Hamiltonian-connected self-complementary graphs

Luis D. Carrillo

MSU-IIT, Tibanga, Iligan City, 9200, Philippines

Received 27 November 1990
Revised 17 January 1992

Abstract

A self-complementary graph having a complementing permutation \( \sigma = [1, 2, 3, \ldots, 4k] \), consisting of one cycle, and having the edges \((1, 2)\) and \((1, 3)\) is strongly Hamiltonian iff it has an edge between two even-labelled vertices. Some of these strongly Hamiltonian self-complementary graphs are also shown to be Hamiltonian connected.

1. Introduction

**Definition 1.1.** A graph \( G = \langle V(G), E(G) \rangle \) is said to be self-complementary (SC) if there is a permutation \( \sigma \) on \( V(G) \) such that \((x, y) \in E(G)\) iff \((x, y) \not\in E(G)\). This permutation \( \sigma \) is called a complementing permutation (CP). The graph \( \tilde{G} \) in which \( V(\tilde{G}) = V(G) \) and \((x, y) \in E(\tilde{G})\) iff \((x, y) \not\in E(G)\) is called the complement of \( G \).

**Definition 1.2.** A graph is said to be Hamiltonian if it has a Hamiltonian cycle. If, in addition, every edge is contained in a Hamiltonian cycle, then it is said to be strongly Hamiltonian. Furthermore, if every pair of vertices are endpoints of a Hamiltonian path, then it is said to be Hamiltonian connected.

The self-complementary graphs \( G \) investigated in this paper are those with the following properties:

**(P1)** \( G \) has a CP \( \sigma = [1, 2, 3, \ldots, 4k] \), consisting of one cycle.

**(P2)** \( G \) has edges \((1, 2)\) and \((1, 3)\).

\( G \) obviously must have an even edge, i.e. an edge between two even-labelled vertices, to be strongly Hamiltonian; otherwise, it is almost constricted in the sense of Nash-Williams [4] and as such no odd edge can be contained in a Hamiltonian cycle. The details for the sufficiency of an even edge to make it strongly Hamiltonian are given in [1].

**Correspondence to:** Luis D. Carrillo, MSU-IIT, Tibanga, Iligan City, 9200, Philippines
Partition the vertex set $V(G)$ into $L_1 \cup L_2 \cup L_3 \cup L_4$, where $L_i = \{ x \in V(G) : x \equiv i (\text{mod} \ 4) \}$ for $i = 1, 2, 3, 4$. Following the observations of Clapham [2] and Gibbs [3], self-complementary graphs having properties (P1) and (P2) can be classified by means of the set $N_{L_i}(1)$, the set of elements in $L_2$ adjacent to vertex 1. The two subclasses discussed here are those in which $N_{L_1}(1) = L_2$ and $N_{L_2}(1) = \{2\}.$

Note that $N_{L_1}(1)$, by property (P2), contains vertex 2, so it can be chosen in $2^{k-1}$ ways, specifically, any subset of $L_2 \setminus \{2\}$, and then augmented by vertex 2.

Let the enumeration $\Phi$ of the possible neighbors of vertex 1 in $L_2$ be such that $\Phi(1) = L_2$ and $\Phi(2) = \{2\}$. Then associate with the enumeration $\Phi$ the following classes of self-complementary graphs with properties (P1) and (P2): $G_1(4k)$, the class where $N_{L_1}(1) = L_2$; $G_2(4k)$, the class where $N_{L_2}(1) = \{2\}$. Then the class $G_n(4k), n \neq 1, 2$, refers to a class where $N_{L_2}(1)$ is neither $L_2$ nor $\{2\}$.

2. Hamiltonian-connected self-complementary graphs, I

Rao [5] introduced an SC graph $G = G^*(4k)$ which is defined as follows:

(a) $V(G) = \{1, 2, 3, \ldots, 4k\}$;

(b) $(x, y) \in E(G)$ iff

1. $x, y \equiv 1, 3 (\text{mod} \ 4),$  
2. $x \equiv 1 (\text{mod} \ 4)$ and $y \equiv 2 (\text{mod} \ 4)$ or  
3. $x \equiv 3 (\text{mod} \ 4)$ and $y \equiv 0 (\text{mod} \ 4)$.

This graph is in $G_1(4k)$. In view of [4], it has no Hamiltonian cycle.

Now let $OE(G)$ be the set of odd edges of $G = G^*(4k)$. Let $(x, y) \in OE(G) \setminus C$, where $C = \{(1, 3)^{\sigma l} : l \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers. Remove the set 

$\{(x, y)^{\sigma 2l} : l \in \mathbb{N}\}$

and replace it by the set

$\{(x, y)^{\sigma 2l+1} : l \in \mathbb{N}\}$.

The resulting graph is still in $G_1(4k)$, and in view of [1] it is already a strongly Hamiltonian self-complementary graph.

