Balanced star decompositions of regular multigraphs and $\lambda$-fold complete bipartite graphs

Hung-Chih Lee$^a$, Chiang Lin$^{b,\ast}$

$^a$Department of Information Management, Ling Tung College, Taichung, Taiwan 408, ROC
$^b$Department of Mathematics, National Central University, Chung-Li, Taiwan 320, ROC

Received 20 November 2003; received in revised form 9 March 2005; accepted 5 April 2005
Available online 2 September 2005

Abstract

Let $S_k$ denote the star with $k$ edges. A balanced $S_k$-decomposition of a multigraph $G$ is a family $\mathcal{D}$ of subgraphs of $G$ whose edge sets form a partition of the edge set of $G$ such that each member of $\mathcal{D}$ is isomorphic to $S_k$, and every vertex of $G$ belongs to the same number of members in $\mathcal{D}$. In this paper, we obtain the following results:

(1) A necessary and sufficient condition for an $r$-regular multigraph to have a balanced $S_k$-decomposition.
(2) A necessary and sufficient condition for the $\lambda$-fold complete bipartite graph $\lambda K_{m,n}$ to have a balanced $S_k$-decomposition.

© 2005 Elsevier B.V. All rights reserved.

Keywords: Balanced star decompositions; Regular multigraphs; $\lambda$-fold complete bipartite graphs

1. Introduction and preliminaries

For an integer $k \geq 1$, let $S_k$ denote the star with $k$ edges. For $k \geq 2$, the vertex of degree $k$ in $S_k$ is called the center of $S_k$ and any vertex of degree 1 is called an endvertex of $S_k$. Let $G$ be a multigraph. A decomposition of $G$ is a family of subgraphs of $G$ whose edge sets form a partition of the edge set of $G$. For $k \geq 1$, an $S_k$-decomposition of $G$ is a decomposition...
of $G$ of which each member is isomorphic to $S_k$. An $S_k$-decomposition of $G$ is \textit{balanced} if every vertex of $G$ belongs to the same number of members in the decomposition. For $k \geq 2$, an $S_k$-decomposition of $G$ is \textit{center balanced} if every vertex of $G$ is the center of the same number of members in the decomposition.

Suppose that $G$ is a multigraph. Let $x$ and $y$ be distinct vertices of $G$. We use $\deg_G(x)$ to denote the number of edges incident with $x$, and $e_G(x, y)$ to denote the number of edges joining $x$ and $y$. When the multigraph $G$ is clear from the context, $\deg x$ is used instead of $\deg_G(x)$. If $\deg_G(x)$ is a constant for all vertices $x$ in $G$, then $G$ is a \textit{regular multigraph}; furthermore, if the constant is $r$, we say that $G$ is $r$-\textit{regular}.

For a graph $G$ and a positive integer $\lambda$, we use $\lambda G$ to denote the multigraph obtained from $G$ by replacing each edge $e$ of $G$ by $\lambda$ edges with the same ends as $e$. In this paper, $\lambda$ is always a positive integer. Let $K_m$ denote the complete graph on $m$ vertices, let $K_m(n)$ denote the complete $m$-partite graph of which each part has $n$ vertices, and let $K_{m,n}$ denote the complete bipartite graph with parts of cardinalities $m$ and $n$. Obviously $\lambda K_m(n)$ is a regular multigraph, and $\lambda K_m(1) = K_m$. A necessary and sufficient condition for $\lambda K_m(n)$ to have a balanced $S_k$-decomposition was obtained by Huang [4]. A more general result for $\lambda K_m(n)$ was obtained by Ushio [9]. To generalize these results (Corollaries 2.4, 2.5 in this paper), we consider in Section 2 the balanced $S_k$-decomposition of regular multigraphs and have Theorem 2.2 as the main result. In Section 3 we consider the balanced $S_k$-decomposition of $\lambda K_{m,n}$. It is easy to see that a multigraph has a balanced $S_1$-decomposition if and only if it is regular. Thus every regular multigraph has a balanced $S_1$-decomposition, and $\lambda K_{m,n}$ has a balanced $S_1$-decomposition if and only if $m = n$. We consider the balanced $S_k$-decomposition for $k \geq 2$. Hereafter we let $k$ be an integer $\geq 2$ unless otherwise stated.

Suppose that $G$ is a multigraph, $\mathcal{D}$ is an $S_k$-decomposition of $G$, and $x$ is any vertex of $G$. In what follows, let $u_{\mathcal{D}}(x)$ be the number of $S_k$’s in $\mathcal{D}$ of which $x$ is the center, $v_{\mathcal{D}}(x)$ be the number of $S_k$’s in $\mathcal{D}$ of which $x$ is an endvertex and $t_{\mathcal{D}}(x)$ be the number of $S_k$’s in $\mathcal{D}$ of which $x$ is a vertex.

