using algorithms from combinatorial optimization based on the max-flow min-cut theorem and on results on minimal-cost flows. In Section 8 we propose a structural version of the disturbance-decoupling problem and the so-called modified disturbance-decoupling problem, and we apply our main results to obtain a graph-theoretic characterization for the solvability of each of the two problems. In Section 9 we offer some remarks and comments.

2. STRUCTURED SYSTEMS

In this section we introduce structured systems, and we describe how these systems can be parametrized. Therefore, we consider the finite-dimensional linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1.1a}$$

$$y(t) = Cx(t), \tag{1.1b}$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, and output $y(t) \in \mathbb{R}^p$, and with A, B, and C real matrices of dimensions $n \times n$, $n \times m$, and $p \times n$, respectively. To give an indication of what we mean by structured systems, we assume that the system (1.1) is a series interconnection of the following two subsystems:

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t), \qquad \dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t),$$

$$y_1(t) = C_1 x_1(t), \qquad y_2(t) = C_2 x_2(t),$$

where $u_2(t) = y_1(t)$, $u(t) = u_1(t)$, and $y(t) = y_2(t)$, and all vectors and matrices have appropriate dimensions. After interconnection of the two subsystems it follows that

$$A=\begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}, \qquad B=\begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \text{and} \quad C=\begin{bmatrix} 0 & C_2 \end{bmatrix}.$$

The zeros in the above representation of A, B, and C are matrices with entries that are fixed zeros. This means that these entries are always zero, no matter what the entries are in the matrices A_1, B_1, C_1, A_2, B_2, and C_2. In this paper we call such fixed zeros in A, B and C *structural zeros*. Entries in A, B, and C that are not structural zeros we call *structural unknowns*, and we assume that the values of these entries are unknown and are independent of each other.

In this paper we say that a matrix is *structured* if each of its entries is either a structural zero or a structural unknown, and we call a system of the type (1.1) a *structured system* if the matrices A, B, and C are structured.

Given a structured system of the type (1.1), we denote the number of structural unknowns in A, B, and C by k, and we parametrize the set of all nominal systems that correspond to the same structured system by a parameter $\lambda \in \mathbb{R}^k$. To do this, we number the structural unknowns in A, B, and C from 1 to k, and we write λ_i at the ith structural unknown. We denote the nominal values of A, B and C at the parameter value $\lambda \in \mathbb{R}^k$ by A_λ, B_λ, and C_λ. Below we give an example of a structured system of the type (1.1), together with a possible parametrization.

EXAMPLE. $k=9$, $n=3$, $m=2$, $p=2$; 0 denotes a structural zero, and x a structural unknown:

$$A=\begin{bmatrix} 0 & x & 0 \\ x & 0 & 0 \\ 0 & x & x \end{bmatrix}, \qquad B=\begin{bmatrix} x & 0 \\ 0 & x \\ x & 0 \end{bmatrix}, \qquad C=\begin{bmatrix} 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}.$$

$$A_\lambda=\begin{bmatrix} 0 & \lambda_2 & 0 \\ \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & \lambda_4 \end{bmatrix}, \qquad B_\lambda=\begin{bmatrix} \lambda_5 & 0 \\ 0 & \lambda_7 \\ \lambda_6 & 0 \end{bmatrix}, \qquad C_\lambda=\begin{bmatrix} 0 & \lambda_8 & 0 \\ 0 & 0 & \lambda_9 \end{bmatrix}.$$

3. STRUCTURE AT INFINITY

In the present section we introduce the rank and the orders of the zeros at infinity of proper rational matrices. However, we start with a brief introduction on rational functions.

We call a function a *rational function* if it can be written as the quotient of two polynomials with real coefficients. Given such a representation and using the usual notion of degree for nonzero polynomials, we define the *degree* of a nonzero rational function to be the degree of the numerator polynomial minus the degree of the denominator polynomial. For rational functions identically equal to zero, we define the degree to be $-\infty$. Note that polynomials are rational functions and that for polynomials this new notion of degree coincides with the usual notion. We call a rational function *proper* if its degree is negative or zero, and *strictly proper* if its degree is negative. This means that, if written as the quotient of two polynomials, a rational function is proper if the degree of the numerator polynomial is not larger than the degree of the denominator polynomial, and strictly proper if the degree of the numerator polynomial is less the degree of the denominator polynomial.

We call a matrix a *rational matrix* if its entries are rational functions, a *proper rational matrix* if its entries are proper rational functions, and a *strictly proper rational matrix* if its entries are strictly proper rational functions. We say that a rational matrix has *rank* r if there is an rth-order minor of the matrix that is *unequal* to zero, while every $r+1$th-order minor of the matrix is *identically equal* to zero. We say that a square proper rational matrix is a *bicausal rational matrix* if the matrix is invertible and if its inverse is a proper rational matrix (cf. Hautus and Heymann [10]). Bicausal rational $t \times t$ matrices are the units in the ring of proper rational $t \times t$ matrices. It can be shown that a proper rational matrix is bicausal if and only if the determinant of its value at infinity is unequal to zero. Using bicausal rational matrices, we can state the following theorem concerning a factorization of proper rational matrices (cf. Descusse and Dion [4], Hautus [8]; also compare with the Smith form for polynomial matrices).

THEOREM 3.1. *Given a proper rational matrix $T(s)$, there exists a factorization*

$$T(s) = V(s)\begin{bmatrix} \Gamma(s) & 0 \\ 0 & 0 \end{bmatrix} U(s),$$

with $U(s)$ and $V(s)$ bicausal rational matrices of suitable dimensions and $\Gamma(s) = \mathrm{diag}(s^{-t_1}, s^{-t_2}, \dots, s^{-t_r})$, where $r = \operatorname{rank} T(s)$ and $t_1, t_2, \dots, t_r$ are integers that satisfy $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_r$.

The integers $t_1, t_2, \dots, t_r$ are known as the *orders of the zeros at infinity* of $T(s)$, and are uniquely determined. We say that two proper rational matrices with the same rank also have zeros at infinity of the same orders if the list of the orders of the zeros at infinity for both matrices is the same. Using the Cauchy-Binet formula, we can prove the following characterization of the orders of the zeros at infinity of a proper rational matrix, where we denote $m_i = \Sigma_{j=1}^{i} t_j$ for $i = 1, 2, \ldots, r$ (cf. Gantmacher [6, Chapter 6.3], where a similar result for polynomial matrices is proved).

LEMMA 3.2. *Let $T(s)$ be a proper rational matrix with a factorization as given in Theorem 3.1. Then for any $i = 1, 2, \ldots, r$, every ith-order minor of $T(s)$ is a proper rational function with a degree $\delta \leqslant -m_i$, and there exists at least one ith-order minor of $T(s)$ with a degree δ such that the equality holds, i.e., $\delta = -m_i$.*

Lemma 3.2 implies that the number m_i equals the exponent of the greatest power of s by which any ith-order minor of the proper rational matrix $T(s)$ can be multiplied such that the product remains proper. Clearly, we could have used the latter characterization to give an alternative definition of the orders of the zeros at infinity in which the use of a factorization of a proper rational matrix is avoided. In fact, in the next section, we more or less use this alternative approach to introduce the *generic* orders of the zeros at infinity of the transfer matrix of a structured system.