Illustration

Note that the second graph is not Hamiltonian connected because vertices 1 and 5 cannot be endpoints of a Hamiltonian path.
The replacement of a set of odd edges by even edges can be obtained in three typical ways by the following constructions.

**Construction 2.1.** Let \( \emptyset \neq A \subseteq \{1, 2, \ldots, |k/2|\} \). Obtain the class \( G_i'(4k) \) from \( G*(4k) \) by replacing the set of edges

\[
\{(1, 4a + 1)_{2^{t}}: a \in A, t \in \mathbb{N}\}
\]

by the set of edges

\[
\{(2, 4a + 2)_{2^{t}}: a \in A, t \in \mathbb{N}\}.
\]

A graph in this class contains edges between vertices in \( L_2 \) but no edge between \( L_2 \) and \( L_4 \).

**Construction 2.2.** Let \( \emptyset \neq B \subseteq \{1, 2, \ldots, |k/2|\} \). Obtain the class \( G_i''(4k) \) from \( G*(4k) \) by replacing the set of edges

\[
\{(1, 4b + 3)_{2^{t}}: b \in B, t \in \mathbb{N}\}
\]

by the set of edges

\[
\{(2, 4b + 4)_{2^{t}}: b \in B, t \in \mathbb{N}\}.
\]

A graph in this class contains edges between \( L_2 \) and \( L_4 \) but not edges between vertices in \( L_2 \).

**Construction 2.3.** Let \( \emptyset \neq A, B \subseteq \{1, 2, \ldots, |k/2|\} \). Obtain the class \( G_i'''(4k) \) from \( G*(4k) \) by replacing the edges

\[
\{(1, 4a + 1)_{2^{t}}: a \in A, t \in \mathbb{N}\} \quad \text{and} \quad \{(1, 4b + 3)_{2^{t}}: b \in B, t \in \mathbb{N}\}
\]

by the set of edges

\[
\{(2, 4a + 2)_{2^{t}}: a \in A, t \in \mathbb{N}\} \quad \text{and} \quad \{(2, 4b + 4)_{2^{t}}: b \in B, t \in \mathbb{N}\}.
\]

A graph in this class contains edges between vertices in \( L_2 \) and edges between \( L_2 \) and \( L_4 \).

**Remark 2.4.** For \( k=3 \), \( G_i'(4k), G_i''(4k) \) and \( G_i'''(4k) \) have one element each and all three graphs can be verified to be Hamiltonian connected. These graphs are illustrated in Fig. 1.

**Lemma 2.5.** For \( k \geq 4 \), any element in \( G_i'(4k), G_i''(4k) \) or \( G_i'''(4k) \) is Hamiltonian connected.

**Proof.** In view of [1], it only remains to show that every pair of nonadjacent vertices are endpoints of a Hamiltonian path.
Fig. 1.
For $G$ in $G'_1(4k)$, every vertex in $L_1$ is adjacent to every vertex in $L_3$. Hence, any nonadjacent pair of vertices is an automorphic image of one of the pairs $1$ and $4a + 1$, $1$ and $4b + 4$, $2$ and $4a + 2$ or $2$ and $4b + 4$, where $1 \leq a, b \leq k - 1$. Together with the fact that every vertex of $L_1$ is adjacent to every vertex of $L_2$, a Hamiltonian path whose end points are any of the nonadjacent pairs above can easily be constructed.

For $G$ in $G''_1(4k)$, the subgraph induced by $L_1 (\cong L_3)$ is a complete graph of order $k$. Therefore, a HP whose endpoints are nonadjacent pairs of vertices of the form $1$ and $4b + 3$, $1$ and $4c + 3$, $1$ and $4c + 4$ with $(2, 4x + 2) \in E(G)$ for some $x$, $2$ and $4d + 2$ with $(2, 4y + 2) \in E(G)$ for some $y$, or $2$ and $4e + 4$. For these nonadjacent pairs, corresponding Hamiltonian paths are constructed below:

**Case 1:** $(1, 4a + 1) \notin E(G)$. Span $L_1 \cup L_2 \setminus \{4a + 1\}$ by a path with end vertices $1$ and $u \in L_1$. Also span $L_3 \cup L_4$ by a path with end vertices $u + 2$ and $4a + 3$. Then connect these paths by the edge $(u, u + 2)$ and add the edge $(4a + 1, 4a + 3)$ to obtain a HP with end vertices $1$ and $4a + 1$.

**Case 2:** $(1, 4c + 3) \notin E(G)$. Span $L_1 \cup L_2$ by a path with end vertices $1$ and $2$. Also span $L_3 \cup L_4$ by a path with end vertices $4c + 3$ and $4c + 4$. Then connect these paths by the edge $(2, 4c + 4)$.

