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Discrete Mathematics 127 (1994) 75–93
North-Holland

Hamiltonian-connected self-complementary graphs

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Received 27 November 1990
Revised 17 January 1992

Abstract

A self-complementary graph having a complementing permutation $\sigma=[1,2,3,\ldots,4k]$, consisting of one cycle, and having the edges (1, 2) and (1, 3) is strongly Hamiltonian iff it has an edge between two even-labelled vertices. Some of these strongly Hamiltonian self-complementary graphs are also shown to be Hamiltonian connected.

1. Introduction

Definition 1.1. A graph $G=\langle V(G), E(G)\rangle$ is said to be *self-complementary* (SC) if there is a permutation σ on $V(G)$ such that $(x,y)\sigma=(x\sigma,y\sigma)\notin E(G)$ iff $(x,y)\in E(G)$. This permutation σ is called a *complementing permutation* (CP). The graph $\bar{G}$ in which $V(G)=V(G)$ and $(x,y)\in E(\bar{G})$ iff $(x,y)\notin E(G)$ is called the *complement of G*.

Definition 1.2. A graph is said to be *Hamiltonian* if it has a *Hamiltonian cycle*. If, in addition, every edge is contained in a Hamiltonian cycle, then it is said to be *strongly Hamiltonian*. Furthermore, if every pair of vertices are endpoints of a *Hamiltonian path*, then it is said to be *Hamiltonian connected*.

The self-complementary graphs G investigated in this paper are those with the following properties:

(P1) G has a CP $\sigma=[1,2,3,\ldots,4k]$, consisting of one cycle.

(P2) G has edges (1, 2) and (1, 3).

G obviously must have an even edge, i.e. an edge between two even-labelled vertices, to be strongly Hamiltonian; otherwise, it is almost constricted in the sense of Nash-Williams [4] and as such no odd edge can be contained in a Hamiltonian cycle. The details for the sufficiency of an even edge to make it strongly Hamiltonian are given in [1].

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Partition the vertex set $V(G)$ into $L_1 \cup L_2 \cup L_3 \cup L_4$, where $L_i = \{x \in V(G): x \cong i \pmod 4\}$ for $i = 1, 2, 3, 4$. Following the observations of Clapham [2] and Gibbs [3], self-complementary graphs having properties (P1) and (P2) can be classified by means of the set $N_{L_2}(1)$, the set of elements in L_2 adjacent to vertex 1. The two subclasses discussed here are those in which $N_{L_2}(1) = L_2$ and $N_{L_2}(1) = \{2\}$.

Note that $N_{L_2}(1)$, by property (P2), contains vertex 2, so it can be chosen in 2^{k-1} ways, specifically, any subset of $L_2 \setminus \{2\}$, and then augmented by vertex 2.

Let the enumeration Φ of the possible neighbors of vertex 1 in L_2 be such that $\Phi(1) = L_2$ and $\Phi(2) = \{2\}$. Then associate with the enumeration Φ the following classes of self-complementary graphs with properties (P1) and (P2): $G_1(4k)$, the class where $N_{L_2}(1) = L_2$; $G_2(4k)$, the class where $N_{L_2}(1) = \{2\}$. Then the class $G_n(4k)$, $n \neq 1, 2$, refers to a class where $N_{L_2}(1)$ is neither L_2 nor $\{2\}$.

2. Hamiltonian-connected self-complementary graphs, I

Rao [5] introduced an SC graph $G = G^*(4k)$ which is defined as follows:

(a) $V(G) = \{1, 2, 3, \ldots, 4k\}$;

(b) $(x, y) \in E(G)$ iff

  (1) $x, y \cong 1, 3 \pmod 4$,

  (2) $x \cong 1 \pmod 4$ and $y \cong 2 \pmod 4$ or

  (3) $x \cong 3 \pmod 4$ and $y \cong 0 \pmod 4$.

This graph is in $G_1(4k)$. In view of [4], it has no Hamiltonian cycle.

Now let $\mathrm{OE}(G)$ be the set of odd edges of $G = G^*(4k)$. Let $(x, y) \in \mathrm{OE}(G) \setminus C$, where $C = \{(1, 3)\sigma^{2l}: l \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers. Remove the set

$$\{(x, y)\sigma^{2l}: l \in \mathbb{N}\}$$

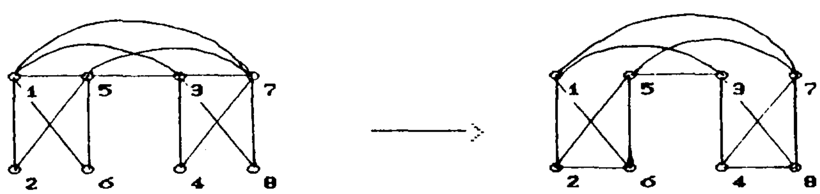
and replace it by the set

$$\{(x, y)\sigma^{2l+1}: l \in \mathbb{N}\}.$$

The resulting graph is still in $G_1(4k)$, and in view of [1] it is already a strongly Hamiltonian self-complementary graph.

Illustration

$G^*(8)$:



Note that the second graph is not Hamiltonian connected because vertices 1 and 5 cannot be endpoints of a Hamiltonian path.

The replacement of a set of odd edges by even edges can be obtained in three typical ways by the following constructions.

Construction 2.1. Let $\emptyset \neq A \subseteq \{1, 2, \ldots, \lfloor k/2 \rfloor\}$. Obtain the class $G_1'(4k)$ from $G^*(4k)$ by replacing the set of edges

$$\{(1, 4a+1)\sigma^{2l}\colon a \in A,\ l \in \mathbb{N}\}$$

by the set of edges

$$\{(2, 4a+2)\sigma^{2l}\colon a \in A,\ l \in \mathbb{N}\}.$$

A graph in this class contains edges between vertices in L_2 but no edge between L_2 and L_4.

Construction 2.2. Let $\emptyset \neq B \subseteq \{1, 2, \ldots, \lfloor k/2 \rfloor\}$. Obtain the class $G_1''(4k)$ from $G^*(4k)$ by replacing the set of edges

$$\{(1, 4b+3)\sigma^{2l}\colon b \in B,\ l \in \mathbb{N}\}$$

by the set of edges

$$\{(2, 4b+4)\sigma^{2l}\colon b \in B,\ l \in \mathbb{N}\}.$$

A graph in this class contains edges between L_2 and L_4 but not edges between vertices in L_2.

Construction 2.3. Let $\emptyset \neq A, B \subseteq \{1, 2, \ldots, \lfloor k/2 \rfloor\}$. Obtain the class $G_1'''(4k)$ from $G^*(4k)$ by replacing the edges

$$\{(1, 4a+1)\sigma^{2l}\colon a \in A,\ l \in \mathbb{N}\} \quad \text{and} \quad \{(1, 4b+3)\sigma^{2l}\colon b \in B,\ l \in \mathbb{N}\}$$

by the set of edges

$$\{(2, 4a+2)\sigma^{2l}\colon a \in A,\ l \in \mathbb{N}\} \quad \text{and} \quad \{(2, 4b+4)\sigma^{2l}\colon b \in B,\ l \in \mathbb{N}\}.$$

A graph in this class contains edges between vertices in L_2 and edges between L_2 and L_4.

Remark 2.4. For $k=3$, $G_1'(4k), G_1''(4k)$ and $G_1'''(4k)$ have one element each and all three graphs can be verified to be Hamiltonian connected. These graphs are illustrated in Fig. 1.

Lemma 2.5. *For $k \geqslant 4$, any element in $G_1'(4k), G_1''(4k)$ or $G_1'''(4k)$ is Hamiltonian connected.*

Proof. In view of [1], it only remains to show that every pair of nonadjacent vertices are endpoints of a Hamiltonian path.