We now state a theorem that we need in Section 8 to obtain suitable conditions for the solvability of the disturbance decoupling problem and the modified disturbance decoupling problem (cf. Emre and Hautus [5], Newman [16] and Verghese [21]).

THEOREM 3.3. *Let $T(s)$ and $S(s)$ be proper rational matrices that have the same number of rows. Then there exists a proper rational matrix $X(s)$ of suitable dimensions such that $T(s)X(s) = S(s)$ if and only if the rank and the sum of the orders of the zeros at infinity of the matrix $T(s)$ and of the compound matrix $[T(s) \;\; S(s)]$ are the same.*

4. GENERIC STRUCTURE AT INFINITY

In this section we introduce the generic rank and the generic orders of the zeros at infinity of the transfer matrix of a structured system of the type (1.1), by giving new meanings to the integers r, t_i, and m_i for $i = 1, 2, \ldots, r$. We start with the introduction of the generic rank.

Given a structured system of the type (1.1), parametrized by $\lambda \in \mathbb{R}^k$, we denote

$$K_\lambda(s) = C_\lambda(sI - A_\lambda)^{-1}B_\lambda, \qquad M_\lambda(s) = \begin{bmatrix} A_\lambda - sI & B_\lambda \\ C_\lambda & 0 \end{bmatrix}, \tag{4.1}$$

and we define

$$r = \max_{\lambda \in \mathbb{R}^k} \{\operatorname{rank} K_\lambda(s)\}, \qquad R = \{\lambda \in \mathbb{R}^k | \operatorname{rank} K_\lambda(s) < r\}. \tag{4.2}$$

Note that if $r = 0$, then $R = \varnothing$, where $\varnothing$ denotes the empty set.

Following Wonham [24], we call a subset L in $\mathbb{R}^k$ an *algebraic variety* in $\mathbb{R}^k$ if L can be described as the locus of common zeros of a finite number of polynomials $\psi_1, \psi_2, \ldots, \psi_t$ in the indeterminate $\tau = (\tau_1, \tau_2, \ldots, \tau_k)$, i.e., $L = \{(\tau_1, \tau_2, \ldots, \tau_k) \in \mathbb{R}^k \,|\, \psi_i(\tau_1, \tau_2, \ldots, \tau_k) = 0$ for all $i = 1, 2, \ldots, t\}$. We say that an algebraic variety L in $\mathbb{R}^k$ is *proper* if $L \neq \mathbb{R}^k$. Now we can state the following (cf. van der Woude [25]).

THEOREM 4.1. *R is a proper algebraic variety in $\mathbb{R}^k$.*

Proof. If $r = 0$, then $R = \varnothing$, and R clearly is a proper algebraic variety. If $r > 0$, then using the identity

$$M_\lambda(s) = \begin{bmatrix} I & 0 \\ C_\lambda(A_\lambda - sI)^{-1} & I \end{bmatrix} \begin{bmatrix} A_\lambda - sI & 0 \\ 0 & K_\lambda \end{bmatrix} \begin{bmatrix} I & (A_\lambda - sI)^{-1}B_\lambda \\ 0 & I \end{bmatrix}, \tag{4.3}$$

it follows that $\operatorname{rank} K_\lambda(s) = \operatorname{rank} M_\lambda(s) - n$. From the definition of R and our notion of rank it is now clear that

$$R = \{\lambda \in \mathbb{R}^k | \text{every } n+r\text{th order minor of } M_\lambda(s) \text{ is identically equal to zero}\}.$$

Next observe that any minor of the matrix $M_\lambda(s)$ is a polynomial in the indeterminate s with coefficients that are polynomials in $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Furthermore, recall that a polynomial in the indeterminate s is identically equal to zero if and only if all its coefficients are zero. Therefore, it follows that R is the locus of common zeros of a finite number of polynomials in λ. By the definition of r it is clear that $R \neq \mathbb{R}^k$. So, also if $r > 0$, the set R is a proper algebraic variety in $\mathbb{R}^k$. ■

The above theorem implies that $\operatorname{rank} K_\lambda(s) = r$ for almost all $\lambda \in \mathbb{R}^k$, where "almost all" is to be interpreted as "everywhere except for a proper algebraic variety." Hence, we can think of r as the *generic rank* of $K(s)$, where $K(s)$ formally denotes the transfer matrix of the structured system, i.e. $K(s) = C(sI - A)^{-1}B$.

By Lemma 3.2 we know that in the unstructured case the orders of the zeros at infinity of a transfer matrix are closely related to the degrees of its minors. Hence, for the introduction of the generic orders of the zeros at infinity of $K(s)$, it seems natural that we first consider a *square* structured system, i.e. $m = p$, and that we introduce the generic degree of the determinant of the transfer matrix of such a system. To do this, we consider a square structured system of the type (1.1), parametrized by $\lambda \in \mathbb{R}^k$, and we assume that the generic rank of its transfer matrix is r.

If $m = p = r$, we define

$$q = \max_{\lambda \in \mathbb{R}^k} \{\deg\det K_\lambda(s)\}, \qquad Q = \{\lambda \in \mathbb{R}^k \mid \deg\det K_\lambda(s) < q\}, \quad (4.4a)$$

and if $m = p > r$, which means that $\det K_\lambda(s) = 0$ for all $\lambda \in \mathbb{R}^k$ and all s, we define

$$q = -\infty, \qquad Q = \varnothing. \quad (4.4b)$$

In the above, deg stands for degree and det for determinant. Now we can state the following.

THEOREM 4.2. *Q is contained in a proper algebraic variety in $\mathbb{R}^k$.*

Proof. If $m = p > r$, then $Q = \varnothing$, and Q clearly is contained in a proper algebraic variety. If $m = p = r$, then it easily follows from (4.3) and (4.4a) that

$$q + n = \max_{\lambda \in \mathbb{R}^k} \{\deg\det M_\lambda(s)\} \ (\geqslant 0), \quad (4.5a)$$

and that

$$Q = \{\lambda \in \mathbb{R}^k \mid \deg\det M_\lambda(s) < q + n\}. \quad (4.5b)$$

As in the proof of Theorem 4.1, it is clear that $\det M_\lambda(s)$ is a polynomial in the indeterminate s with coefficients that are polynomials in $\lambda \in \mathbb{R}^k$. By the above description of Q, it is therefore clear that Q is contained in the algebraic variety in $\mathbb{R}^k$ defined as the set of $\lambda \in \mathbb{R}^k$ for which the coefficient of s^{q+n} in the $q+n$th order polynomial $\det M_\lambda(s)$ is equal to zero. By the definition of q it moreover follows that this algebraic variety is proper. So, also if $m = p = r$, the set Q is contained in a proper algebraic variety. ∎

Theorem 4.2 implies that for a square structured system of the type (1.1), parametrized by $\lambda \in \mathbb{R}^k$, we have that if $m = p > r$, then $\deg\det K_\lambda(s) = q = -\infty$ for all $\lambda \in \mathbb{R}^k$, and that if $m = p = r$, then $\deg\det K_\lambda(s) = q$, with $-\infty < q < 0$, for almost all $\lambda \in \mathbb{R}^k$. Hence, for a square structured system of the type (1.1), we can think of q as the *generic degree* of the determinant of $K(s)$.