**Case 3:** $(1, 4b + 4) \notin E(G)$ and $(2, 4x + 2) \in E(G)$. Span $L_1 \cup L_2$, using the edge $(2, 4x + 2)$, by a path with end vertices $1$ and $u \in L_1$ and span $L_3 \cup L_4$ by a path with end vertices $u + 2$ and $4b + 4$. Then connect these paths by the edge $(u, u + 2)$.

**Case 4:** $(2, 4y + 2) \in E(G)$ and $(2, 4y + 2) \in E(G)$. Span $L_1 \cup L_2 \setminus \{4d + 1, 4d + 2\}$ by a path with end vertices $2$ and $4x + 1$. Also span $L_3 \cup L_4$, using the edge $(2, 4y + 2)$, by a path with end vertices $4x + 3$ and $4d + 3$. Then connect these paths by the edge $(4x + 1, 4x + 3)$ and add the path $(4d + 3, 4d + 1, 4d + 2)$.

**Case 5:** $(2, 4e + 4) \notin E(G)$. Span $L_1 \cup L_2$ by a path with end vertices $2$ and $4x + 1$ and span $L_3 \cup L_4$ by a path with end vertices $4x + 3$ and $4e + 4$. Then connect these paths by the edge $(4x + 1, 4x + 3)$.

**Theorem 2.6.** Let $G$ be a self-complementary graph with properties (P1) and (P2). If $G$ is such that $N_{L_2}(1) = L_2$, then $G$ is Hamiltonian connected if it is strongly Hamiltonian and $k \geq 3$.

3. Hamiltonian-connected graphs, II

The classes $G'_1(4k), G''_1(4k)$ and $G'''_1(4k)$ obtained from $G^*(4k)$ have the property that $N_{L_2}(1) = L_2$. Now obtain the graphs $G'_2(4k), G''_2(4k)$ and $G'''_2(4k)$ from $G'_1(4k), G''_1(4k)$ and $G'''_1(4k)$, respectively, by removing the edges $(1, 4b + 2), b = 1, 2, 3, \ldots, k - 1$, and their automorphic images under the automorphism $\sigma^2$, $l \in \mathbb{N}$, and then replacing them by the edges $(1, 4k - 4b), 1 \leq b \leq k - 1$, with their automorphic images. Graphs in
these classes are strongly Hamiltonian self-complementary graphs with properties (P1) and (P2). However, they have the property that \( N_{\sigma}(1) = \{2\} \).

Now if \((2, 4b + 2) \in E(G)\), where \( G \in G'_2(4k) \), then \( L_2 \) and \( L_4 \) can be partitioned into disjoint cycles as follows.

**Step 1:** Define \( C_2^g = \{(2, 4b + 2)^{4nb} : 0 \leq n \leq k' - 1\} \), where \( b = db' \), \( k = dk' \) and \( d = \gcd(k, b) \).

**Step 2:** Define \( C_2^p = C_2^g \sigma^{4p} \), where \( 0 \leq p \leq d' - 1 \).

**Step 3:** Define \( C_2^p = C_2^g \sigma^{4p + 2} \), where \( 0 \leq p \leq d - 1 \).

Clearly, \( L_2 \) is a disjoint union of the cycles \( C_2^g \) and \( L_4 \) is the disjoint union of the cycles \( C_2^p \). Let \( x \in C_2^p, y \in C_2^p \) and \( x \equiv y \pmod{4k} \). Then \( x = 4nb + 4p + 2 \) and \( y = 4mb + 4q + 2 \) for some \( n, m \). Hence, \( x - y \equiv 0 \pmod{4k} \) implies that \( d \) divides \( p - q \) and \( k' \) divides \( n - m \) as observed in the equations

\[
4(n - m)db' + 4(p - q) = 4dk',
\]

\[
db' = b \quad \text{and} \quad dk' = k.
\]

These cycles are said to be *generated* by the edge \((2, 4b + 2)\) via the CP \( \sigma \). These cycles are degenerate if \( k = 2b \).

If \((2, 4b + 4) \in E(G)\), where \( G \in G'_2(4k) \), then \( L_2 \cup L_4 \) can be partitioned into disjoint cycles as follows.

**Step 1:** Define \( C_0 = \{(2, 4b + 4)^{4b+2} : 0 \leq n \leq k^* - 1\} \), where \( 2b + 1 = db^*, 2k = dk^*, d' = \gcd(2k, 2b + 1) \).

**Step 2:** Define \( C_p = C_0 \sigma^{4p} \), where \( 0 \leq p \leq d' - 1 \). The cycles \( C_p \) can easily be shown to be disjoint and span \( L_2 \cup L_4 \). These cycles are degenerate if \( k = 2b + 1 \).