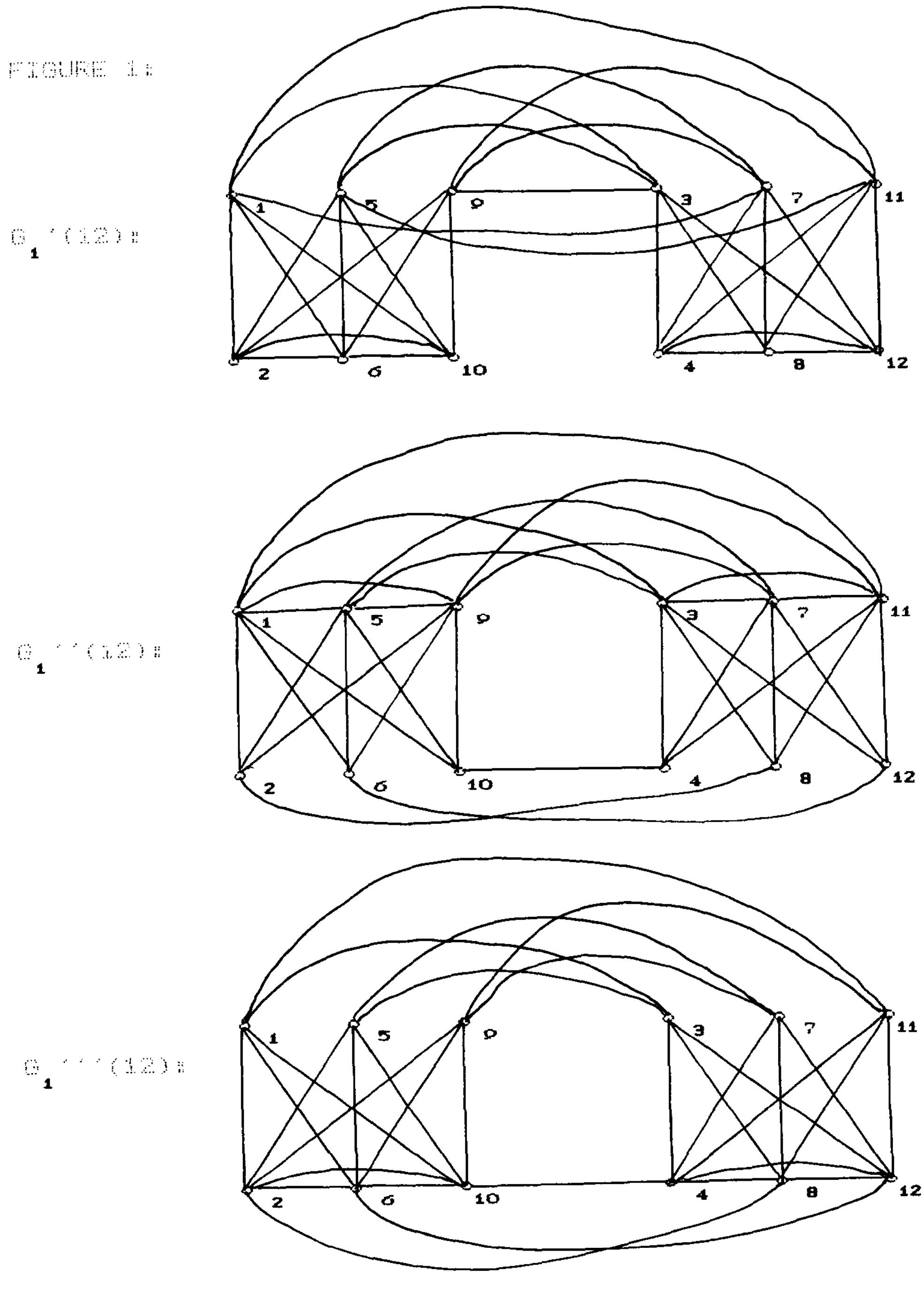


FIGURE 1:
G1'(12):
1 5 9 3 7 11
2 6 10 4 8 12
G1''(12):
1 5 9 3 7 11
2 6 10 4 8 12
G1'''(12):
1 5 9 3 7 11
2 6 10 4 8 12

Fig. 1.

For G in $G_1'(4k)$, every vertex in L_1 is adjacent to every vertex in L_3. Hence, any nonadjacent pair of vertices is an automorphic image of one of the pairs 1 and $4a+1$, 1 and $4b+4$, 2 and $4a+2$ or 2 and $4b+4$, where $1 \leqslant a, b \leqslant k-1$. Together with the fact that every vertex of L_1 is adjacent to every vertex of L_2, a Hamiltonian path whose end points are any of the nonadjacent pairs above can easily be constructed.

For G in $G_1''(4k)$, the subgraph induced by L_1 ($\cong L_3$) is a complete graph of order k. Therefore, a HP whose endpoints are nonadjacent pairs of vertices of the form 1 and $4b+3$, 1 and $4b+4$ (whether 2 and $4b+4$ are adjacent or not), 2 and $4a+2$ or 2 and $4b+4$ can easily be constructed.

For G in $G_1'''(4k)$, nonadjacent pairs are of the form 1 and $4a+1$, 1 and $4c+3$, 1 and $4b+4$ with $(2, 4x+2) \in E(G)$ for some x, 2 and $4d+2$ with $(2, 4y+2) \in E(G)$ for some y, or 2 and $4e+4$. For these nonadjacent pairs, corresponding Hamiltonian paths are constructed below:

Case 1: $(1, 4a+1) \notin E(G)$. Span $L_1 \cup L_2 \backslash \{4a+1\}$ by a path with end vertices 1 and $u \in L_1$. Also span $L_3 \cup L_4$ by a path with end vertices $u+2$ and $4a+3$. Then connect these paths by the edge $(u, u+2)$ and add the edge $(4a+1, 4a+3)$ to obtain a HP with end vertices 1 and $4a+1$.

Case 2: $(1, 4c+3) \notin E(G)$. Span $L_1 \cup L_2$ by a path with end vertices 1 and 2. Also span $L_3 \cup L_4$ by a path with end vertices $4c+3$ and $4c+4$. Then connect these paths by the edge $(2, 4c+4)$.

Case 3: $(1, 4b+4) \notin E(G)$ and $(2, 4x+2) \in E(G)$. Span $L_1 \cup L_2$, using the edge $(2, 4x+2)$, by a path with end vertices 1 and $u \in L_1$ and span $L_3 \cup L_4$ by a path with end vertices $u+2$ and $4b+4$. Then connect these paths by the edge $(u, u+2)$.

Case 4: $(2, 4d+2) \notin E(G)$ and $(2, 4y+2) \in E(G)$. Span $L_1 \cup L_2 \backslash \{4d+1, 4d+2\}$ by a path with end vertices 2 and $4x+1$. Also span $L_3 \cup L_4$, using the edge $(2, 4y+2)\sigma^{4x+2}$, by a path with end vertices $4x+3$ and $4d+3$. Then connect these paths by the edge $(4x+1, 4x+3)$ and add the path $(4d+3, 4d+1, 4d+2)$.

Case 5: $(2, 4e+4) \notin E(G)$. Span $L_1 \cup L_2$ by a path with end vertices 2 and $4x+1$ and span $L_3 \cup L_4$ by a path with end vertices $4x+3$ and $4e+4$. Then connect these paths by the edge $(4x+1, 4x+3)$. $\square$

Theorem 2.6. *Let G be a self-complementary graph with properties* (P1) *and* (P2). *If G is such that $N_{L_2}(1) = L_2$, then G is Hamiltonian connected iff it is strongly Hamiltonian and $k \geqslant 3$.*

3. Hamiltonian-connected graphs, II

The classes $G_1'(4k)$, $G_1''(4k)$ and $G_1'''(4k)$ obtained from $G^*(4k)$ have the property that $N_{L_2}(1) = L_2$. Now obtain the graphs $G_2'(4k)$, $G_2''(4k)$ and $G_2'''(4k)$ from $G_1'(4k)$, $G_1''(4k)$ and $G_1'''(4k)$, respectively, by removing the edges $(1, 4b+2)$, $b = 1, 2, 3, \ldots, k-1$, and their automorphic images under the automorphism σ^{2l}, $l \in \mathbb{N}$, and then replacing them by the edges $(1, 4k-4b)$, $1 \leqslant b \leqslant k-1$, with their automorphic images. Graphs in these classes are strongly Hamiltonian self-complementary graphs with properties (P1) and (P2). However, they have the property that $N_{L_2}(1)=\{2\}$.