In the remainder of the present section we return to a general structured system of the type (1.1), and we do not assume any more that the system is square. Then, using the above, we can introduce the generic orders of the zeros at infinity of the transfer matrix $K(s)$. To that end, we note that for every $\lambda \in \mathbb{R}^k$, any minor of $K_\lambda(s)$ corresponds to the determinant of the transfer matrix of a square subsystem of the system (1.1) at the parameter value λ. Therefore, it is clear that we can consider the generic degree of such a minor to be the generic degree of the determinant of the transfer matrix of the corresponding square structured subsystem. Since there are only a finite number of minors of the same order, we can take the maximum of the generic degrees of all these minors. We define m_i as *minus* the maximum of the generic degrees of all ith-order minors of $K(s)$, where $1 \leqslant i \leqslant r$, with r the generic rank of $K(s)$. We can now easily prove that

$$m_i = -\max_{\lambda \in \mathbb{R}^k} \left\{ \max\{\deg K_\lambda^i(s) \mid K_\lambda^i(s) \text{ is an } i\text{th-order minor of } K_\lambda(s)\}\right\} \tag{4.6}$$

for $i = 1, 2, \ldots, r$, and that $0 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_r$. In addition, we can prove in the same way as in Theorems 4.1 and 4.2 that the set of parameter values $\lambda \in \mathbb{R}^k$ for which all ith-order minors of $K_\lambda(s)$ have degree less than $-m_i$ is contained in a proper algebraic variety in $\mathbb{R}^k$.

Now, in the spirit of Theorem 3.1 and Lemma 3.2, we define for $i = 1, 2, \ldots, r$ the integers t_i, given by

$$t_i = m_i - m_{i-1} \tag{4.7}$$

with $m_0 = 0$, to be the *generic orders of the zeros at infinity* of $K(s)$.

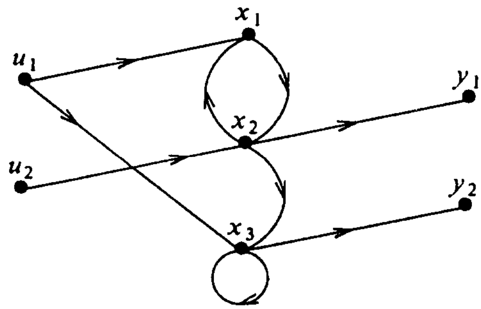

FIG. 1.

5. GRAPHS

In the previous section we introduced the generic rank and the generic orders of the zeros at infinity of the transfer matrix $K(s)$. In the next section we describe how these notions can be related to the structure of systems of the type (1.1). For this purpose, we represent a structured system of the type (1.1) by a *directed graph*. This graph, denoted $G(V, E)$, consists of a vertex set V with $n+m+p$ vertices and an edge set E of k directed edges (ordered pairs). The set V is defined as $V = U \cup X \cup Y$, where $U = \{u_1, u_2, \dots, u_m\}$, $X = \{x_1, x_2, \dots, x_n\}$, $Y = \{y_1, y_2, \dots, y_p\}$, and $\cup$ denotes the union. The set E is defined as $E = \{(u_j, x_i) \mid b_{i,j} \neq 0\} \cup \{(x_j, x_i) \mid a_{i,j} \neq 0\} \cup \{(x_j, y_i) \mid c_{i,j} \neq 0\}$. Here, for instance, the ordered pair (u_j, x_i) represents a directed edge from the vertex u_j to the vertex x_i, and $b_{i,j} \neq 0$ means that the entry $b_{i,j}$ in the matrix B is a structural unknown entry. U, X, and Y are called the sets of input vertices, state vertices, and output vertices, respectively. In Figure 1 we have depicted the graph $G(V, E)$ corresponding to the structured system in the example of Section 2.

In Section 2 we considered a structured system of the type (1.1), and we assumed that the system contained k structural unknowns, numbered from 1 to k. We parametrized all nominal systems that correspond to the same structured system with a parameter $\lambda \in \mathbb{R}^k$ by writing λ_i at the ith structural unknown. Using this numbering, we can also number the edges in E from 1 to k, and we can introduce the nominal (or weighted) directed graph $G_\lambda(V, E)$ at the parameter value $\lambda \in \mathbb{R}^k$ by weighting the ith edge of the graph $G(V, E)$ by λ_i. In Figure 2 we have depicted the graph $G_\lambda(V, E)$ that is obtained from the graph in Figure 1 by weighting the edges in accordance to the parametrization described in the example of Section 2.

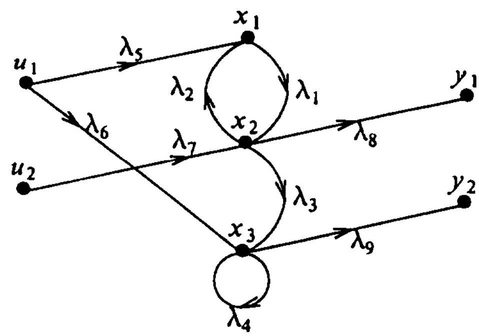

FIG. 2.

Given the graph $G(V, E)$ [or $G_\lambda(V, E)$], we say that there is a *self-loop* at the vertex $v \in V$ if $(v, v) \in E$. We say that there exists a *path* from the vertex v to the vertex v' if there are vertices $w_1, w_2, \ldots, w_\tau$ in V such that $v = w_1$, $v' = w_\tau$, and $(w_i, w_{i+1}) \in E$ for $i = 1, 2, \ldots, \tau - 1$. If, in addition, $v \in U$ and $v' \in Y$, we say that there is a path from U to Y. If we have a path from v to v' with $v = v'$, we say that the path is *closed*. If a path consists of distinct vertices, we say that the path is *simple*, and if a path is both simple and closed, we call it a *cycle*. Clearly, a self-loop is a cycle. We say that an l-tuple of paths (cycles) in $G(V, E)$ are *disjoint* if each pair of paths (cycles) of the l-tuple have no vertices in common.

The weighted graph $G_\lambda(V, E)$ can be considered to be a special case of a so-called *Coates graph* associated to a real square matrix (cf. Chen [2]). For a real nominal $\tau \times \tau$ matrix M that has l nonzero entries, the associated Coates graph, denoted G_M, is a graph with a vertex set V_M of τ vertices and an edge set E_M of l directed and weighted edges. If the vertex set is given by $V_M = \{v_1, \ldots, v_\tau\}$, then the edge set E_M consists of edges weighted $m_{i,j}$ and directed from v_j to v_i precisely if $m_{i,j} \neq 0$, i.e., $E_M = \{(v_j, v_i) \mid m_{i,j} \neq 0\}$.