**Lemma 3.1.** For \( k \geq 4 \), any element \( G \in G'_2(4k) \) is Hamiltonian connected.

**Proof.** Note here that \((4a + 1, 4b + 3) \in E(G)\) for all \( a, b = 0, 1, 2, \ldots, k - 1 \). With these edges, the required Hamiltonian path for every nonadjacent pair of vertices can be obtained as automorphic images of the Hamiltonian paths constructed below:

1. \((1, 4b + 1) \notin E(G)\). Construct a required Hamiltonian path by the following steps:
   1a) Partition \( L_2 \) and \( L_4 \) into cycles generated by the edge \((2, 4b + 2)\). Correspondingly partition \( L_1 \) and \( L_3 \) in such a way that if \( x \) is in a partition of \( L_2 \) or \( L_4 \), then \( x - 1 \) is in the corresponding partition of \( L_1 \) or \( L_3 \).
   1b) Let \( x \in L_3 \) be such that \((2, x) \in E(G)\). Starting from vertex 1, span by a path this partition of \( L_1 \), except \( 4b + 1 \), and a partition of \( L_3 \) not containing \( x \), if any, together with its cycle partition of \( L_4 \) and ending at a vertex in \( L_3 \). This is illustrated in Fig. 2.
   1c) From the last vertex in \( L_3 \), span one partition each of \( L_1, L_2, L_3 \) and \( L_4 \) at a time, each time ending at a vertex in \( L_3 \), and finally end at \( x \).
(1d) Complete the Hamiltonian path by connecting \( x \) to 2, spanning the last cycle of \( I_2 \), ending at \( 4b + 2 \), then go finally to \( 4b + 1 \).

(2) \((1, 4a + 2) \not\in E(G)\) for \(1 \leq a \leq k - 1\). Let \((2, 4b + 2) \in E(G)\) and do the partition as in (1). Let \( y (\neq 1) \in L_1 \) be such that \((y + 1, 4a + 2)\) is an edge in a partition of \( L_2 \). Construct a required Hamiltonian path under the following cases.
Case 1: Vertices 1 and y are in the same partition.

Case 1.1: Vertex $y \neq 4a + 1$.

(2a.1.1) Subsumed in this subcase is the fact that each partition has more than two vertices. From vertex 1 span the partition containing it, except y, and a partition in $L_3$ together with its corresponding partition of $L_4$, ending at a vertex in $L_3$. This is illustrated in Fig. 3.

(2b.1.1) From the last vertex in $L_3$, span a partition each from $L_1, L_2, L_3$ and $L_4$ at a time, and each time ending at a vertex in $L_3$, until all of $L_3$ is spanned.

(2c.1.1) Complete the Hamiltonian path by joining the last vertex in $L_3$ to y, then to $y+1$, spanning the last cycle-partition of $L_2$, then ending at $4a + 2$.

Fig. 3.
Case 1.2: $y = 4a + 1$. This degenerate case happens only when $a = b = k/2$, i.e. a partition of $L_2$ is an edge, and $(2, 4b + 2) = (2, 2k + 2)$. In this case, span $V(G)$ by the paths

$$(1, 2, 2k + 2, 2k + 1)\sigma^{2l} \text{ for } 0 \leq l \leq k - 1,$$

remove the edge $(2, 2k + 2)$ and add the edges $(2, 2k + 3), (2k - 1, 2k + 1), (3, 5)\sigma^{4m}$ and $(2k + 5, 2k + 7)\sigma^{4m}$ for $m = 0, 1, 2, \ldots, (k - 4)/2$ to obtain a Hamiltonian path with end vertices 1 and $2k + 2$. This is illustrated in Fig. 4.

Case 2: Vertices 1 and $y$ are not in the same partition.

(2a.2) In this case choose $y$ so that $y \neq 4a + 1$ and $(y + 1, 4a + 2) \in E(G)$. Then from vertex 1, span the partition containing it and a partition in $L_3$ together with their corresponding partitions in $L_2$ and $L_4$, ending at a vertex in $L_3$. This is illustrated in Fig. 5.

\[ A \text{ is even} \]
(2b.2) Span a partition each of $L_1, L_2, L_3$ and $L_4$ at a time, each time ending at a vertex in $L_3$, until only the partition containing $y$ is left unspanned in $L_1$.

(2c.2) Complete the Hamiltonian path by joining the last vertex in $L_3$ to $z (\neq y) \in L_1$, then proceed to a vertex in $L_3$, then to a vertex in $L_4$, spanning the rest of $L_4$, then go back to a vertex in $L_3$, spanning the rest of $L_1$ and $L_3$, ending at $y$. 
Finally, connect $y$ to $y + 1$ and span the rest of $L_2$, ending at $4a + 2$.

(3) $(1, 4k) \notin E(G)$

(3a) Let $(2,4b + 2) \in E(G)$ and partition $V(G)$ as in (1a) and let $x (\neq 4k - 1) \in L_3$ be such that $(x + 1, 4k)$ is an edge of a cycle-partition.