By the definitions, we have

\begin{equation}
    u_{\mathcal{D}}(x) + v_{\mathcal{D}}(x) = t_{\mathcal{D}}(x). \tag{1}
\end{equation}

Counting the edges incident with $x$, we have

\begin{equation}
    u_{\mathcal{D}}(x)k + v_{\mathcal{D}}(x) = \deg_G(x). \tag{2}
\end{equation}

From (1) and (2), we obtain

\begin{equation}
    u_{\mathcal{D}}(x) = \frac{\deg_G(x) - t_{\mathcal{D}}(x)}{k - 1}, \tag{3}
\end{equation}

equivalently,

\begin{equation}
    t_{\mathcal{D}}(x) = \deg_G(x) - (k - 1)u_{\mathcal{D}}(x). \tag{4}
\end{equation}

Also, from the definitions, $\mathcal{D}$ is center balanced if and only if $u_{\mathcal{D}}(x)$ is a constant for all $x \in V(G)$. And $\mathcal{D}$ is balanced if and only if $t_{\mathcal{D}}(x)$ is a constant for all $x \in V(G)$. 
We need some more notations and results for our discussions. Suppose that \( G \) is a multidigraph. Let \( x \) and \( y \) be distinct vertices of \( G \). We use \( \deg^+_G(x) \) to denote the number of edges oriented from \( x \), \( \deg^-_G(x) \) the number of edges oriented to \( x \), and \( e^+_G(x, y) \) the number of edges oriented from \( x \) to \( y \). When the multidigraph \( G \) is clear from the context, \( \deg^+_x \), \( \deg^-_x \), and \( e^+_G(x, y) \) are used instead of \( \deg^+_G(x) \), \( \deg^-_G(x) \), and \( e^+_G(x, y) \), respectively.

A multistar is a star with multiple edges allowed.

**Proposition 1.1** ([Lin et al. [5, Proposition 1.3]]). Suppose that \( H \) is a multistar. Then \( H \) has an \( S_k \)-decomposition if and only if there exists a nonnegative integer \( l \) such that \( |E(H)| = l|k \) and \( e_H(w, x) \leq l \) where \( w \) is the center of \( H \) and \( x \) is any endvertex.

The following corollary follows immediately from Proposition 1.1.

**Corollary 1.2.** Let \( G \) be a multigraph with vertex set \( \{x_1, x_2, \ldots, x_n\} \) and \( l_1, l_2, \ldots, l_n \) be nonnegative integers. Then \( G \) has an \( S_k \)-decomposition such that in this decomposition there are \( l_i \) stars with centers at \( x_i \) for \( i = 1, 2, \ldots, n \) if and only if there exists an orientation \( \vec{G} \) of \( G \) such that for \( 1 \leq i, j \leq n \) with \( i \neq j \), \( \deg^+_{\vec{G}}(x_i) = l_i k \) and \( e^+_{\vec{G}}(x_i, x_j) \leq l_i \).

Letting \( l_1 = l_2 = \cdots = l_n \) in Corollary 1.2, we obtain a result on center balanced \( S_k \)-decompositions as follows:

**Corollary 1.3.** Let \( G \) be a multigraph. Then \( G \) has a center balanced \( S_k \)-decomposition if and only if there is an orientation \( \vec{G} \) of \( G \) such that for some integer \( l \), \( \deg^+_G(x) = l|k \) for every vertex \( x \) and \( e^+_G(x, y) \leq l \) for every pair of vertices \( x, y \).

2. Balanced \( S_k \)-decomposition of regular multigraphs

This section is devoted to prove a necessary and sufficient condition for regular multigraphs to have balanced \( S_k \)-decompositions. We begin with the following lemma:

**Lemma 2.1.** Let \( \mathcal{D} \) be an \( S_k \)-decomposition of a regular multigraph \( G \). Then \( \mathcal{D} \) is balanced if and only if \( \mathcal{D} \) is center balanced.

**Proof.** Let \( G \) be an \( r \)-regular multigraph and let \( \mathcal{D} \) be an \( S_k \)-decomposition of \( G \).

**Necessity:** Suppose that \( \mathcal{D} \) is balanced. Let \( r \) be the number of \( S_k \)'s in \( \mathcal{D} \) to which each vertex of \( G \) belongs. Let \( x \in V(G) \). By (3), we have
\[
u_\mathcal{D}(x) = (\deg_G(x) - t_\mathcal{D}(x))/(k - 1) = (r - t)/(k - 1),
\] which is a constant. Hence \( \mathcal{D} \) is center balanced.