We define paths, cycles, disjoint paths, and disjoint cycles for Coates graphs in the same way as for $G(V, E)$, and we define a *cycle family* for a Coates graph to be a number of *disjoint* cycles such that each vertex of the graph belongs to precisely one cycle, in which case we say that the cycle family *spans* the graph. We define the weight of a cycle family to be the product of the weights of the edges that constitute the cycle family. If Cy denotes a cycle family, we denote its weight by $W(\text{Cy})$, and we denote by $n(\text{Cy})$ the total number of disjoint cycles the cycle family consists of. Now we can state the following classical result (cf. Chen [2, Theorem 3.1]), where we

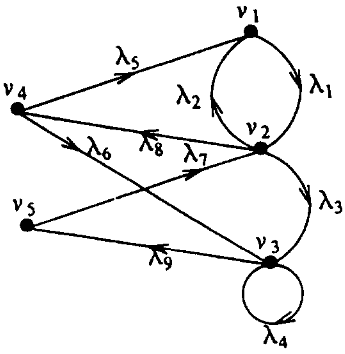

FIG. 3.

denote by Cf the set of all cycle families in the Coates graph G_M associated to the $\tau \times \tau$ matrix M, and where, as before, det stands for determinant.

THEOREM 5.1. $\det M = (-1)^{\tau} \sum_{\mathrm{Cy} \in \mathrm{Cf}} (-1)^{n(\mathrm{Cy})} W(\mathrm{Cy}).$

As an example, we have depicted in Figure 3 the Coates graph $G_{M_\lambda(0)}$ corresponding to the square matrix $M_\lambda(0)$ defined in (4.1), with A_λ, B_λ, and C_λ as described in the example of Section 2.

In the remainder of the present section we consider a structured system of the type (1.1), parametrized by $\lambda \in \mathbb{R}^k$, for which the number of inputs and the number of outputs are equal, i.e., $m = p$. Hence, the system is square and also the nominal matrix $M_\lambda(0)$ is square. We now can obtain the Coates graph $G_{M_\lambda(0)}$ corresponding to the matrix $M_\lambda(0)$ directly from the graph $G_\lambda(V, E)$. We can do this by identifying in the vertex set V the ith input vertex with the ith output vertex, for $i = 1, 2, \ldots, m$. We then obtain a graph with a vertex set consisting of $n + m$ vertices, and with a weighted edge set similar to $E_{M_\lambda(0)}$. Conversely, if we have a Coates graph $G_{M_\lambda(0)}$ with $V_{M_\lambda(0)} = \{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+m}\}$, we can obtain the graph $G_\lambda(V, E)$ by replacing each vertex v_{n+i} with two vertices u_i and y_i for $i = 1, 2, \ldots, m$, and by replacing each edge of the form $(v_{n+i}, v) \in E_{M_\lambda(0)}$ with the edge (u_i, v), and each edge of the form $(v', v_{n+i}) \in E_{M_\lambda(0)}$ with the edge (v', y_i), where $i = 1, 2, \ldots, m$. We then obtain a graph with a vertex set consisting of $n + m + m$ vertices and with an edge set similar to E.

Furthermore, with the vertex set of the Coates graph $G_{M_\lambda(0)}$ given as $V_{M_\lambda(0)} = \{v_1, v_2, \ldots, v_{n+m}\}$, we can make the following observation. If there exists a *cycle* in $G_{M_\lambda(0)}$ that contains exactly μ vertices of the set

$\{v_{n+1},\ldots,v_{n+m}\}$, then there exists in $G_\lambda(V,E)$ [and also in $G(V,E)$] a μ-tuple of disjoint paths from U to Y. To see this, we may assume without loss of generality that the μ vertices of the cycle in $\{v_{n+1},\ldots,v_{n+m}\}$ are in fact the vertices $v_{n+1},v_{n+2},\ldots,v_{n+\mu}$, and that in the cycle v_{i+n} precedes v_{i+n+1} for $i=1,2,\ldots,\mu-1$, and that $v_{n+\mu}$ precedes v_{n+1}. Here we say that the vertex v *precedes* the vertex v' if there is a part of the cycle that constitutes a simple path from v to v' that besides v and v' does not contain any other vertex of the set $\{v_{n+1},v_{n+2},\ldots,v_{n+m}\}$. Then it easily follows that in the graph $G_\lambda(V,E)$ the paths from the vertex u_i to the vertex y_{i+1}, for $i=1,2,\ldots,\mu-1$, together with the path from the vertex u_μ to the vertex y_1, constitute a μ-tuple of disjoint paths in $G_\lambda(V,E)$ from U to Y. Using the same reasoning, it follows that if there is a *cycle family* in $G_{M_\lambda(0)}$ that contains (necessarily) all the vertices of the set $\{v_{n+1},\ldots,v_{n+m}\}$, then there is an m-tuple of disjoint paths in $G_\lambda(V,E)$ [and in $G(V,E)$] from U to Y.

6. MAIN RESULTS

In this section we state the main results of this paper. The results describe relations between the graph $G(V,E)$ associated to a structured system of type (1.1), and the generic rank and the generic orders of the zeros at infinity of the corresponding transfer matrix. We recall that X, U, and Y denote the sets of state vertices, input vertices, and output vertices, respectively, of the graph $G(V,E)$. As a first result we state the following theorem, in which we use r as defined in (4.2) (cf. van der Woude [25]).

THEOREM 6.1. *The largest number of disjoint paths in $G(V,E)$ from U to Y is equal to r.*

Proof. We start the proof by considering the case that $r>0$. By the definition of r and the notion of rank introduced in Section 3, it follows that there is a parameter $\bar\lambda\in\mathbb{R}^k$ for which there is an rth-order minor of $K_\lambda(s)$ unequal to zero. Without loss of generality we may assume that this rth-order minor is $\det C'_{\bar\lambda}(sI-A_{\bar\lambda})^{-1}B'_{\bar\lambda}$, where B'_λ denotes the first r columns of B_λ, C'_λ denotes the first r rows of C_λ, and we have substituted $\lambda=\bar\lambda$. Since the minor is nonzero, there exists a real number $\bar s$ such that $\det C'_{\bar\lambda}(\bar sI-A_{\bar\lambda})^{-1}B'_{\bar\lambda}\neq 0$ and $\det(A_{\bar\lambda}-\bar sI)\neq 0$. Using (4.3), it now follows that $\det M'_{\bar\lambda}(\bar s)\neq 0$, where

$$M'_\lambda(s)=\begin{bmatrix} A_\lambda-sI & B'_\lambda \\ C'_\lambda & 0\end{bmatrix}.$$

By Theorem 5.1 this implies that in the Coates graph associated to the nominal matrix $M'_{\bar\lambda}(\bar s)$ there is at least one spanning cycle family. Now we let $\hat A$, $\hat B$, and $\hat C$ be structured matrices for which $A_{\bar\lambda} - \bar s I$, $B'_{\bar\lambda}$, and $C'_{\bar\lambda}$, respectively, can occur as the nominal values. Clearly, we can take $\hat A = A + E$, $\hat B = B'$, and $\hat C = C'$, where B' denotes the first r columns of B, C' denotes the first r rows of C, and E denotes a structured matrix with only structural unknowns on its diagonal. By the remarks at the end of the previous section it now follows that in the graph of the structured system described by $\hat A$, $\hat B$, and $\hat C$ there exists an r-tuple of disjoint paths from the set of input vertices to the set of output vertices. Since in the context of disjoint paths the self-loops introduced by E are of no interest, it follows that in the graph of the structured system described by A, B', and C' there are r disjoint paths from the set of input vertices to the set of output vertices. Because B' is a part of B and C' is a part of C, it now follows that $n_+ \geqslant r > 0$, where we have denoted by n_+ the largest number of disjoint paths in $G(V, E)$ from U to Y. The latter also implies that if $n_+ = 0$, which means that there is no path in $G(V, E)$ from U to Y, then $r = 0$.