If there is only one partition of $L_2$, then from vertex 1 go to vertex 2 and span $L_2$, then connect to $W (\neq x)$ in $L_3$ and go alternatively between $L_1$ and $L_3$, ending at a vertex $x$. Complete the Hamiltonian path by connecting $x$ to $x + 1$ and spanning $L_4$ up to $4k$.

(3b) If there are $l > 1$ partitions of $L_2$, span $l - 1$ partitions each of $L_1, L_2, L_3$ and $L_4$ from vertex 1 and ending at a vertex in $L_3$, leaving out among others the partition containing $x$. This is illustrated in Fig. 6.

(3c) Connect the last vertex in $L_3$ to a vertex in $L_2$, then span the rest of $L_2$, then go to a vertex in $L_1$. From there, go alternatively between $L_1$ and $L_3$, ending at a vertex $x$.

(3d) Complete the Hamiltonian path by connecting $x$ to $x + 1$ and span the rest of $L_4$ up to $4k$.

(4) $(2,4a + 2) \notin E(G)$ and $(2,4b + 2) \in E(G)$.

Case 1. Vertices 2 and $4a + 2$ are not in the same cycle generated by $(2,4b + 2)$.

(4a.1) From vertex 2, span the partition containing it, then proceed to a vertex in $L_1$; then span the partition of $L_1$ containing it and a partition of $L_3$ together with its corresponding partition in $L_4$, ending at a vertex in $L_1$.

(4b.1) From the last vertex in $L_1$, span one partition each of $L_1, L_2, L_3$ and $L_4$ at a time, each time ending at a vertex in $L_1$, until only the partition containing $4a + 2$ is left in $L_2$.

(4c.1) Connect the last vertex in $L_1$ to a vertex in $L_4$ and span the rest of $L_4$. Then proceed to a vertex in $L_3$ and go back and forth between $L_1$ and $L_3$, ending at $x$ in $L_1$, where $(x + 1,4a + 2)$ is an edge in a cycle-partition of $L_2$.

(4d.1) Complete the Hamiltonian path by connecting $x$ to $x + 1$, then spanning the rest of $L_2$ up to $4a + 2$.

Case 2. Vertices 2 and $4a + 2$ are in the same cycle generated by $(2,4b + 2)$.

(4a.2) Let $y$ be adjacent to $4a + 2$ in a cycle-partition such that the paths from 2 to $y$ and from $4b + 2$ to $4a + 2$ span this cycle. Further, let $z \in L_3$ be such that $(z,4b + 2) \in E(G)$. Then from 2, span part of the cycle to $y$, connect $y$ to $y - 1$ and span this partition of $L_1$ and a partition of $L_3$ not containing $z$, if any, together with its corresponding partition in $L_4$, ending at a vertex in $L_1$.

If there is only one partition of $L_2$, let $w \in L_4$ be such that $(y - 1, w) \in E(G)$. From $y - 1$, proceed instead to $w$ and span $L_4$, ending at some vertex $x (\neq z + 1)$, then connect to $x - 1$ and alternate between $L_1$ and $L_3$, ending at $z$. Finally, connect to $4b + 2$ and span the rest of $L_2$ up to $4a + 2$.

(4b.2) From the last vertex in $L_1$, span one partition each of $L_1, L_2, L_3$ and $L_4$ at a time, each time ending at $L_1$, until only one partition each of $L_1, L_2, L_3$ and $L_4$ and the path from $4b + 2$ to $4a + 2$ are left unspanned.

(4c.2) From the last vertex in $L_1$, go to a vertex in $L_4$, span the rest of $L_4$, then connect to a vertex in $L_3$. From there, span the rest of $L_3$ and $L_1$ with the
corresponding partition in $L_2$, ending at $z \in L_3$. Finally, connect $z$ to $4b + 2$ and span the rest of $L_2$ up to $4a + 2$.

(5) $(2, 4a + 4) \notin E(G)$. 

(5a) Let $(2, 4b + 2) \in E(G)$ and partition $V(G)$ as in (1a). Let $x \in L_1$ and $y \in L_4$ be such that $(x, y) \in E(G)$, $(y, 4a + 4)$ is an edge in a cycle-partition and $x$ is not in the partition.
containing 1, if there is more than one cycle generated by \( (2, 4b + 2) \). From vertex 2, span the partition containing it, then connect to a vertex in the corresponding partition of \( L_1 \). Span this partition of \( L_1 \), together with a partition of \( L_3 \) not containing \( y - 1 \), and its corresponding partition in \( L_4 \), then end at a vertex in \( L_1 \).

(5b) From the last vertex in \( L_1 \), span one partition each from \( L_1, L_2, L_3 \) and \( L_4 \) at a time, each time ending at a vertex in \( L_1 \), until only one partition each of \( L_1, L_2, L_3 \) and \( L_4 \) is left.