**Sufficiency:** Suppose that \( \mathcal{D} \) is center balanced. Let \( u \) be the number of \( S_k \)'s in \( \mathcal{D} \) of which each vertex of \( G \) is the center. Let \( x \in V(G) \). By (4), \( t_\mathcal{D}(x) = \deg_G(x) - (k - 1)\nu_\mathcal{D}(x) = r - (k - 1)u \), which is a constant. Hence \( \mathcal{D} \) is balanced. \( \square \)

Now we prove the main result of this section.
Theorem 2.2. Let $G$ be an $r$-regular multigraph ($r \geq 1$). Then the following conditions are equivalent.

(A) $G$ has a balanced $S_k$-decomposition.
(B) $G$ has a center balanced $S_k$-decomposition.
(C) $r \equiv 0 \pmod {2k}$ and $e_G(x, y) \leq r/k$ for all $x, y \in V(G)$ with $x \neq y$.

Proof. By Lemma 2.1, (A) and (B) are equivalent. We now show that (B) $\iff$ (C).

(B) $\implies$ (C): Suppose that $|V(G)| = n$. Then $|E(G)| = nr/2$. Let $\mathcal{D}$ be a center balanced $S_k$-decomposition of $G$, and let $u$ be the number of $S_k$'s in $\mathcal{D}$ of which each vertex of $G$ is the center. Then $|\mathcal{D}| = |E(G)|/k = nr/2k$, and hence $u = |\mathcal{D}| / n = r/2k$. Thus $r \equiv 0 \pmod {2k}$.

For all $x, y \in V(G)$ with $x \neq y$, each edge joining $x$ and $y$ belongs to an $S_k$ in $\mathcal{D}$ with center at either $x$ or $y$. Thus $e_G(x, y) \leq 2u = r/k$.

(C) $\implies$ (B): By the assumption, $r = 2kl$ for some positive integer $l$, and $e_G(x, y) \leq r/k = 2l$ for all $x, y \in V(G)$ with $x \neq y$. To show the existence of a center balanced $S_k$-decomposition of $G$, by the sufficiency of Corollary 1.3, it suffices to show that there exists an orientation $\tilde{G}$ of $G$ such that $\deg_{\tilde{G}}^+(x) = kl$ for every vertex $x$ and $\deg_{\tilde{G}}^+(x, y) \leq l$ for every pair of vertices $x$ and $y$. Now we construct such $\tilde{G}$.

Let $G_1$ be a spanning subgraph of $G$ such that for every pair of vertices $x, y \in V(G)$ there are $e_G(x, y)$ edges joining $x$ and $y$ if $e_G(x, y)$ is even, and there are $e_G(x, y) - 1$ edges joining $x$ and $y$ if $e_G(x, y)$ is odd. Let $G_2 = G - E(G_1)$. Note that there are even number of edges joining every pair of vertices in $G_1$, and that $G_2$ is a simple graph. For every $x \in V(G_2)$,

$$\deg_{G_2}(x) = \deg_G(x) - \deg_{G_1}(x) = r - \sum_{y \neq x} e_{G_1}(x, y).$$

Since $r = 2kl$ is even and each $e_{G_1}(x, y)$ is even, we see that $\deg_{G_2}(x)$ is even. By a well known theorem of Euler, each nontrivial component of $G_2$ has an Euler tour. (An Euler tour in a graph is a closed walk which traverses each edge of the graph exactly once. See, for example [1, pp. 51–52] for this result.) Now we orient the edges of $G$. Begin with those of $G_2$. The edges in each nontrivial component of $G_2$ are oriented as follows. Suppose that $x_1, x_2, x_3, \ldots, x_p, x_1$ is an Euler tour of a nontrivial component of $G_2$. Then, for $i = 1, 2, \ldots, p$, we orient the edge $x_i x_{i+1}$ from $x_i$ to $x_{i+1}$, where $x_{p+1} = x_1$. Thus, for each $x \in V(G_2)$ there are the same number of edges in $G_2$ oriented from $x$ as those oriented to $x$. As to the edges in $G_1$, among the edges joining every pair of vertices $x$ and $y$, half of them are oriented from $x$ to $y$, and the other half from $y$ to $x$. Let $\tilde{G}$ be the digraph thus obtained from $G$.