Next we consider the case that we have an n_+-tuple of disjoint paths in $G(V, E)$ and in $G_\lambda(V, E)$ from U to Y with $n_+ > 0$, where n_+ is as defined above, and we concentrate on the subgraph built up from the vertices and edges in the n_+-tuple of paths only. It is easy to see that this subgraph corresponds to n_+ totally decoupled structured single-input single-output systems that each have a transfer function with a generic rank equal to 1. The n_+ subsystems can be obtained from the original system by specifying that some of the structural unknowns are in fact zero. This comes down to saying that the parameter λ, which parametrizes $G_\lambda(V, E)$ and also the system (1.1), is restricted to some proper subset L in $\mathbb{R}^k$. Therefore, since $L \subseteq \mathbb{R}^k$, it is clear that $0 < n_+ = \max_{\lambda \in L} \operatorname{rank} K_\lambda(s) \leqslant r$. This also implies that if $r = 0$, meaning that $K_\lambda(s) = 0$ for all $\lambda \in \mathbb{R}^k$ and all s, then $n_+ = 0$.

The proof of the present theorem can now be completed by combining all the obtained relations between r and n_+. ■

From Section 4 it is immediate that Theorem 6.1 implies that the generic rank of $K(s)$ is equal to the largest number of disjoint paths in $G(V, E)$ from U to Y. Hence, we have obtained a graph-theoretic characterization of the generic rank of the transfer matrix $K(s)$. To obtain a graph-theoretic characterization of the generic orders of the zeros at infinity of the transfer matrix $K(s)$, we need the following theorem, formulated for a square structured system with a transfer matrix that has a generic rank r equal to $m = p$. In the theorem we use q as defined in (4.4a).

THEOREM 6.2. *If $m = p = r$, then the smallest number of state vertices in any r-tuple of disjoint paths in $G(V, E)$ from U to Y is equal to $-q$.*

Proof. By the definition (4.4a) of q, there exists a $\bar{\lambda} \in \mathbb{R}^k$ for which $\deg\det K_{\bar{\lambda}}(s) = q$ with $-\infty < q < 0$. Then, from (4.3) it follows that $\deg\det M_{\bar{\lambda}}(s) = n + q$ with $0 \leqslant n + q < n$. Hence, $\det M_{\bar{\lambda}}(s)$ is equal to an $n + q$th-order polynomial in the indeterminate s. By Theorem 5.1 this means that in the Coates graph $G_{M_{\bar{\lambda}}(s)}$ there is at least one cycle family with a weight that is equal to an $n + q$th-order polynomial in the indeterminate s. Now note that a factor $s + \alpha$ in the product making up the weight of a cycle family precisely corresponds to a self-loop at one of the vertices $v_1, \dots, v_n$ in the vertex set $V_{M_{\bar{\lambda}}(s)} = \{v_1, \dots, v_{n+m}\}$ of $G_{M_{\bar{\lambda}}(s)}$. This implies that the above cycle family consists of at least $n + q + 1$ disjoint cycles, of which exactly $n + q$ are self-loops at $n + q$ vertices in the subset $\{v_1, \dots, v_n\}$. The other $n - (n + q) = -q$ vertices in the subset $\{v_1, \dots, v_n\}$ appear in the remaining cycles of the cycle family. These remaining cycles cannot be self-loops, and have weights that are independent of the indeterminate s. Also these cycles contain all the vertices of the set $\{v_{n+1}, \dots, v_{n+m}\}$. By the remarks at the end of the previous section, it now follows that these cycles correspond with r disjoint paths in the graph $G(V, E)$ from U to Y. It is clear that these r disjoint paths contain at most $-q$ state vertices. Hence, $n_- \leqslant -q$, where we have denoted by n_- the smallest number of state vertices in any r-tuple of disjoint paths in $G(V, E)$ from U to Y.

Conversely, suppose that for a given $\bar{\lambda} \in \mathbb{R}^k$, there is a set of r *simple* and disjoint paths in $G_{\bar{\lambda}}(V, E)$ from U to Y, and that the r paths contain n_- state vertices with n_- as defined above. Then consider the Coates graph associated to the matrix $M_{\bar{\lambda}}(s)$ obtained by identifying the vertices u_i with y_i for $i = 1, 2, \dots, r$. The r-tuple disjoint paths in $G_{\bar{\lambda}}(V, E)$ induce a number of disjoint cycles in $G_{M_{\bar{\lambda}}(s)}$. It is clear that for almost all values of s there is a self-loop with a nonzero weight at each of the vertices $v_1, \dots, v_n$ in the vertex set $\{v_1, \dots, v_{n+m}\}$ of $G_{M_{\bar{\lambda}}(s)}$. So clearly, at the $n - n_-$ vertices in $\{v_1, \dots, v_n\}$ that do not appear in the disjoint cycles induced by the r disjoint paths in $G_{\bar{\lambda}}(V, E)$ from U to Y, there is a self-loop with a nonzero weight for almost all values of s. The product of the weights of these self-loop is a polynomial of degree $n - n_-$ in the value of s. Since the weights of the disjoint cycles induced in $G_{M_{\bar{\lambda}}(s)}$ by the r disjoint paths in $G_{\bar{\lambda}}(V, E)$ from U to Y are independent of the value of s (the paths are simple), it follows that the cycle family constituted by the self-loops and the disjoint cycles has a weight that is a polynomial of degree $n - n_-$ in the value s. Because each cycle family contributes to $\det M_{\bar{\lambda}}(s)$, this implies that $n - n_- \leqslant n + q$, which in turn implies that $-q \leqslant n_-$. ■

The above theorem was formulated for a square structured system. We now return to a general system of the type (1.1) that is not necessarily square. If the transfer matrix of the system $K(s)$ has a generic rank equal to

r, then for $i = 1, 2, \ldots, r$, every ith-order minor of $K(s)$ has a generic rank less than or equal to $-m_i$, and there exists at least one ith-order minor of $K(s)$ that has a generic degree equal to $-m_i$. This immediately follows from the properties of the numbers m_i defined in (4.6). Since each minor of $K(s)$ corresponds to the determinant of the transfer matrix of a square subsystem, the next theorem immediately follows from Theorem 6.2 and the way in which the numbers m_i were introduced in Section 4. In the theorem we assume that the generic rank of $K(s)$ is equal to r, and we assume that the generic orders of the zeros at infinity are defined by (4.7).