Now if $(2,4b+2)\in E(G)$, where $G\in G_2'(4k)$, then L_2 and L_4 can be partitioned into disjoint cycles as follows.

Step 1: Define $C_0^2=\{(2,4b+2)\ \sigma^{4nb}:\ 0\leqslant n\leqslant k'-1\}$, where $b=db'$ $k=dk'$ and $d=gcd(k,b)$.

Step 2: Define $C_p^2=C_0^2\sigma^{4p}$, where $0\leqslant p\leqslant d-1$.

Step 3: Define $C_p^4=C_0^2\sigma^{4p+2}$, where $0\leqslant p\leqslant d-1$.

Clearly, L_2 is a disjoint union of the cycles C_p^2 and L_4 is the disjoint union of the cycles C_p^4. Let $x\in C_p^2$, $y\in C_q^2$ and $x\cong y(\text{mod }4k)$. Then $x=4nb+4p+2$ and $y=4mb+4q+2$ for some n,m. Hence, $x-y\cong 0(\text{mod }4k)$ implies that d divides $p-q$ and k' divides $n-m$ as observed in the equations

$$4(n-m)db'+4(p-q)=4ldk',$$

$$db'=b \quad\text{and}\quad dk'=k.$$

These cycles are said to be *generated* by the edge $(2,4b+2)$ via the CP σ. These cycles are degenerate if $k=2b$.

If $(2,4b+4)\in E(G)$, where $G\in G_2''(4k)$, then $L_2\cup L_4$ can be partitioned into disjoint cycles as follows.

Step 1: Define $C_0=\{(2,4b+4)\sigma^{(4b+2)n}:\ 0\leqslant n\leqslant k^*-1\}$, where $2b+1=d'b^*$, $2k=d'k^*, d'=\gcd(2k,2b+1)$.

Step 2: Define $C_p=C_0\sigma^{4p}$, where $0\leqslant p\leqslant d'-1$. The cycles C_p can easily be shown to be disjoint and span $L_2\cup L_4$. These cycles are degenerate if $k=2b+1$.

Lemma 3.1. *For $k\geqslant 4$, any element $G\in G_2'(4k)$ is Hamiltonian connected.*

Proof. Note here that $(4a+1,4b+3)\in E(G)$ for all $a,b=0,1,2,\dots,k-1$. With these edges, the required Hamiltonian path for every nonadjacent pair of vertices can be obtained as automorphic images of the Hamiltonian paths constructed below:

(1) $(1,4b+1)\notin E(G)$. Construct a required Hamiltonian path by the following steps:

(1a) Partition L_2 and L_4 into cycles generated by the edge $(2,4b+2)$. Correspondingly partition L_1 and L_3 in such a way that if x is in a partition of L_2 or L_4, then $x-1$ is in the corresponding partition of L_1 or L_3.

(1b) Let $x\in L_3$ be such that $(2,x)\in E(G)$. Starting from vertex 1, span by a path this partition of L_1, except $4b+1$, and a partition of L_3 not containing x, if any, together with its cycle partition of L_4 and ending at a vertex in L_3. This is illustrated in Fig. 2.

If there is only one partition each, traverse the edge from 1 to vertex y $(\neq x)\in L_3$, then go to $y+1\in L_4$, span the rest of L_4, then go to a vertex in L_3 and go alternatively between L_1 and L_3, leaving out $4b+1$, and end at x. Then from x traverse the edge to 2, span L_2 up to $4b+2$ and finally end at $4b+1$.

(1c) From the last vertex in L_3, span one partition each of L_1, L_2, L_3 and L_4 at a time, each time ending at a vertex in L_3, and finally end at x.

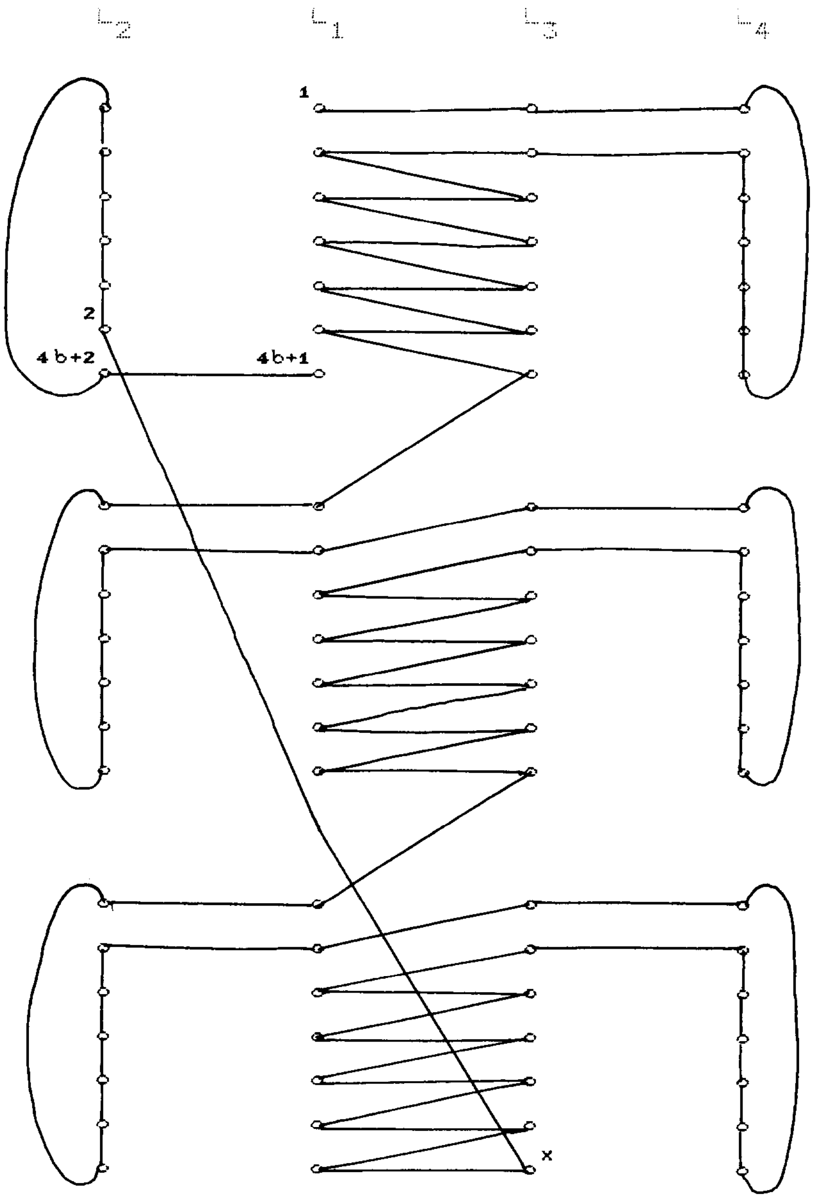


Fig. 2.

(1d) Complete the Hamiltonian path by connecting x to 2, spanning the last cycle of L_2, ending at $4b+2$, then go finally to $4b+1$.

(2) $(1,4a+2)\notin E(G)$ for $1\leqslant a\leqslant k-1$. Let $(2,4b+2)\in E(G)$ and do the partition as in (1). Let $y\,(\neq 1)\in L_1$ be such that $(y+1,4a+2)$ is an edge in a partition of L_2. Construct a required Hamiltonian path under the following cases.

Case 1: Vertices 1 and y are in the same partition.

Case 1.1: Vertex $y \neq 4a+1$.

(2a.1.1) Subsumed in this subcase is the fact that each partition has more than two vertices. From vertex 1 span the partition containing it, except y, and a partition in L_3 together with its corresponding partition of L_4, ending at a vertex in L_3. This is illustrated in Fig. 3.

(2b.1.1) From the last vertex in L_3, span a partition each from L_1, L_2, L_3 and L_4 at a time, and each time ending at a vertex in L_3, until all of L_3 is spanned.