In case there is only one partition, from 2 span the rest of \( L_2 \), then go to a vertex in \( L_3 \), then alternate between \( L_1 \) and \( L_3 \), ending at \( x + L_1 \). Finally, connect \( x \) to \( y \) and span the rest of \( L_4 \) up to vertex \( 4a + 4 \).

(5c) From the last vertex in \( L_1 \), connect to a vertex in \( L_3 \), then to a vertex in \( L_2 \), spanning the rest of \( L_2 \), then go to a vertex in \( L_1 \) and alternate between \( L_1 \) and \( L_3 \), ending at \( x + L_1 \). Finally, connect \( x \) to \( y \) and span the rest of \( L_4 \) up to vertex \( 4a + 4 \) \( \square \)

Lemma 3.2. For \( k \geq 4 \), if \( G \in G_2^v(4k) \), then \( G \) is Hamiltonian connected.

Proof. Here, the subgraph induced by \( L_1 \) (\( \cong L_3 \)) is a complete graph. A Hamiltonian path whose endpoints are nonadjacent vertices can be obtained as an automorphic image of one of the HP's constructed below:

(1) \((1, 4b + 3) \notin E(G)\).

(1a) Partition \( L_2 \cup L_4 \) into cycles generated by the edge \((2, 4b + 4)\).

Case 1: There is an odd number of cycles.

Case 1.1: There is only one cycle.

(1b.1.1) From vertex 1, span \( L_1 \), then join the last vertex to a vertex in \( L_4 \), then span \( L_2 \cup L_4 \), ending at a vertex in \( L_2 \). Then connect to a vertex in \( L_3 \) (\( \neq 4b + 3 \)) and span \( L_3 \), ending at \( 4b + 3 \).

Case 1.2: There are at least three cycles.

(1b.1.2) Connect vertex 1 to a vertex in \( L_4 \) and span the cycle containing it, ending at a vertex in \( L_2 \).

(1c.1.2) Connect the last vertex in \( L_2 \) to a vertex in \( L_3 \) (\( \neq 4b + 3 \)), then go back to a vertex in \( L_2 \), spanning the cycle containing it and ending at a vertex in \( L_4 \). Proceed to a vertex in \( L_1 \), return to a vertex in \( L_4 \), spanning the cycle containing it and ending at a vertex in \( L_2 \). Repeat the process until only one cycle is left, the last vertex spanned being in \( L_4 \).

(1d.1.2) From the last vertex in \( L_4 \), go to a vertex in \( L_1 \), span the rest of \( L_1 \), then proceed to a vertex in \( L_4 \), spanning the last cycle, ending at a vertex in \( L_2 \).

(1e.1.2) Complete the Hamiltonian path by proceeding to a vertex in \( L_3 \) and spanning the rest of \( L_3 \) up to the vertex \( 4b + 3 \).

Case 2. There is an even number of cycles.

(1b.2) From vertex 1, go instead to vertex 2, then span the cycle containing it, ending at a vertex in \( L_4 \). Then proceed as in case 1.2 to obtain the required Hamiltonian path.
(2) $(1, 4a + 2) \notin E(G)$. Let $(2, 4b + 4) \in E(G)$ and partition $L_2 \cup L_4$ into cycles generated by $(2, 4b + 4)$.

Case 1: $(2, 4b + 4)$ generates only one cycle.

(2a.1) Let $x \in L_1, y \in L_3$ be such that $(x, 4a + 4b + 4)$ and $(y, 4a + 2) \in E(G)$. Span $L_1$ by starting at $1$ and ending at $x$. Then go to $4a + 4b + 4$ and go through the vertices of the cycle up to $4a - 4b$, leaving out $4a + 2$. Then proceed to a vertex in $L_3$, then span $L_3$, ending at $y$. Finally, connect $y$ to $4a + 2$. If $a + b + 1 - k$, reverse the roles of $4a + 4b + 4 = 4k$ and $4a - 4b$.

Case 2: $(2, 4b + 4)$ generates at least two cycles.

Case 2.1: There is an odd number of cycles and each cycle has more than two vertices.

(2a.2.1) Let $y$ be as in (2a.1). From $1$ go to $4a + 4b + 4$ and span the cycle up to $4a - 4b$, leaving out $4a + 2$. Then go to $4a - 4b - 1$, then to a vertex in $L_2$, spanning the cycle containing it and ending at a vertex in $L_4$.

(2b.2.1) From this vertex of $L_4$ go to a vertex in $L_1$, then return to $L_4$, spanning a cycle and ending at a vertex in $L_2$. Repeat the process until only one cycle is left, with the last vertex traversed in $L_4$.

(2c.2.1) Connect the last vertex in $L_4$ to a vertex in $L_1$, span the rest of $L_3$, then go to a vertex in $L_4$, spanning the last cycle, and ending at a vertex in $L_2$.

(2d.2.1) Finally, continue on to a vertex in $L_3$, spanning the rest of $L_3$ up to vertex $y$, and then to $4a + 2$.