THEOREM 6.3. *For $i = 1, 2, \ldots, r$, the smallest number of state vertices in any i-tuple of disjoint paths in $G(V, E)$ from U to Y is equal to $m_i = \sum_{j=1}^{i} t_j$.*

To illustrate the results of this section, we return to the structured system in the example of Section 2. From the graph in Figure 1 and Theorem 6.1, it follows that the generic rank r of the transfer matrix $K(s)$ equals 2. In addition, it follows from Theorem 6.3 that the generic orders of the zeros at infinity satisfy $t_1 = 1$ and $t_2 = 1$. The generic rank and the generic orders of the zeros at infinity could also have been determined by considering

$$K_\lambda(s) = \frac{1}{s^3 - s^2\lambda_4 - s\lambda_1\lambda_2 + \lambda_1\lambda_2\lambda_4}$$

$$\times \begin{bmatrix} (s - \lambda_4)\lambda_1\lambda_5\lambda_8 & s(s - \lambda_4)\lambda_7\lambda_8 \\ (s^2 - \lambda_1\lambda_2)\lambda_6\lambda_9 + \lambda_1\lambda_3\lambda_5\lambda_9 & s\lambda_3\lambda_7\lambda_9 \end{bmatrix},$$

but it is clear that computing and manipulating $K_\lambda(s)$ may be more cumbersome than working with the simple graph in Figure 1. However, for systems larger than the one in the example of Section 2, the graphs may become more complicated, and it may not be possible to determine the generic rank and the generic orders of the zeros at infinity by hand. In such cases, we can use some efficient algorithms from combinatorial optimization. We discuss these algorithms in the next section.

To conclude this section, we note that its results are closely related to results in Reinschke [17, Section 32]. There, starting from the point of view of closed-loop systems, graph-theoretic characterizations of the rank and the orders of the zeros at infinity are given in terms of so-called feedback edges and feedback cycles.

7. COMPUTATIONAL ASPECTS

In the previous section we derived a graph-theoretic characterization of the generic rank and the generic orders of the zeros at infinity of the transfer matrix of a structured system. In the present section, we describe how these generic rank and generic orders can be computed by means of algorithms from combinatorial optimization.

Therefore, in addition to the graph $G(V, E)$, we introduce a second type of graph corresponding to a structured system of the type (1.1). This new type of graph, denoted $G(\tilde{V}, \tilde{E})$, consists of a vertex set $\tilde{V}$ and an edge set $\tilde{E}$. The set $\tilde{V}$ is given by $\tilde{V} = \{a\} \cup \{u_1, u_2, \dots, u_m\} \cup \{x_1^I, x_2^I, \dots, x_n^I\} \cup \{x_1^O, x_2^O, \dots, x_n^O\} \cup \{y_1, y_2, \dots, y_p\} \cup \{b\}$, and the edge set $\tilde{E}$ is given by $\{(a, u_i) \mid i = 1, \dots, m\} \cup \{(u_j, x_i^I) \mid b_{i,j} \neq 0\} \cup \{(x_j^O, x_i^I) \mid a_{i,j} \neq 0\} \cup \{(x_j^O, y_i) \mid c_{i,j} \neq 0\} \cup \{(y_j, b) \mid j = 1, \dots, p\} \cup \{(x_i^I, x_i^O) \mid i = 1, \dots, n\}$. Again, for instance, the ordered pair (u_j, x_i^I) represents a directed edge from the vertex u_j to the vertex x_i^I, and $b_{i,j} \neq 0$ means that the entry $b_{i,j}$ in the matrix B is a structural unknown entry. We call the vertices a and b in $G(\tilde{V}, \tilde{E})$ the *source* and the *sink*, respectively. It is easy to see that any i-tuple of disjoint paths in $G(V, E)$ from U to Y is in one-to-one correspondence with an i-tuple of paths in $G(\tilde{V}, \tilde{E})$ from a to b, in which each pair of paths, apart from a and b, have no vertices in common (compare Figure 1 with Figure 4).

In the remainder of this section we think of the graph $G(\tilde{V}, \tilde{E})$ as a network in which there is a flow from the source a to the sink b. We only allow nonnegative flows in the directions of the edges of the network, and we assume that all edges have a (maximal flow) capacity equal to 1. In Figure 4 we have depicted the graph $G(\tilde{V}, \tilde{E})$ associated to the structured system of the example in Section 2. The number above each edge denotes its capacity.

Using algorithms based on the celebrated max-flow min-cut theorem, we can compute the maximal flow in the network $G(\tilde{V}, \tilde{E})$ from a to b (cf. Lawler [11, Section 4.3]). Moreover, using standard results we can prove the following (cf. Lawler [11]).

THEOREM 7.1. *The maximal flow in $G(\tilde{V}, \tilde{E})$ from a to b is equal to the largest number of disjoint paths in $G(V, E)$ from U to Y.*

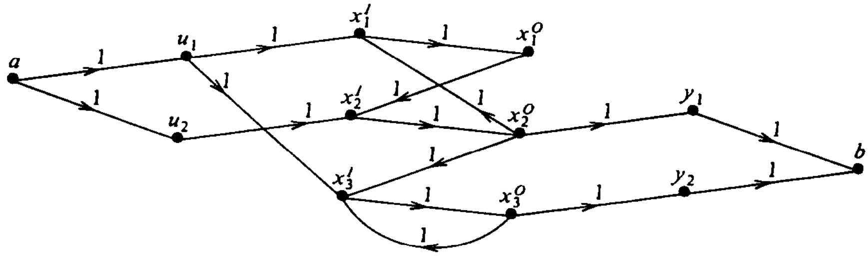

FIG. 4.

Hence, by algorithms based on the max-flow min-cut theorem, we can compute the largest number of disjoint paths in $G(V, E)$ from U to Y, and consequently, we can compute the generic rank of the transfer matrix of the underlying structured system. As before, we denote this generic rank by r.

To compute the generic orders of the zeros at infinity of the transfer matrix of the structured system, we proved in the previous section that we have to compute, for $i = 1, \ldots, r$, the smallest number of state vertices appearing in any i-tuple of disjoint paths in $G(V, E)$ from U to Y. These numbers can also be computed by means of a well-known algorithm from combinatorial optimization. To do this, we modify to the graph $G(\tilde{V}, \tilde{E})$ by also attaching costs to flows along the edges. To flows along each of the n edges from x_i^I to x_i^O we attach a cost factor 1, and to flows along all the other edges we attach a cost factor 0. The actual costs of a flow along an edge are then given by the product of the cost factor and the strength of the flow, and the cost associated to a flow in the network is given by the sum of the costs of the flows along the edges. More precisely, the above means that if, for $i = 1, 2, \ldots, n$, the flow along the edge $(x_i^I, x_i^O) \in \tilde{E}$ has a strength α_i, then the cost associated to the total flow in $G(\tilde{V}, \tilde{E})$ from a to b is equal to $\Sigma_{i=1}^n(\alpha_i \times 1) = \Sigma_{i=1}^n \alpha_i$. In Figure 5 we have depicted the graph of Figure 4 where in addition to the capacity also a cost factor is attached to each edge. The two numbers above each edge denote the capacity and the cost factor in that order.

Let the maximal flow in $G(\tilde{V}, \tilde{E})$ from a to b have strength r. Then it is easy to see that if the smallest number of state vertices in any r-tuple of disjoint paths in $G(V, E)$ from U to Y is equal to l, the minimal cost associated to a maximal flow in $G(\tilde{V}, \tilde{E})$ from a to b is less than or equal to l. The converse is also true. In fact, using standard results we can prove the following (cf. Lawler [11]).

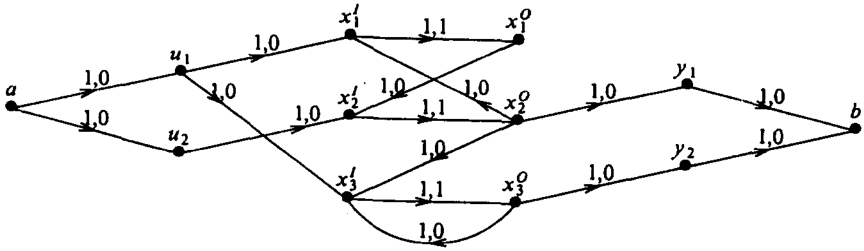

FIG. 5.