(2c.1.1) Complete the Hamiltonian path by joining the last vertex in L_3 to y, then to $y+1$, spanning the last cycle-partition of L_2, then ending at $4a+2$.

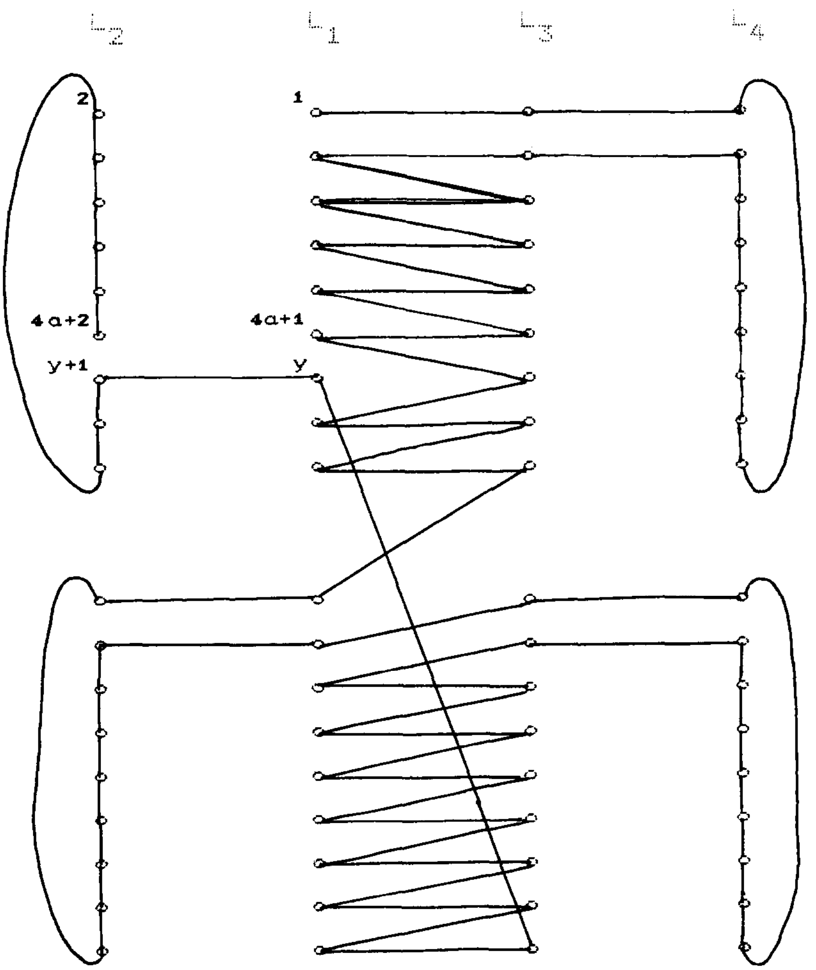


Fig. 3.

Case 1.2: $y=4a+1$. This degenerate case happens only when $a=b=k/2$, i.e. a partition of L_2 is an edge, and $(2,4b+2)=(2,2k+2)$. In this case, span $V(G)$ by the paths

$$(1,2,2k+2,2k+1)\sigma^{2l} \quad \text{for } 0\leqslant l\leqslant k-1,$$

remove the edge $(2,2k+2)$ and add the edges $(2,2k+3),(2k-1,2k+1),(3,5)\sigma^{4m}$ and $(2k+5,2k+7)\sigma^{4m}$ for $m=0,1,2,\ldots,(k-4)/2$ to obtain a Hamiltonian path with end vertices 1 and $2k+2$. This is illustrated in Fig. 4.

Case 2: Vertices 1 and y are not in the same partition.

(2a.2) In this case choose y so that $y\neq 4a+1$ and $(y+1,4a+2)\in E(G)$. Then from vertex 1, span the partition containing it and a partition in L_3 together with their corresponding partitions in L_2 and L_4, ending at a vertex in L_3. This is illustrated in Fig. 5.

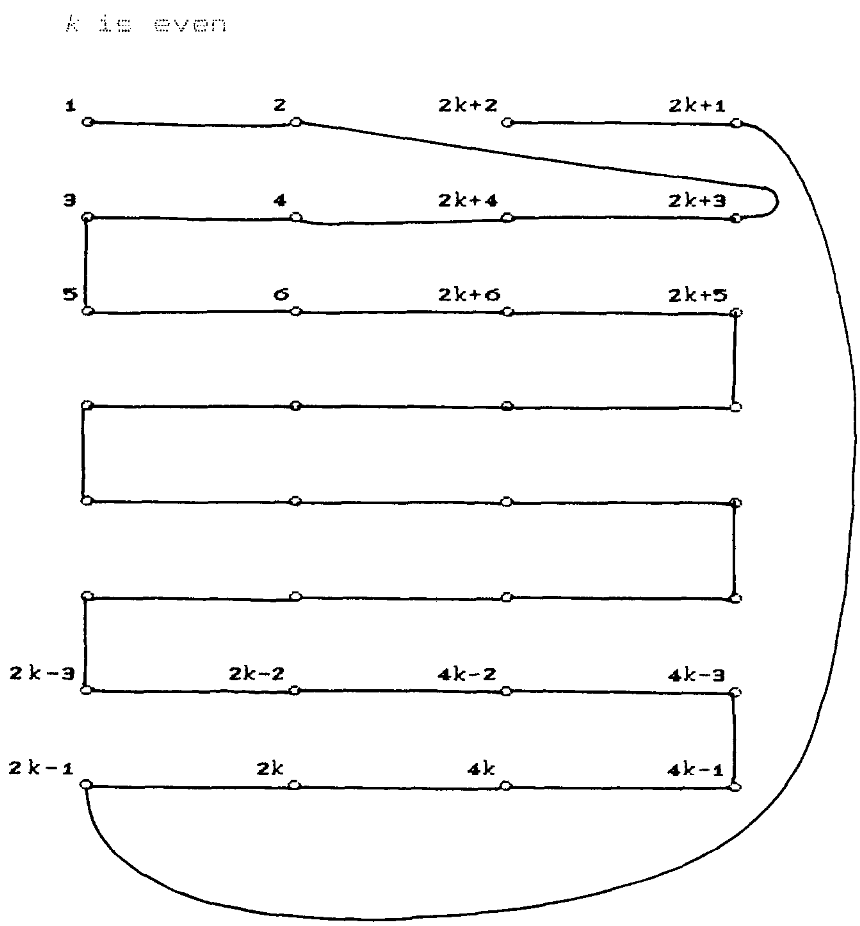


Fig. 4.