We have, for each $x \in V(\tilde{G})$

$$\deg_{\tilde{G}}^+(x) = \deg_{G_1}(x)/2 + \deg_{G_2}(x)/2 = \deg_G(x)/2 = r/2 = kl.$$

And for every $x, y \in V(\tilde{G})$ with $x \neq y$, 

$$e_{\tilde{G}}(x, y) \leq l.$$
if $e_G(x, y)$ is even, then $e_G^+(x, y) = e_G(x, y)/2 \leq l$; 
if $e_G(x, y)$ is odd, then $e_G(x, y) \leq 2l - 1$, and hence

$$e_G^+(x, y) \leq e_G(x, y)/2 + e_G(x, y) = (e_G(x, y) - 1)/2 + 1 \leq (2l - 2)/2 + 1 = l.$$ 

This completes the proof. \[QED\]

An immediate corollary follows from Theorem 2.2.

**Corollary 2.3.** Let $G$ be a $d$-regular simple graph ($d \geq 1$). Then $\lambda G$ has a balanced $S_k$-decomposition if and only if $\lambda d \equiv 0 \pmod{2k}$ and $k \leq d$.

We now apply Corollary 2.3 to the following regular graphs: complete graphs, balanced complete $m$-partite graphs, cubes, circulant graphs and crowns. The definitions of balanced complete $m$-partite graphs, cubes, circulant graphs and crowns will be given below. First since the complete graph $K_m$ is $(m - 1)$-regular, we have:

**Corollary 2.4** (Bosák [2, p. 107], Huang [4], Ushio [9]). $\lambda K_m$ has a balanced $S_k$-decomposition if and only if $\lambda (m - 1) \equiv 0 \pmod{2k}$ and $k \leq m - 1$.

Recall that $K_m(n)$ is the complete $m$-partite graph of which each part has $n$ vertices. This graph is called a balanced complete $m$-partite graph. Obviously $K_m(n)$ is $(m - 1)n$-regular.

**Corollary 2.5** (Bosák [2, p. 105], Ushio [9]). $\lambda K_m(n)$ has a balanced $S_k$-decomposition if and only if $\lambda (m - 1)n \equiv 0 \pmod{2k}$ and $k \leq (m - 1)n$.

For a positive integer $n$, let $Q_n$ denote the $n$-dimensional cube, i.e., $Q_n$ is the graph with vertex set $\{(a_1, a_2, \ldots, a_n) : a_i = 0$ or $1, \ i = 1, 2, \ldots, n\}$ such that two vertices are adjacent if and only if they differ in exactly one component. Obviously $Q_n$ is $n$-regular.

**Corollary 2.6.** $\lambda Q_n$ has a balanced $S_k$-decomposition if and only if $\lambda n \equiv 0 \pmod{2k}$ and $k \leq n$.

For an integer $n \geq 3$, let $n_1, n_2, \ldots, n_p$ be positive integers with $n_1 < n_2 < \cdots < n_p \leq n/2$. Then the circulant graph $C_n(n_1, n_2, \ldots, n_p)$ is the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_j : i, j \equiv i + n_t \pmod{n}, 1 \leq t \leq p\}$. It is obvious that when $n$ is even and $n_p = n/2$, $C_n(n_1, n_2, \ldots, n_p)$ is $(2p - 1)$-regular; otherwise it is $2p$-regular.

**Corollary 2.7.** $\lambda C_n(n_1, n_2, \ldots, n_p)$ has a balanced $S_k$-decomposition if and only if

$$\begin{cases} 
\lambda (2p - 1) \equiv 0 \pmod{2k}, & k \leq 2p - 1 \text{ when } n \text{ is even and } n_p = n/2, \\
\lambda p \equiv 0 \pmod{k}, & k \leq 2p \text{ otherwise}. 
\end{cases}$$
For a positive integer \( n \), let \( n_1, n_2, \ldots, n_p \) be positive integers with \( n_1 < n_2 < \cdots < n_p \leq n \). Then the crown \( CR_n(n_1, n_2, \ldots, n_p) \) is the graph with vertex set \( \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\} \) and edge set \( \{a_ib_j : i = 1, 2, \ldots, n, j \equiv i + n_t (\text{mod } n), 1 \leq t \leq p\} \). It is easy to see that \( CR_n(n_1, n_2, \ldots, n_p) \) is \( p \)-regular.

**Corollary 2.8.** \( \lambda CR_n(n_1, n_2, \ldots, n_p) \) has a balanced \( S_k \)-decomposition if and only if \( \lambda p \equiv 0 \pmod{2^k} \) and \( k \leq p \).

**Remark.** In the following we list the known results about \( S_k \)-decompositions (not necessarily balanced) of the above regular graphs. Here \( k \geq 1 \).