Case 2.2: There is an odd number of cycles and each cycle is an edge (degenerate case).

(2a.2.2) Let $x$ be as in (2a.1). Then from vertex $1$ go to a vertex in $L_4 (\neq 4a + 4b + 4)$, then span the edge. Go to a vertex in $L_3$, continue to a vertex in $L_2 (\neq 4a + 2)$, then span the edge and go to a vertex in $L_1$. Repeat the process until only two edges are left with the last vertex spanned in $L_2$.

(2b.2.2) Proceed to a vertex in $L_3$, span the rest of $L_3$ and go to a vertex in $L_2$, then span the edge and proceed to $L_1$, spanning the rest of $L_1$ and ending at vertex $x$. Finally, join $x$ to $4a + 4b + 4$ and then go to $4a + 2$.

Case 2.3: There is an even number of cycles and each cycle has more than two vertices.

(2a.2.3) Proceed as in case 2.1, but instead connect the vertex $4a - 4b$ to a vertex in $L_1$. The effect is to end up at $y$. Proceed finally to $4a + 2$.

(3) $(1, 4k) \notin E(G)$. Let $(2, 4b + 4) \in E(G)$ and partition $L_2 \cup L_4$ as in (1a). Then proceed as in (2) but interchange the roles of $x$ and $y$, and replace the roles of $4a + 2, 4a + 4b + 4,$ and $4a - 4b$ by $4k, 4b + 2$, and $4k - 4b - 2$, respectively, and start from vertex $1$, then to $3$ and let $3$ take the role of $1$, except in Case 1.2, where $4k - 1$ takes the role of $y$.

(4) $(2, 4a + 2) \notin E(G) \forall 1 \leq a \leq k - 1$.

Case 1: Vertices $4a + 2$ and $2$ are in the same cycle generated by $(2, 4b + 4)$.

Case 1.1: There is only one cycle.

(4a.1.1) From vertex $2$ span the cycle through $4b + 4$ up to $4a - 4b$, then connect the latter to a vertex in $L_3$, then span $L_3$ and connect the last vertex to $4a + 4b + 4$, etc.
leaving out $4a+2$. Then span the rest of the cycle up to $4k-4b$, connect this to a vertex in $L_1$, span $L_1$, then connect the last vertex to $4a+2$.

**Case 1.2:** There are at least two cycles.

(4a.1.2) From vertex 2, span the vertices as in (4a.1.1) up to $4k-4b$, but traverse only one vertex in $L_3$. Then proceed as in (2b.2.1) until only one cycle is left. If the last vertex is in $L_2$, connect this to a vertex in $L_3$, span $L_3$ then return to a vertex in $L_2$ and span the last cycle to a vertex in $L_4$. Then go to a vertex in $L_1$, span the rest of $L_1$, ending at $4a+1$. Finally, go to $4a+2$. On the other hand, if the last vertex is in $L_4$, connect this to a vertex in $L_1$, span the rest of $L_1$, except $4a+1$, then go back to a vertex in $L_4$ and span the last cycle, ending at a vertex in $L_2$. Finally, connect this to a vertex in $L_3$ $(\neq 4a+3)$, span the rest of $L_3$ up to $4a+3$ and then connect $4a+1$ to vertex $4a+2$.

**Case 2:** Vertices $4a+2$ and 2 are not in the same cycle.

(4a.2) Span from 2 the cycle containing it up to $4b+4$, then go to $4b+3$, then to $4a+4b+4$, spanning the cycle up to $4a-4b$, leaving out $4a+2$.

(4b.2) If there are only two cycles, go from vertex $4a-4b$ to a vertex in $L_3$, span the rest of $L_3$, then connect to a vertex in $L_1$, span the rest of $L_1$ ending at $x$, where $(x,4a+2)\in E(G)$, and, finally, connect the last vertex to $4a+2$. If there are more than two cycles, proceed as in (4a.1.2).

(5) $(2,4a+4)\notin E(G)$. Let $(2,4b+4)\in E(G)$. Then proceed as in (4) but let $4a+4$ take the role of $4a+2$, taking into consideration the adjacencies between $L_1$ and $L_2\cup L_4$. □

**Lemma 3.3.** For $k \geq 4$, if $G \in G_k(4k)$, then $G$ is Hamiltonian connected.

**Proof.** From the Hamiltonian paths constructed below, a Hamiltonian path whose end vertices are nonadjacent can be obtained as an automorphic image of one of them.

(1) $(1,4a+1)\notin E(G)$: Partition $L_2$ and $L_4$ into cycles generated by $(2,4a+2)$. Then, using the edge $(2,4b+4)$, obtain a cycle from a cycle of $L_2$ and a cycle of $L_4$ for each pair of cycles connected by some edge of the form $(2,4b+4)a_4$. This is illustrated in Fig. 7.