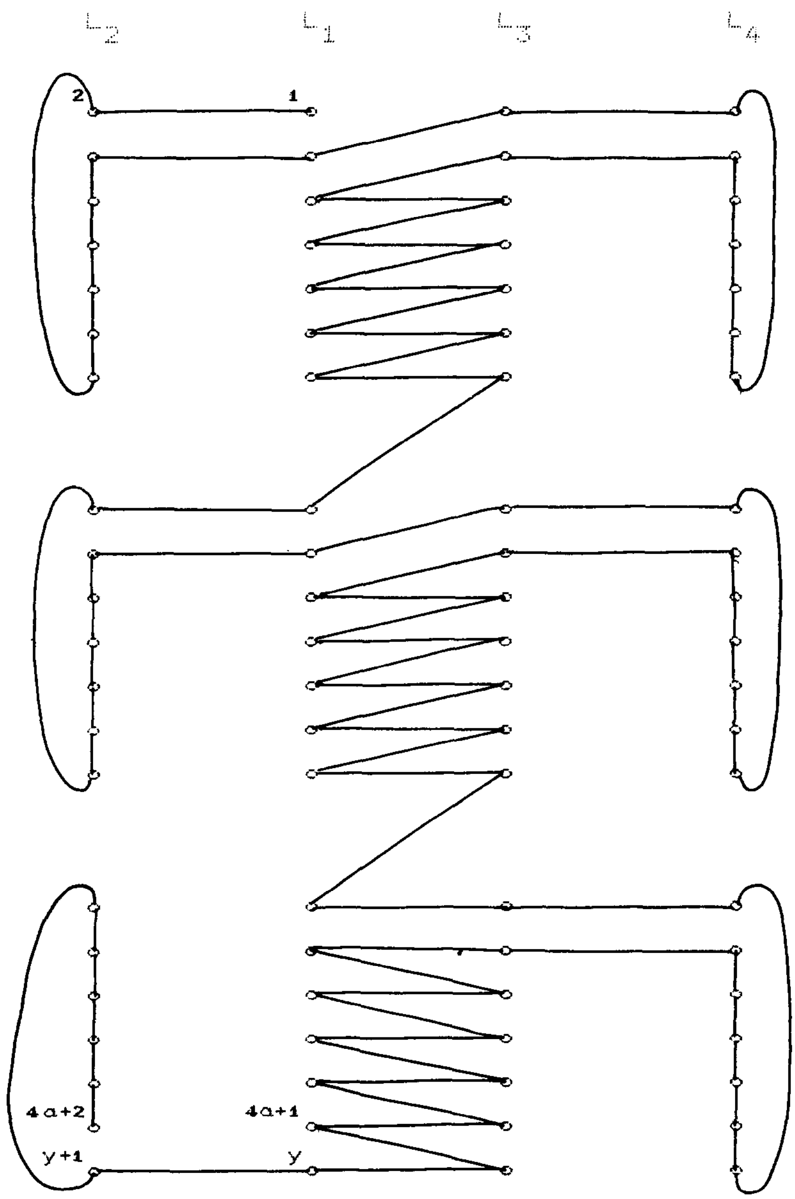


Fig. 5.

(2b.2) Span a partition each of L_1, L_2, L_3 and L_4 at a time, each time ending at a vertex in L_3, until only the partition containing y is left unspanned in L_1.

(2c.2) Complete the Hamiltonian path by joining the last vertex in L_3 to $z\,(\neq y) \in L_1$, then proceed to a vertex in L_3, then to a vertex in L_4, spanning the rest of L_4, then go back to a vertex in L_3, spanning the rest of L_1 and L_3, ending at y.

Finally, connect y to $y+1$ and span the rest of L_2, ending at $4a+2$.

(3) $(1,4k)\notin E(G)$

(3a) Let $(2,4b+2)\in E(G)$ and partition $V(G)$ as in (1a) and let x $(\neq 4k-1)\in L_3$ be such that $(x+1,4k)$ is an edge of a cycle-partition.

If there is only one partition of L_2, then from vertex 1 go to vertex 2 and span L_2, then connect to W $(\neq x)$ in L_3 and go alternatively between L_1 and L_3, ending at a vertex x. Complete the Hamiltonian path by connecting x to $x+1$ and spanning L_4 up to $4k$.

(3b) If there are $l>1$ partitions of L_2, span $l-1$ partitions each of L_1, L_2, L_3 and L_4 from vertex 1 and ending at a vertex in L_3, leaving out among others the partition containing x. This is illustrated in Fig. 6.

(3c) Connect the last vertex in L_3 to a vertex in L_2, then span the rest of L_2, then go to a vertex in L_1. From there, go alternatively between L_1 and L_3, ending at a vertex x.

(3d) Complete the Hamiltonian path by connecting x to $x+1$ and span the rest of L_4 up to $4k$.

(4) $(2,4a+2)\notin E(G)$ and $(2,4b+2)\in E(G)$.

Case 1. Vertices 2 and $4a+2$ are not in the same cycle generated by $(2,4b+2)$.

(4a.1) From vertex 2, span the partition containing it, then proceed to a vertex in L_1; then span the partition of L_1 containing it and a partition of L_3 together with its corresponding partition in L_4, ending at a vertex in L_1.

(4b.1) From the last vertex in L_1, span one partition each of L_1, L_2, L_3 and L_4 at a time, each time ending at a vertex in L_1, until only the partition containing $4a+2$ is left in L_2.

(4c.1) Connect the last vertex in L_1 to a vertex in L_4 and span the rest of L_4. Then proceed to a vertex in L_3 and go back and forth between L_1 and L_3, ending at x in L_1, where $(x+1,4a+2)$ is an edge in a cycle-partition of L_2.

(4d.1) Complete the Hamiltonian path by connecting x to $x+1$, then spanning the rest of L_2 up to $4a+2$.

Case 2: Vertices 2 and $4a+2$ are in the same cycle generated by $(2,4b+2)$.

(4a.2) Let y be adjacent to $4a+2$ in a cycle-partition such that the paths from 2 to y and from $4b+2$ to $4a+2$ span this cycle. Further, let $z\in L_3$ be such that $(z,4b+2)\in E(G)$. Then from 2, span part of the cycle to y, connect y to $y-1$ and span this partition of L_1 and a partition of L_3 not containing z, if any, together with its corresponding partition in L_4, ending at a vertex in L_1.

If there is only one partition of L_2, let $w\in L_4$ be such that $(y-1,w)\in E(G)$. From $y-1$, proceed instead to w and span L_4, ending at some vertex x $(\neq z+1)$, then connect to $x-1$ and alternate between L_1 and L_3, ending at z. Finally, connect to $4b+2$ and span the rest of L_2 up to $4a+2$.

(4b.2) From the last vertex in L_1, span one partition each of L_1, L_2, L_3 and L_4 at a time, each time ending at L_1, until only one partition each of L_1, L_2, L_3 and L_4 and the path from $4b+2$ to $4a+2$ are left unspanned.

(4c.2) From the last vertex in L_1, go to a vertex in L_4, span the rest of L_4, then connect to a vertex in L_3. From there, span the rest of L_3 and L_1 with the

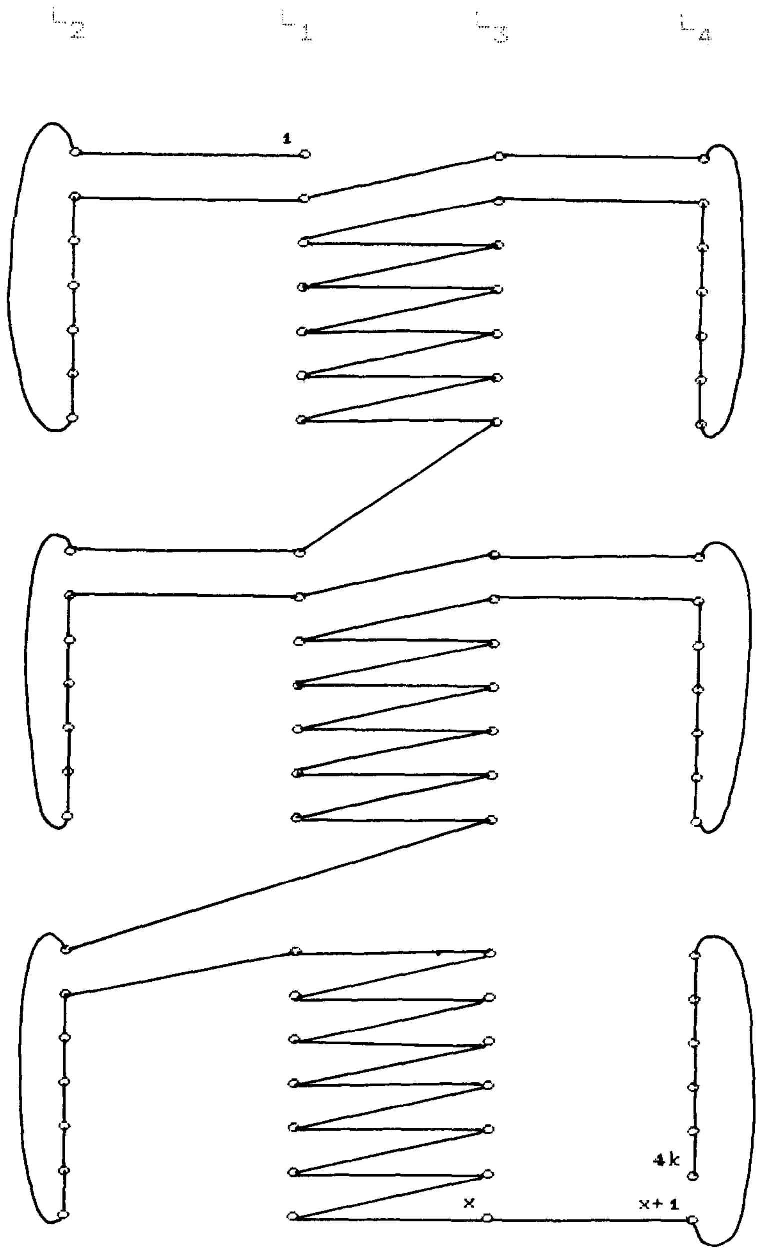


Fig. 6.

corresponding partition in L_2, ending at $z \in L_3$. Finally, connect z to $4b+2$ and span the rest of L_2 up to $4a+2$.