**Proposition 2.9** (Bosák [2, p. 104], Tarsi [6]). Let \( m \geq 2 \) be an integer. Then \( \lambda K_m \) has an \( S_k \)-decomposition if and only if
\[
\lambda m(m - 1) \equiv 0 \pmod{2k}
\]
and
\[
m \geq \begin{cases} 
2k & \lambda = 1, \\
k + 1 & \lambda \text{ is even}, \\
k + 1 + k/\lambda & \lambda \geq 3 \text{ is odd}.
\end{cases}
\]

**Proposition 2.10** (Bosák [2, p. 105], Ushio et al. [10]). Let \( m \geq 2 \) be an integer. Then \( K_m(n) \) has an \( S_k \)-decomposition if and only if \( m(m - 1)n^2 \equiv 0 \pmod{2k} \) and \( mn \geq 2k \).

The following result about \( S_k \)-decompositions of \( n \)-dimensional cubes was obtained by Bryant et al. [3]. Their proof used the existence of Hamming codes of length \( 2^m - 1 \) and constructions based on properties of finite vector space.

**Proposition 2.11** (Bryant et al. [3, Theorem 1]). The \( n \)-dimensional cube \( Q_n \) has an \( S_k \)-decomposition if and only if \( n^2 \equiv 0 \pmod{k} \) and \( k \leq n \).

Investigations of \( S_k \)-decompositions of circulant graphs and crowns appeared in [5].

**Proposition 2.12** (Lin et al. [5, Theorem 3.4]). Let \( p, n \) be positive integers with \( n \geq 3, p < n/2 \). The circulant graph \( C_n(1, 2, 3, \ldots, p) \) has an \( S_k \)-decomposition if and only if \( np \equiv 0 \pmod{k} \) and \( k \leq p + 1 \).

**Proposition 2.13** (Lin et al. [5, Corollary 3.2]). Let \( \lambda, p, n \) be positive integers with \( n \geq 3, p < n/2 \). The circulant multigraph \( \lambda C_n(1, 2, 3, \ldots, p) \) has an \( S_k \)-decomposition if \( \lambda np \equiv 0 \pmod{k} \) and \( k \leq p \).

**Proposition 2.14** (Lin et al. [5, Theorem 2.2]). Let \( p, n \) be positive integers with \( p \leq n \). The multicrown \( \lambda CR_n(1, 2, 3, \ldots, p) \) has an \( S_k \)-decomposition if and only if \( \lambda np \equiv 0 \pmod{k} \) and \( k \leq p \).

Further investigations of \( S_k \)-decompositions of \( \lambda K_m(n), \lambda Q_n, \lambda C_n(1, 2, \ldots, p), C_n(n_1, n_2, \ldots, n_p) \), and \( CR_n(n_1, n_2, \ldots, n_p) \) are worthwhile.
3. Balanced $S_k$-decomposition and center balanced $S_k$-decomposition of $\lambda$-fold complete bipartite graphs

The multigraph $\lambda K_{m,n}$ is called a $\lambda$-fold complete bipartite graph. In this section, we consider the balanced $S_k$-decompositions and the center balanced $S_k$-decompositions of $\lambda K_{m,n}$. In the following discussions, let $(M, N)$ be the bipartition of $\lambda K_{m,n}$ with $|M| = m$ and $|N| = n$. Let us begin with a lemma about $S_k$-decomposition of $\lambda K_{m,n}$.

**Lemma 3.1.** Let $m \geq n \geq 1$, $m \geq k \geq 2$ be integers. Suppose that $u_1$, $u_2$ are nonnegative integers and $\lambda$ is a positive integer such that

$$mu_1k + nu_2k = \lambda mn.$$  \hfill (5)

And in addition to (5),

$$u_1 = 0 \quad \text{if } k > n.$$  \hfill (6)

Then there exists an $S_k$-decomposition $\mathcal{D}$ of $\lambda K_{m,n}$ such that for each $x \in M$, there are $u_1 S_k$’s in $\mathcal{D}$ with centers at $x$, and for each $y \in N$, there are $u_2 S_k$’s in $\mathcal{D}$ with centers at $y$.