THEOREM 7.2. *For $i = 1, 2, \ldots, r$, the minimal cost associated to a flow of strength i in $G(\bar{V}, \bar{E})$ from a to b is equal to the smallest number of state vertices appearing in any i-tuple of disjoint paths in $G(V, E)$ from U to Y.*

Hence, based on Theorem 7.2, we can apply a well-known algorithm that computes the minimal costs of successive flows in $G(\bar{V}, \bar{E})$ from a to b, from a flow of strength zero up to the maximal flow r (cf. Lawler [11, Section 4.7]). The obtained sequence of minimal costs can then be used to compute the generic orders of the zeros at infinity of $K(s)$, using Theorem 6.3 and (4.7).

8. APPLICATION

In the present section we propose structural versions of the well-known disturbance-decoupling problem and the so-called modified disturbance-decoupling problem, and we apply our main results to obtain graph-theoretic conditions for the solvability of each of the two problems. To formulate the problems, we consider the following extension of the system (1.1):

$$\dot{x}(t) = Ax(t) + Bu(t) + Gd(t), \tag{8.1a}$$

$$y(t) = Cx(t). \tag{8.1b}$$

Here $x(t)$, $u(t)$, $y(t)$, A, B, and C are as in the description of the system (1.1), $d(t) \in \mathbb{R}^l$ denotes the disturbance input, and G is an $n \times l$ matrix. Like A, B, and C, we assume that G is a structured matrix. We denote the total number of structural unknowns in A, B, C, and G by k'. Parametrizing the structural unknowns and collecting all parameters in the vector $\lambda' \in \mathbb{R}^{k'}$, we denote by $A_{\lambda'}$, $B_{\lambda'}$, $C_{\lambda'}$, and $G_{\lambda'}$ the nominal values of A, B, C, and G for a given $\lambda' \in \mathbb{R}^{k'}$. Note that the compound matrix $[B \; G]$ can be considered to be the input matrix for the system (8.1) in the same way as the matrix B is the input matrix for the system (1.1). As with the system (1.1), we can associate graphs $G(V', E')$ and $G_{\lambda'}(V', E')$ to the system (8.1). The graph $G(V', E')$ consists of a vertex set $V' = V \cup D$ and an edge set $E' = E \cup \{(d_j, x_i) \mid g_{i,j} \neq 0\}$. Here $D = \{d_1, d_2, \dots, d_l\}$, called the set of disturbance vertices; and V and E are the vertex set and the edge set, respectively, of the graph $G(V, E)$ associated to the system (1.1). The graph $G_{\lambda'}(V', E')$ is obtained from $G(V', E')$ by weighting each edge in E' with the appropriate component of $\lambda' \in \mathbb{R}^{k'}$.

Following Emre and Hautus [5], we say that for a given $\lambda' \in \mathbb{R}^{k'}$ the *modified disturbance decoupling problem* for the system (8.1) is solvable if there is a real $m \times n$ matrix F and a real $m \times l$ matrix H, representing a feedback law $u(t) = Fx(t) + Hd(t)$, such that $C_{\lambda'}(sI - [A_{\lambda'} + B_{\lambda'}F])^{-1}(G_{\lambda'} + B_{\lambda'}H) = 0$ (see also Wonham [24, Exercise 4.10]). Using the results of Emre and Hautus [5], it can be shown that for a given $\lambda' \in \mathbb{R}^{k'}$ the modified disturbance-decoupling problem for the system (8.1) is solvable if and only if there exists a *proper* rational $m \times l$ matrix $X(s)$ such that $K_{\lambda'}(s)X(s) = L_{\lambda'}(s)$. Here we have denoted $K_{\lambda'}(s) = C_{\lambda'}(sI - A_{\lambda'})^{-1}B_{\lambda'}$ and $L_{\lambda'}(s) = C_{\lambda'}(sI - A_{\lambda'})^{-1}G_{\lambda'}$.

From Theorem 3.3 it is now immediate that for a given $\lambda' \in \mathbb{R}^{k'}$ the modified disturbance-decoupling problem for the system (8.1) is solvable if and only if the rank and the sum of the orders of the zeros at infinity of $K_{\lambda'}(s)$ and $[K_{\lambda'}(s) \; L_{\lambda'}(s)]$ are equal.

In the spirit of the present paper, we say that the modified disturbance-decoupling problem for the structured system (8.1) is *generically solvable* if the set of parameter values $\lambda' \in \mathbb{R}^{k'}$ for which the rank and/or the sum of the orders of the zeros at infinity of $K_{\lambda'}(s)$ and of $[K_{\lambda'}(s) \; L_{\lambda'}(s)]$ are *not* equal is contained in a proper variety in $\mathbb{R}^{k'}$. The following theorem is now an immediate consequence of Theorems 6.1 and 6.3.

THEOREM 8.1. *The modified disturbance-decoupling problem for the structured system* (8.1) *is generically solvable if and only if*

(a) *the largest number of disjoint paths in $G(V, E)$ from U to Y is equal to the largest number of disjoint paths in $G(V', E')$ from $U \cup D$ to Y, say r, and*

(b) *the smallest number of state vertices (vertices in the set X) in any r-tuple of disjoint paths in $G(V, E)$ from U to Y is equal to the smallest number of state vertices in any r-tuple of disjoint paths in $G(V', E')$ from $U \cup D$ to Y.*

We continue with the disturbance-decoupling problem. Following Wonham [24], we say that for a given $\lambda' \in \mathbb{R}^{k'}$ the disturbance-decoupling problem for the system (8.1) is solvable if there is a real $m \times n$ matrix F, representing a feedback law $u(t) = Fx(t)$, such that $C_{\lambda'}(sI - [A_{\lambda'} + B_{\lambda'}F])^{-1}G_{\lambda'} = 0$.

By the results of Hautus [9], it follows that the disturbance-decoupling problem for the system (8.1) for a given $\lambda' \in \mathbb{R}^{k'}$ is solvable if and only if there exists a *strictly proper* rational $m \times l$ matrix $X(s)$ such that $K_{\lambda'}(s)X(s) = L_{\lambda'}(s)$. Clearly, the latter is equivalent to the existence of a *proper* rational matrix $X'(s)$ such that $K_{\lambda'}(s)\Delta s^{-1}X'(s) = L_{\lambda'}(s)$, where Δ is an arbitrary constant nonsingular diagonal matrix. Therefore, to derive conditions for the solvability of a structural version of the disturbance-decoupling problem, it turns out to be useful to extend the structured system (8.1) as follows:

$$\dot{u}(t) = Nw(t), \tag{8.2}$$

where N denotes a square structured matrix with only structural unknowns on the diagonal. The compound system made up of (8.1) and (8.2) is again a structured system and is described by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ N \end{bmatrix} w(t) + \begin{bmatrix} G \\ 0 \end{bmatrix} d(t), \tag{8.3a}$$

$$y(t) = Cx(t). \tag{8.3b}$$

Note that the control input $u(t)$ of the system (8.1) is part of the state of the system (8.3), and that $w(t)$ is the control input of (8.3). We denote the total number of structural unknowns in A, B, C, G, and N by k'', i.e., $k'' = k' + n$. Parametrizing the structural unknowns and collecting all the parameters in $\lambda'' \in \mathbb{R}^{k''}$, we denote by $A_{\lambda''}$, $B_{\lambda''}$, $C_{\lambda''}$, $G_{\lambda''}$, and $N_{\lambda''}$ the nominal values of A, B, C, G, and N for a given $\lambda'' \in \mathbb{R}^{k''}$. Furthermore, as before, we denote $K_{\lambda''}(s) = C_{\lambda''}(sI - A_{\lambda''})^{-1}B_{\lambda''}$ and $L_{\lambda''}(s) = C_{\lambda''}(sI - A_{\lambda''})^{-1}G_{\lambda''}$.