(5) $(2, 4a+4) \notin E(G)$.

(5a) Let $(2, 4b+2) \in E(G)$ and partition $V(G)$ as in (1a). Let $x \in L_1$ and $y \in L_4$ be such that $(x, y) \in E(G)$, $(y, 4a+4)$ is an edge in a cycle-partition and x is not in the partition containing 1, if there is more than one cycle generated by $(2, 4b+2)$. From vertex 2, span the partition containing it, then connect to a vertex in the corresponding partition of L_1. Span this partition of L_1, together with a partition of L_3 not containing $y-1$, and its corresponding partition in L_4, then end at a vertex in L_1.

(5b) From the last vertex in L_1, span one partition each from L_1, L_2, L_3 and L_4 at a time, each time ending at a vertex in L_1, until only one partition each of L_1, L_2, L_3 and L_4 is left.

In case there is only one partition, from 2 span the rest of L_2, then go to a vertex in L_3, then alternate between L_1 and L_3, ending at $x \in L_1$. Finally, connect x to y and span the rest of L_4 to $4a+4$.

(5c) From the last vertex in L_1, connect to a vertex in L_3, then to a vertex in L_2, spanning the rest of L_2, then go to a vertex in L_1 and alternate between L_1 and L_3, ending at x. Finally, connect x to y and span the rest of L_4 up to vertex $4a+4$. □

Lemma 3.2. *For* $k \geq 4$, *if* $G \in G_2''(4k)$, *then* G *is Hamiltonian connected.*

Proof. Here, the subgraph induced by L_1 ($\cong L_3$) is a complete graph. A Hamiltonian path whose endpoints are nonadjacent vertices can be obtained as an automorphic image of one of the HP's constructed below:

(1) $(1, 4b+3) \notin E(G)$.

(1a) Partition $L_2 \cup L_4$ into cycles generated by the edge $(2, 4b+4)$.

Case 1: There is an odd number of cycles.

Case 1.1: There is only one cycle.

(1b.1.1) From vertex 1, span L_1, then join the last vertex to a vertex in L_4, then span $L_2 \cup L_4$, ending at a vertex in L_2. Then connect to a vertex in L_3 ($\neq 4b+3$) and span L_3, ending at $4b+3$.

Case 1.2: There are at least three cycles.

(1b.1.2) Connect vertex 1 to a vertex in L_4 and span the cycle containing it, ending at a vertex in L_2.

(1c.1.2) Connect the last vertex in L_2 to a vertex in L_3 ($\neq 4b+3$), then go back to a vertex in L_2, spanning the cycle containing it and ending at a vertex in L_4. Proceed to a vertex in L_1, return to a vertex in L_4, spanning the cycle containing it and ending at a vertex in L_2. Repeat the process until only one cycle is left, the last vertex spanned being in L_4.

(1d.1.2) From the last vertex in L_4, go to a vertex in L_1, span the rest of L_1, then proceed to a vertex in L_4, spanning the last cycle, ending at a vertex in L_2.

(1e.1.2) Complete the Hamiltonian path by proceeding to a vertex in L_3 and spanning the rest of L_3 up to the vertex $4b+3$.

Case 2. There is an even number of cycles.

(1b.2) From vertex 1, go instead to vertex 2, then span the cycle containing it, ending at a vertex in L_4. Then proceed as in case 1.2 to obtain the required Hamiltonian path.

(2) $(1, 4a+2) \notin E(G)$. Let $(2, 4b+4) \in E(G)$ and partition $L_2 \cup L_4$ into cycles generated by $(2, 4b+4)$.

Case 1: $(2, 4b+4)$ generates only one cycle.

(2a.1) Let $x \in L_1$, $y \in L_3$ be such that $(x, 4a+4b+4)$ and $(y, 4a+2) \in E(G)$. Span L_1 by starting at 1 and ending at x. Then go to $4a+4b+4$ and go through the vertices of the cycle up to $4a-4b$, leaving out $4a+2$. Then proceed to a vertex in L_3, then span L_3, ending at y. Finally, connect y to $4a+2$. If $a+b+1=k$, reverse the roles of $4a+4b+4=4k$ and $4a-4b$.

Case 2: $(2, 4b+4)$ generates at least two cycles.

Case 2.1: There is an odd number of cycles and each cycle has more than two vertices.

(2a.2.1) Let y be as in (2a.1). From 1 go to $4a+4b+4$ and span the cycle up to $4a-4$b, leaving out $4a+2$. Then go to $4a-4b-1$, then to a vertex in L_2, spanning the cycle containing it and ending at a vertex in L_4.

(2b.2.1) From this vertex of L_4 go to a vertex in L_1, then return to L_4, spanning a cycle and ending at a vertex in L_2. Repeat the process until only one cycle is left, with the last vertex traversed in L_4.

(2c.2.1) Connect the last vertex in L_4 to a vertex in L_1, span the rest of L_1 then go to a vertex in L_4, spanning the last cycle, and ending at a vertex in L_2.

(2d.2.1) Finally, continue on to a vertex in L_3, spanning the rest of L_3 up to vertex y, and then to $4a+2$.

Case 2.2: There is an odd number of cycles and each cycle is an edge (degenerate case).

(2a.2.2.) Let x be as in (2a.1). Then from vertex 1 go to a vertex in L_4 ($\neq 4a+4b+4$), then span the edge. Go to a vertex in L_3, continue to a vertex in L_2 ($\neq 4a+2$), then span the edge and go to a vertex in L_1. Repeat the process until only two edges are left with the last vertex spanned in L_2.

(2b.2.2) Proceed to a vertex in L_3, span the rest of L_3 and go to a vertex in L_2, then span the edge and proceed to L_1, spanning the rest of L_1 and ending at vertex x. Finally, join x to $4a+4b+4$ and then go to $4a+2$.

Case 2.3: There is an even number of cycles and each cycle has more than two vertices.

(2a.2.3) Proceed as in case 2.1, but instead connect the vertex $4a-4b$ to a vertex in L_1. The effect is to end up at y. Proceed finally to $4a+2$.

(3) $(1, 4k) \notin E(G)$. Let $(2, 4b+4) \in E(G)$ and partition $L_2 \cup L_4$ as in (1a). Then proceed as in (2) but interchange the roles of x and y, and replace the roles of $4a+2$, $4a+4b+4$, and $4a-4b$ by $4k$, $4b+2$, and $4k-4b-2$, respectively, and start from vertex 1, then to 3 and let 3 take the role of 1, except in Case 1.2, where $4k-1$ takes the role of y.

(4) $(2, 4a+2) \notin E(G)\ \forall\, 1 \leqslant a \leqslant k-1$.

Case 1: Vertices $4a+2$ and 2 are in the same cycle generated by $(2, 4b+4)$.

Case 1.1: There is only one cycle.

(4a.1.1) From vertex 2 span the cycle through $4b+4$ up to $4a-4b$, then connect the latter to a vertex in L_3, then span L_3 and connect the last vertex to $4a+4b+4$, leaving out $4a+2$. Then span the rest of the cycle up to $4k-4b$, connect this to a vertex in L_1, span L_1, then connect the last vertex to $4a+2$.