**Proof.** Let $M = \{x_0, x_1, \ldots, x_{m-1}\}$, $N = \{y_0, y_1, \ldots, y_{n-1}\}$. By the sufficiency of Corollary 1.2, it suffices to show that there exists an orientation of $\lambda K_{m,n}$ such that, with this orientation, for $x \in M$, $y \in N$

$$\deg^+ x = u_1k,$$  \hfill (7)

$$e^+(x, y) \leq u_1,$$  \hfill (8)

$$\deg^+ y = u_2k,$$  \hfill (9)

$$e^+(y, x) \leq u_2.$$  \hfill (10)

For this end, we orient the edges of $\lambda K_{m,n}$. First consider the case $u_2 = 0$. We have, by (5), $u_1k = \lambda n$ and hence $u_1 > 0$, which implies, by (6), $k \leq n$; in turn, it implies $u_1 \geq \lambda$. We orient all edges in $\lambda K_{m,n}$ from $M$ to $N$. Then for $x \in M$, $y \in N$, we have $\deg^+ x = \lambda n = u_1k$, $e^+(x, y) = \lambda \leq u_1$, $\deg^+ y = 0 = u_2k$, and $e^+(y, x) = 0 \leq u_2$. Thus (7)–(10) hold.

Now consider the case $u_2 > 0$. First, for $j = 0, 1, 2, \ldots, n-1$, the edges $y_{j}x_{j+u_2k}$, $y_{j}x_{j+u_2k+1}$, $y_{j}x_{j+u_2k+2}$, $\ldots$, $y_{j}x_{j+(u_2k-1)}$ are all oriented outward from $y_j$ where the subscripts of $x$ are taken modulo $m$. Note that from each $y_j$, we orient $u_2k$ edges. Since the degree of each $y_j$ in $\lambda K_{m,n}$ is $\lambda m$ and by (5), $u_2k \leq \lambda m$, these assure us that there are enough edges for the above orientation. Note also that if $u_2k > m$, oriented edges with multiplicity greater than one occur. The edges which are not oriented yet are all oriented from $M$ to $N$.

We check that the orientation satisfies (7)–(10). From the construction of the orientation, it is easy to see that for all $y_j \in N$, and all $x_j, x_{j'} \in M$, we have

$$\deg^+ y_j = u_2k,$$  \hfill (11)

$$|e^+(y_j, x_j) - e^+(y_j, x_{j'})| \leq 1,$$  \hfill (12)

$$|\deg^- x_j - \deg^- x_{j'}| \leq 1.$$  \hfill (13)

As (11) is the same as (9), we only need to check (7), (8), and (10).
Since $\deg^+ y_j = \sum_{i=0}^{m-1} e^+(y_j, x_i)$, it follows from (11), (12) that for $0 \leq i \leq m - 1$
\[
\left\lfloor \frac{u_2 k}{m} \right\rfloor \leq e^+(y_j, x_i) \leq \left\lceil \frac{u_2 k}{m} \right\rceil.
\]
(14)
Thus $e^+(y_j, x_i) \leq u_2$ since $k \leq m$. This proves (10).

Since $\deg^+ x_i + \deg^- x_i = \lambda n$ for $x_i \in M$, it follows from (13) that $|\deg^+ x_i - \deg^+ x_i'| \leq 1$ for $x_i, x_i' \in M$. Furthermore
\[
\sum_{i=0}^{m-1} \deg^+ x_i = |E(\lambda K_{m,n})| - \sum_{j=0}^{n-1} \deg^+ y_j
= \lambda mn - nu_2 k
= mu_1 k \quad \text{(by (5))}.
\]
Thus $\deg^+ x_i = u_1 k$ for $x_i \in M$. This proves (7).

Lastly we prove (8). We have
\[
e^+(x_i, y_j) = \lambda - e^+(y_j, x_i)
\leq \lambda - \left\lfloor \frac{u_2 k}{m} \right\rfloor \quad \text{(by the left inequality of (14))}
\leq \lambda - \frac{u_2 k}{m}
\leq \frac{u_1 k}{n} \quad \text{(by (5))}
\leq u_1 \quad \text{(by (6))}.
\]
This completes the proof. \qed

We now consider balanced $S_k$-decomposition of $\lambda K_{m,n}$. Recall that if $\mathcal{D}$ is an $S_k$-decomposition of a multigraph $G$, then for each vertex $x$ of $G$, $u_{\mathcal{D}}(x)$ denotes the number of $S_k$’s in $\mathcal{D}$ of which $x$ is the center, and $t_{\mathcal{D}}(x)$ denotes the number of $S_k$’s in $\mathcal{D}$ of which $x$ is a vertex.