It is now easy to see that there is a *strictly proper* rational matrix $X(s)$ such that $K_{\lambda''}(s)X(s) = L_{\lambda''}(s)$ if and only if there is a *proper* rational matrix $X'(s)$ such that $K_{\lambda''}(s)N_{\lambda''}s^{-1}X'(s) = L_{\lambda''}(s)$, for all $\lambda'' \in \mathbb{R}^{k''}$ for which $N_{\lambda''}$ is nonsingular. Furthermore, it is easy to see that $K_{\lambda''}(s)N_{\lambda''}s^{-1}$ and $L_{\lambda''}(s)$ are the transfer matrices of the system (8.3) from the control input to the output and from the disturbance input to the output, respectively, at $\lambda'' \in \mathbb{R}^{k''}$. Moreover, note that the system (8.3) is of the same type as the system (8.1), for which we have formulated a structural version of the modified disturbance-decoupling problem.

Based on the above observations, we say that the disturbance-decoupling problem for the structured system (8.1) is generically solvable if the modified disturbance-decoupling problem for the structured system (8.3) is generically solvable.

We can now apply Theorem 8.1 to the system (8.3) to obtain a graph-theoretic characterization for the generic solvability of the disturbance-decoupling problem for the structured system (8.1) in terms of the graph of the system (8.3). To do this, we have to modify the graphs $G(V', E')$ and $G_\lambda(V', E')$ to make them correspond to the structured system (8.3). For instance, we have to add the set W of "new" input vertices. However, it is easy to see that the characterization obtained in this way is equivalent to the following characterization, which is entirely in terms of the graphs $G(V, E)$ and $G(V', E')$.

THEOREM 8.2. *The disturbance-decoupling problem for the structured system* (8.1) *is generically solvable if and only if*

(a) *the largest number of disjoint paths in $G(V, E)$ from U to Y is equal to the largest number of disjoint paths in $G(V', E')$ from $U \cup D$ to Y, say r, and*

(b) *the smallest number of vertices in $X \cup U$ in any r-tuple of disjoint paths in $G(V, E)$ from U to Y is equal to the smallest number of vertices in $X \cup U$ in any r-tuple of disjoint paths in $G(V', E')$ from $U \cup D$ to Y.*

Note that for the generic solvability of the modified disturbance-decoupling problem only the number of state vertices in a tuple of disjoint paths is relevant, while for the generic solvability of the disturbance-decoupling problem the number of both state and input vertices in a tuple of disjoint paths is relevant.

9. REMARKS AND CONCLUSIONS

In this paper we have studied a general type of structured systems. We introduced the structured systems in Section 2 as systems of which only the zero-nonzero structure is given, where we assumed the nonzeros, called structural unknowns, to be entries of the system matrices of which the values are unknown and independent of each other. Also in Section 2 we described how the structured systems can be parametrized by a parameter $\lambda \in \mathbb{R}^k$, where k denoted the number of structural unknowns.

In Section 4 we used the parametrization to introduce the *generic rank* and *generic orders of the zeros at infinity* of the transfer matrix of a structured system. For the introduction of these notions, we assumed that the parameter space was $\mathbb{R}^k$. However, it is easy to see that we could have restricted ourselves to parameter spaces that are *open nonempty subsets* in $\mathbb{R}^k$. Such parameter spaces are sometimes more realistic, because in practical situations there may be components of the parameter vector λ that only can have values in an (open) subset of $\mathbb{R}$. Then, using the techniques of Section 4, we can show that if the overall parameter space is an open nonempty subset in $\mathbb{R}^k$, the main results of this paper are still valid.

The main results of this paper, presented in Section 6, relate the generic rank and the generic orders of the zeros at infinity of the transfer matrix of a structured system (1.1) to properties of the corresponding graph $G(V, E)$. This graph was introduced in Section 5. We showed that the generic rank of the transfer matrix of structured system can be determined by calculating the largest number of disjoint paths in $G(V, E)$ from the set of input vertices U to the set of output vertices Y. The generic orders of the zeros at infinity of the transfer matrix can be determined by calculating the smallest number of state vertices in any i-tuple of disjoint paths in $G(V, E)$ from U to Y, for $i = 1, 2, \dots, r$, where r is the generic rank of the transfer matrix of the structured system. For simple systems these numbers can be determined by hand; for complicated systems we indicated in Section 7 that these numbers can be determined by means of *max-flow min-cut* and *minimal-cost flow* algorithms (cf. Lawler [11]).

As an application of our results we proposed structural versions of the well-known disturbance-decoupling problem and the so-called modified disturbance-decoupling problem for a structured system of the type (8.1) (cf. Wonham [24], Emre and Hautus [5]), and we derived necessary and sufficient conditions for the structural solvability of the problems in terms of the graphs $G(V, E)$ and $G(V', E')$. Results concerning the solvability of a structural version of dual problems like, for instance, the *disturbance decoupled estimation problem* (cf. Schumacher [18]) can be obtained in a similar way.

In van der Woude [25], we derived conditions for the solvability of a structural version of the *almost disturbance-decoupling problem* (cf. Willems [22]). We also indicated there that with the obtained results, conditions for the solvability of structural versions of the *almost disturbance decoupled estimation problem* and the *almost disturbance-decoupling problem with measurement feedback* (cf. Willems [23]) can be derived in a straightforward way.

Conditions concerning the solvability of structural versions of the *disturbance-decoupling problem with measurement feedback* (cf. Akashi and Imai [1], Schumacher [18]) and the *disturbance-decoupling problem with pole assignment* (cf. Wonham [24]) are topics of future investigation. Clearly, in view of the unstructured case, in the latter problem the notion of structural controllability will play an important role (cf. Glover and Silverman [7], Lin [13], Murota [15], and Shields and Pearson [19]). Also a topic of future investigation is how the results of the present paper can be extended to descriptor or singular systems (cf. Lewis [12]). Finally, another important matter for future investigation is, once the generic solvability of a control problem has been established, is it possible to actually solve the control problem by a structured control law, and how can it be determined (cf. Linneman [14])?

After having submitted this paper for publication, we became aware of a recent paper by Suda, Wan, and Ueno [20] and one by Commault, Dion, and Perez [3]. Both papers contain results similar to the results in the present paper which were obtained simultaneously and independently of ours.

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Received 25 October 1989; final manuscript accepted 20 February 1990