Case 1.2: There are at least two cycles.

(4a.1.2) From vertex 2, span the vertices as in (4a.1.1) up to $4k-4b$, but traverse only one vertex in L_3. Then proceed as in (2b.2.1) until only one cycle is left. If the last vertex is in L_2, connect this to a vertex in L_3, span L_3 then return to a vertex in L_2 and span the last cycle to a vertex in L_4. Then go to a vertex in L_1, span the rest of L_1, ending at $4a+1$. Finally, go to $4a+2$. On the other hand, if the last vertex is in L_4, connect this to a vertex in L_1, span the rest of L_1, except $4a+1$, then go back to a vertex in L_4 and span the last cycle, ending at a vertex in L_2. Finally, connect this to a vertex in L_3 ($\neq 4a+3$), span the rest of L_3 up to $4a+3$ and then connect $4a+1$ to vertex $4a+2$.

Case 2: Vertices $4a+2$ and 2 are not in the same cycle.

(4a.2) Span from 2 the cycle containing it up to $4b+4$, then go to $4b+3$, then to $4a+4b+4$, spanning the cycle up to $4a-4b$, leaving out $4a+2$.

(4b.2) If there are only two cycles, go from vertex $4a-4b$ to a vertex in L_3, span the rest of L_3, then connect to a vertex in L_1, span the rest of L_1 ending at x, where $(x,4a+2)\in E(G)$, and, finally, connect the last vertex to $4a+2$. If there are more than two cycles, proceed as in (4a.1.2).

(5) $(2,4a+4)\notin E(G)$. Let $(2,4b+4)\in E(G)$. Then proceed as in (4) but let $4a+4$ take the role of $4a+2$, taking into consideration the adjacencies between L_1 and $L_2\cup L_4$. □

Lemma 3.3. *For $k\geq 4$, if $G\in G_2'''(4k)$, then G is Hamiltonian connected.*

Proof. From the Hamiltonian paths constructed below, a Hamiltonian path whose end vertices are nonadjacent can be obtained as an automorphic image of one of them.

(1) $(1,4a+1)\notin E(G)$:- Partition L_2 and L_4 into cycles generated by $(2,4a+2)$. Then, using the edge $(2,4b+4)$, obtain a cycle from a cycle of L_2 and a cycle of L_4 for each pair of cycles connected by some edge of the form $(2,4b+4)\sigma^{4l}$. This is illustrated in Fig. 7.

Case 1: A cycle in L_2 has more than two vertices.

(1a.1) First, choose the connected pair of cycles containing $4k-4b-2$ and $4k$. Obtain a path from these cycles by deleting the edges $(4k-4b-2,\ 4k-4a-4b-2)$, $(4k,4a)$ and $(4k,4k-4a)$ and adding the edges $(4k-4b-2,4k)$ and $(4k-4a-4b-2,4k-4a)$. Then extend this to a path with end vertices 1 and $4a+1$ by adding the edges $(4k,4k-1)$ and $(4a,4a-1)$ and the cycle generated by the edge $(1,3)$, where the edges $(1,4k-1)$ and $(4a-1,4a+1)$ are deleted.

(1b.1) Extend this path to a Hamiltonian path by adding the edges (y_i,y_i-1), (y_i+1,y_i-4a) or (y_i+1,y_i+4a), whichever applies, where $y_i\in L_4$ for $i=1,2,\dots,p-1$ and p is the number of cycles in L_2 generated by $(2,4a+2)$, and by deleting the edges (y_i,y_i-4a) or (y_i,y_i+4a), whichever applies.

Case 2: A cycle in L_2 is an edge (degenerate case).

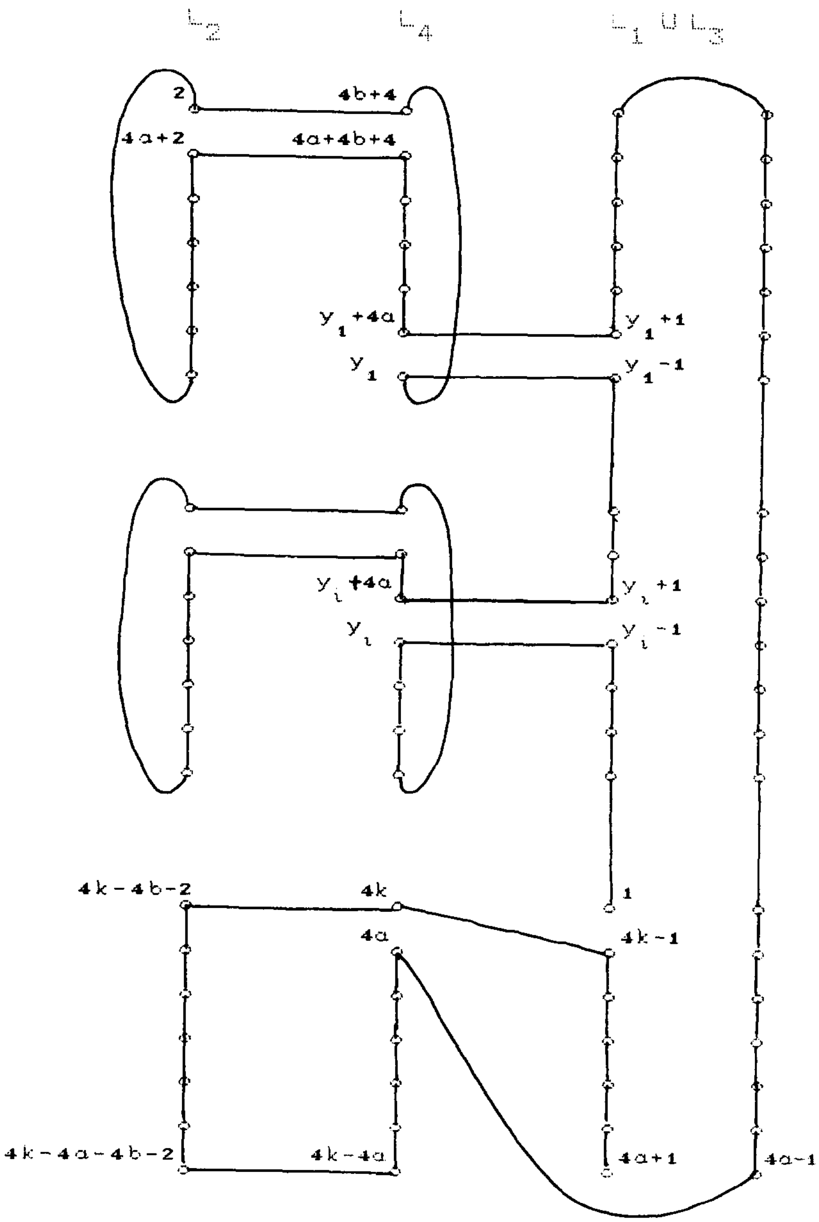


Fig. 7.

In this case the cycles generated by the edge $(2, 4b+4)$ cannot be degenerate because k must be even, i.e. $k=2a$; therefore, the cycles generated by $(2, 4b+4)$ have at least 4 vertices.

(1a.2) Partition $L_2 \cup L_4$ into cycles generated by $(2, 4b+4)$ and suppose that $4a+2=2k+2$ and 2 are in the same cycle. To the cycle generated by $(1,3)$ and the cycle generated by $(2, 4b+4)$ containing 2, add the edges $(2, 2k-1), (2k+2, 4k-1)$ and

$(4b+4, 2k+4b+4)$ and remove the edges $(2, 4b+4), (2k+2, 2k+4b+4), (1, 4k-1)$ and $(2k-1, 2k+1)$ to obtain a path with end vertices 1 and $2k+1=4a+1$.

Note that $(2, 2k-1) \in E(G)$ because $2k-2 \neq 2$; hence, $(1, 2k-2) \notin E(G)$, k being even, and $(1, 2k-2)\sigma = (2, 2k-1)$.

(1b.2) Extend this path to a Hamiltonian path as in the procedure in (1b.1).