**Lemma 3.2.** Let $\mathcal{D}$ be an $S_k$-decomposition of $\lambda K_{m,n}$. Then $\mathcal{D}$ is balanced if and only if
\[
u_{\mathcal{D}}(x) =\begin{cases} \frac{\lambda n (nk - m)}{k(k-1)(m+n)} & \text{if } x \in M, \\ \frac{\lambda m (mk - n)}{k(k-1)(m+n)} & \text{if } x \in N. \end{cases}
\]

**Proof.** Necessity: Since $\mathcal{D}$ is an $S_k$-decomposition of $\lambda K_{m,n}$, we have
$|E(\lambda K_{m,n})| = |\mathcal{D}| k$. Hence $|\mathcal{D}| = \lambda mn / k$. Now $\mathcal{D}$ is balanced. Suppose that each vertex of $\lambda K_{m,n}$ belongs to $t$ $S_k$’s in $\mathcal{D}$. Let
\[
F = \{(x, S) : x \in M \cup N, S \in \mathcal{D}, x \text{ is a vertex of } S\}.
\]
Count the members in $F$. Since $|M \cup N| = m + n$ and each vertex in $M \cup N$ belongs to $t$ $S_k$'s in $D$, we have $|F| = (m + n)t$. On the other hand, since there are $|D|$ $S_k$'s in $D$, and each $S_k$ contains $k + 1$ vertices, we have $|F| = |D|(k + 1)$. Thus $(m + n)t = |D|(k + 1)$, which implies $t = \frac{|D|(k + 1)}{m + n}$.

Hence, by (3), we have

$$t = \frac{|D|(k + 1)}{m + n} = \frac{\lambda mn(k + 1)}{k(m + n)}.$$  

for each $x \in M$,

$$u(x) = \frac{\deg x - t(x)}{k - 1} = \frac{\lambda n - t}{k - 1} = \frac{\lambda n(nk - m)}{k(k - 1)(m + n)}.$$  

Similarly, we have

$$u(x) = \frac{\deg x - t(x)}{k - 1} = \frac{\lambda m - t}{k - 1} = \frac{\lambda m(mk - n)}{k(k - 1)(m + n)}.$$  

Sufficiency: Since $D$ is an $S_k$-decomposition of $\lambda K_{m,n}$, we have, by (4), for each $x \in M$

$$t(x) = \deg x - (k - 1)u(x) = \lambda n - \frac{\lambda n(nk - m)}{k(m + n)} = \frac{\lambda mn(1 + k)}{k(m + n)}.$$  

Similarly, for each $x \in N$

$$t(x) = \deg x - (k - 1)u(x) = \lambda m - \frac{\lambda m(mk - n)}{k(m + n)} = \frac{\lambda mn(1 + k)}{k(m + n)}.$$  

Thus $t(x)$ is a constant for all $x \in M \cup N$, which implies that $D$ is balanced. \hfill \Box

\begin{theorem}
Let $m \geq n \geq 1$ be integers. Then $\lambda K_{m,n}$ has a balanced $S_k$-decomposition if and only if the following conditions hold:

(A) $m \geq k \geq m/n$ and

(B) \[
\begin{cases}
m = nk \\
\lambda n(nk - m) \equiv \lambda m(mk - n) \equiv 0 \pmod{k(k - 1)(m + n)}
\end{cases}
\]  

if $k > n$,

(B) \[
\begin{cases}
m = nk \\
\lambda n(nk - m) \equiv \lambda m(mk - n) \equiv 0 \pmod{k(k - 1)(m + n)}
\end{cases}
\]

if $n \geq k$.
\end{theorem}
Theorem 3.4. Let $D$ be a balanced $S_k$-decomposition of $\lambda K_{m,n}$. By the necessity of Lemma 3.2,

$$u_D(x) = \begin{cases} \frac{\lambda n(nk - m)}{k(k-1)(m+n)} & \text{if } x \in M, \\ \frac{\lambda m(mk - n)}{k(k-1)(m+n)} & \text{if } x \in N. \end{cases}$$

Since $u_D(x) \geq 0$ for $x \in M$, we have $nk - m \geq 0$, which implies $k \geq m/n$. Also since $S_k$ is a subgraph of $\lambda K_{m,n}$, we have $k \leq \max\{m,n\}$, which implies $k \leq m$ for the assumption $m \geq n$. Thus, $m \geq k \geq m/n$. This establishes (A).

Now prove (B). In the case $k > n$, no $S_k$ in $D$ can have its center in $M$. Thus $u_D(x) = 0$ for $x \in M$, which implies $nk = m$.

Now consider the case $n \geq k$. Since $u_D(x)$ is an integer for $x \in M$, we have

$$\lambda n(nk - m) \equiv 0 \pmod{k(k-1)(m+n)}.$$  

Similarly, since $u_D(x)$ is an integer for $x \in N$, we have

$$\lambda m(mk - n) \equiv 0 \pmod{k(k-1)(m+n)}.$$  

Thus $\lambda n(nk - m) \equiv \lambda m(mk - n) \equiv 0 \pmod{k(k-1)(m+n)}$. This completes (B).