In the case where 2 and $4a+2$ are in different cycles, the same edges are removed and added, except that two cycles are involved initially, a cycle containing 2 and a cycle containing $4a+2=2k+2$.

(2) If $(1, 4b+3) \notin E(G)$, partition $L_2 \cup L_4$ into cycles generated by $(2, 4b+4)$.

Case 1: A cycle has more than two vertices.

(2a.1) Connect vertex 1 to $4b+4$ (or $4k-4b$) and then span the cycle containing it, ending at 2. Then connect 2 to the path $(4k-1, 4k-3, 4k-5, \ldots, 4b+3)$.

(2b.1) Remove (x_1, y_1) from the cycle containing 2 such that $(x_1, 4b+1)$ and $(y_1, 3)$ are edges of G. Then add the path $(x_1, 4b+1, 4b-1, 4b-3, \ldots, 3, y_1)$ to obtain a path spanning $OV(G)$ and the cycle containing 2 with end vertices 1 and $4b+3$. Then proceed as in (1b.1) to complete the Hamiltonian path.

Case 2: Each cycle is an edge (degenerate case).

(2a.2) In this case, $(2, 4a+2)$ generates cycles of L_2 and L_4 containing more than two vertices. Then proceed as in (1b.1), but replace the role of $4a$ by either $4k-4a$ or $4a$, whichever is adjacent to $4b+1$, and the role of $4a+1$ by $4b+3$.

(3) If $(1, 4a+2) \notin E(G)$, let $(2, 4b+4) \in E(G)$ and partition $L_2 \cup L_4$ into cycles generated by this edge.

Case 1: A cycle has more than two vertices.

(3a.1) Remove the path $(4a-4b, 4a+2, 4a+4b+4)$ in the cycle containing $4a+2$ and the edges $(1, 3)$ and $(4a+4b+3, 4a+4b+5)$ in the cycle generated by $(1, 3)$, then add the edges $(3, 4a+2), (4a+4b+3, 4a+4b+4)$ and $(4a-4b, 4a+4b+5)$, since $k \neq 2b+1$, to obtain a path with end vertices $4a+2$ and 1. Then proceed as in 1b.1) to complete the Hamiltonian path.

Case 2: A cycle has two vertices.

(3a.2) Remove $(1, 3)$ and $(4t+1, 4a+4b+3)$, where $4t+1$ is adjacent to $4a+4b+4$, from the cycle generated by $(1, 3)$. Add the edge $(3, 4a+2)$ and the path $4t+1, 4a+4b+4, 4a+4b+3)$ to obtain a path with end vertices 1 and $4a+2$. Then attach the rest of the degenerate cycles (y_i, z_i) by removing the edges (x_i, x_i+2), $x_i \in L_1$, $y_i \in L_4$, $z_i \in L_2$, and adding the edges (x_i, y_i) and (z_i, x_i+2) to obtain the required Hamiltonain path.

(4) If $(1, 4k) \notin E(G)$, partition $L_2 \cup L_4$ into cycles generated by $(2, 4b+4)$.

Case 1: A cycle has more than two vertices.

(4a.1) Obtain a path with end vertices 1 and $4k$ from the cycle generated by $(1, 3)$ and the cycle containing $4k$ by removing $(1, 4k-1), (4k-4b-3, x)$ and $(4k-4b-2, 4k, 4b+2)$, where $(x, 4b+2) \in E(G)$ and $x \in L_3$, and then adding the edges $(4k-1, 4k), (x, 4b+2$ and $(4k-4b-3, 4k-4b-2)$.

(4b.1) Complete the Hamiltonian path by following the procedure in (1b.1).

Case 2: A Cycle has only two vertices.

(4a.2) Partition instead L_2 and L_4 into cycles generated by the edge $(2,4c+2)$. Then obtain a path with end vertices 1 and $4k$ from the cycle generated by $(1,3)$ and the cycles containing $4k$ and $4k-4b-2$ by removing $(4k-4b-2,4c-4b-2),(4k,4c)$ and $(1,3)$ and then adding the edges $(4c-4b-2,4c)=(2,4b+4)\sigma^{4c-4b-4}$ and $(3,2k)$. The last edge exists because $2k\neq 2$.

(4b.2) Complete the Hamiltonian path by following the procedure in (1b.1).

(5) If $(2,4a+2)\notin E(G)$, partition $L_2\cup L_4$ into cycles generated by $(2,4b+4)$.

Case 1: A cycle has more than two vertices.

Case 1.1: Vertices 2 and $4a+2$ are in the same cycle.

(5a.1.1) Obtain a path with end vertices 2 and $4a+2$ by removing the edges $(2,4b+4)$ and $(4a+2,4a+4b+4)$ from the cycle containing 2 and the edge $(4a+4b+1,4a+4b+3)$ from the cycle generated by $(1,3)$ and then adding the edges $(4b+4,4a+4b+1)$ and $(4a+4b+3,4a+4b+4)$. If $(4b+4,4a+4b+1)\notin E(G)$, replace $4a+4b+1$ by $4a+4b+5$. Complete the Hamiltonian path by following (1b.1).

Case 1.2: Vertices 2 and $4a+2$ are in different cycles.

(5a.1.2) Obtain a path with end vertices 2 and $4a+2$ by removing $(2,4b+4)$ and $(4a+2,4a+4b+4)$ from the two cycles containing 2 and $4a+2$ and the edge $(4a+4b+1,4a+4b+3)$ from the cycle generated by $(1,3)$ and then adding the edges $(4b+4,4a+4b+1)$ and $(4a+4b+3,4a+4b+4)$. If $(4b+4,4a+4b+1)\notin E(G)$, then replace $4a+4b+1$ by $4a+4b+5$. Finally, complete the Hamiltonian path following the procedure in (1b.1).

Case 2: A cycle has only two vertices.

Treat this case as in (5a.1.1), except that no edge is removed since each partition is already a path.

(6) If $(2,4a+4)\notin E(G)$, partition $L_2\cup L_4$ into cycles generated by $(2,4b+4)$.

Case 1: A cycle has more than two vertices.

Case 1.1: Vertices 2 and $4a+4$ are in the same cycle.

(6a.1.1) Remove the edges $(4a-4b+2,4a+4)$ and $(2,4k-4b)$ from the cycle containing 2 and the edge $(1,3)$ from the cycle generated by $(1,3)$ and then add the edges $(1,4k-4b)$ and $(3,4a-4b+2)$ to obtain a path with end vertices 2 and $4a+4$. The last edge exists since $a\neq b$ and $b<k$. Complete the Hamiltonian path as in (1b.1).

Case 1.2: Vertices 2 and $4a+4$ are in different cycles.

(6a.1.2) Remove $(2,4b+4)$ and $(4a+4,4a-4b+2)$ from the cycles containing 2 and $4a+4$ and the edge $(1,3)$ from the cycle generated by $(1,3)$ and then add $(1,4b+4)$ and $(3,4a-4b+2)$ to obtain a path with end vertices 2 and $4a+4$. Complete the Hamiltonian path as in the procedure in (1b.1).

Case 2: There are two vertices in each cycle.

In this case obtain a Hamiltonian path with end vertices 2 and $4a+4$ as in (5a.2). This completes the proof. □

Theorem 3.4. *Let G be an SC graph having a CP* $\sigma=[1,2,\ldots,4k]$, $k\geqslant 2$, *and having the edges* $(1,2)$ *and* $(1,3)$. *If it is an SHSC graph such that* $N_{L_2}(1)=\{2\}$, *then it is Hamiltonian connected.*

4. Summary and recommendations

Strongly Hamiltonian self-complementary graphs (when $k \geqslant 3$) having properties (P1) and (P2) where $N_{L_2}(1) = L_2$ or $N_{L_2}(1) = \{2\}$ are also Hamiltonian connected. If the Hamiltonian connectedness of the classes $G'_n(4k), G''_n(4k)$ and $G'''_n(4k)$, where n is neither 1 or 2, is decided, then the question as to which strongly Hamiltonian self-complementary graphs with properties (P1) and (P2) are also Hamiltonian connected will have been settled.

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