Sufficiency: We will apply Lemma 3.1. Let

$$u_1 = \frac{\lambda n(nk - m)}{k(k-1)(m+n)}, \quad u_2 = \frac{\lambda m(mk - n)}{k(k-1)(m+n)}.$$  

First show that $u_1$ and $u_2$ are nonnegative integers. In the case $k > n$, since $m = nk$, we have $u_1 = 0$, $u_2 = \lambda n$; $u_1$ and $u_2$ are nonnegative integers. Consider the case $n \geq k$. From the assumption (B), $u_1$ and $u_2$ are integers. Since $k \geq m/n$, we have $nk - m \geq 0$, which also implies $mk - n \geq 0$ since $m \geq n \geq 1$. Thus, $u_1$ and $u_2$ are nonnegative integers. A simple calculation shows that

$$mu_1k + nu_2k = \lambda mn.$$  

By Lemma 3.1, there exists an $S_k$-decomposition $D$ of $\lambda K_{m,n}$ such that for each $x \in M$, there are $u_1$ $S_k$’s in $D$ with centers at $x$, and for each $y \in N$, there are $u_2$ $S_k$’s in $D$ with centers at $y$, i.e., $u_D(x) = u_1$ for $x \in M$ and $u_D(y) = u_2$ for $y \in N$. Thus, by the sufficiency of Lemma 3.2, $D$ is a balanced $S_k$-decomposition of $\lambda K_{m,n}$. This completes the proof.

Now consider center balanced $S_k$-decomposition of $\lambda K_{m,n}$.

Theorem 3.4. Let $m \geq n \geq 1$ be integers. Then $\lambda K_{m,n}$ has a center balanced $S_k$-decomposition if and only if

$$n \geq k \text{ and } \lambda mn \equiv 0 \pmod{m+n}.$$ 

Proof. Necessity: Let $D$ be a center balanced $S_k$-decomposition of $\lambda K_{m,n}$. Since there exists an $S_k$ in $D$ with center in $M$, we have $k \leq n$. Since $|D| = |E(\lambda K_{m,n})|/|E(S_k)| = \lambda mn/k$,
and there are $|\mathcal{D}|/(m+n) S_k$’s with centers at each vertex in $M \cup N$, we see that $|\mathcal{D}|/(m+n) = \lambda mn / k(m+n)$ is an integer. Thus $\lambda mn \equiv 0 \pmod{k(m+n)}$.

**Sufficiency:** Let $u = \lambda mn / (m+n)$. Then $u$ is a positive integer and $muk + nuk = \lambda mn$. By Lemma 3.1, there exists an $S_k$-decomposition $\mathcal{D}$ of $\lambda K_{m,n}$ such that for each $x \in M \cup N$, there exists $u$ $S_k$’s in $\mathcal{D}$ with centers at $x$. Thus $\mathcal{D}$ is center balanced. This completes the proof. □

**Remark.** The results on $S_k$-decompositions of complete bipartite graphs are listed below. Here $k \geq 1$.

**Proposition 3.5** (Yamamoto et al. [11, Theorem 2.2]). Let $m \geq n \geq 1$ be integers. Then $K_{m,n}$ has an $S_k$-decomposition if and only if

\[
m \geq k
\]

and

\[
\begin{cases}
m \equiv 0 \pmod{k} & \text{if } k > n, \\
mn \equiv 0 \pmod{k} & \text{if } n \geq k.
\end{cases}
\]

The following is the generalization of the above result to complete bipartite multigraphs, which was mentioned in [8].

**Proposition 3.6** (Truszczyński [8]). Let $m \geq n \geq 1$ be integers. Then $\lambda K_{m,n}$ has an $S_k$-decomposition if and only if

\[
m \geq k
\]

and

\[
\begin{cases}
\lambda m \equiv 0 \pmod{k} & \text{if } k > n, \\
\lambda mn \equiv 0 \pmod{k} & \text{if } n \geq k.
\end{cases}
\]

As to the generalization of Proposition 3.5 to complete $r$-partite graphs, the readers are referred to [7] for Tazawa’s result. Further investigations of $S_k$-decomposition, balanced $S_k$-decomposition, and center balanced $S_k$-decomposition of complete $r$-partite multigraphs are worth while.

**Acknowledgements**

The authors thank the referees for their comments which improved the readability of this paper.

**References**