**Case 1:** A cycle in $L_2$ has more than two vertices.

(1a.1) First, choose the connected pair of cycles containing $4k-4b-2$ and $4k$.

Obtain a path from these cycles by deleting the edges $(4k-4b-2,4k-4a-4b-2)$, $(4k,4a)$ and $(4k,4k-4a)$ and adding the edges $(4k-4b-2,4k)$ and $(4k-4a-4b-2,4k-4a)$. Then extend this to a path with end vertices 1 and $4a+1$ by adding the edges $(4k,4k-1)$ and $(4a,4a-1)$ and the cycle generated by the edge $(1,3)$, where the edges $(1,4k-1)$ and $(4a-1,4a+1)$ are deleted.

(1b.1) Extend this path to a Hamiltonian path by adding the edges $(y_i, y_i-1)$, $(y_i+1, y_i-4a)$ or $(y_i+1, y_i+4a)$, whichever applies, where $y_i \in L_4$ for $i = 1, 2, \ldots, p-1$ and $p$ is the number of cycles in $L_2$ generated by $(2,4a+2)$, and by deleting the edges $(y_i, y_i-4a)$ or $(y_i, y_i+4a)$, whichever applies.

**Case 2:** A cycle in $L_2$ is an edge (degenerate case).
In this case the cycles generated by the edge $(2, 4b + 4)$ cannot be degenerate because $k$ must be even, i.e. $k = 2a$; therefore, the cycles generated by $(2, 4b + 4)$ have at least 4 vertices.

(1a.2) Partition $L_2 \cup L_4$ into cycles generated by $(2, 4b + 4)$ and suppose that $4a + 2 = 2k + 2$ and 2 are in the same cycle. To the cycle generated by $(1, 3)$ and the cycle generated by $(2, 4b + 4)$ containing 2, add the edges $(2, 2k - 1), (2k + 2, 4k - 1)$ and
Hamiltonian-connected self-complementary graphs

$$(4b + 4, 2k + 4b + 4)$$ and remove the edges $(2, 4b + 4), (2k + 2, 2k + 4b + 4), (1, 4k - 1)$ and $(2k - 1, 2k + 1)$ to obtain a path with end vertices 1 and $2k + 1 = 4a + 1$.

Note that $(2, 2k - 1) \in E(G)$ because $2k - 2 \neq 2$, hence, $(1, 2k - 2) \notin E(G)$, $k$ being even, and $(1, 2k - 2) \sigma = (2, 2k - 1)$.

(1b.2) Extend this path to a Hamiltonian path as in the procedure in (1b.1).

In the case where 2 and $4a + 2$ are in different cycles, the same edges are removed and added, except that two cycles are involved initially, a cycle containing 2 and a cycle containing $4a + 2 = 2k + 2$.

(2) If $(1, 4b + 3) \notin E(G)$, partition $L_2 \cup L_4$ into cycles generated by $(2, 4b + 4)$.

Case 1: A cycle has more than two vertices.

(2a.1) Connect vertex 1 to $4b + 4$ (or $4k - 4b$) and then span the cycle containing it, ending at 2. Then connect 2 to the path $(4k - 1, 4k - 3, 4k - 5, \ldots, 4b + 3)$.

(2b) Remove $(x_1, y_1)$ from the cycle containing 2 such that $(x_1, 4b + 1)$ and $(y_1, 3)$ are edges of $G$. Then add the path $(x_1, 4b + 1, 4b - 1, 4b - 3, \ldots, 3, y_1)$ to obtain a path spanning $OV(G)$ and the cycle containing 2 with end vertices 1 and $4b + 3$. Then proceed as in (1b.1) to complete the Hamiltonian path.

Case 2: Each cycle is an edge (degenerate case).

(2a.2) In this case, $(2, 4a + 2)$ generates cycles of $L_2$ and $L_4$ containing more than two vertices. Then proceed as in (1b.1), but replace the role of $4a$ by either $4k - 4a$ or $4a$, whichever is adjacent to $4b + 1$, and the role of $4a + 1$ by $4b + 3$.

(3) If $(1, 4a + 2) \notin E(G)$, let $(2, 4b + 4) \in E(G)$ and partition $L_2 \cup L_4$ into cycles generated by this edge.

Case 1: A cycle has more than two vertices.

(3a.1) Remove the path $(4a - 4b, 4a + 2, 4a + 4b + 4)$ in the cycle containing $4a + 2$ and the edges $(1, 3)$ and $(4a + 4b + 3, 4a + 4b + 5)$ in the cycle generated by $(1, 3)$, then add the edges $(3, 4a + 2), (4a + 4b + 3, 4a + 4b + 4)$ and $(4a - 4b, 4a + 4b + 5)$, since $k \neq 2b + 1$, to obtain a path with end vertices $4a + 2$ and 1. Then proceed as in 1b.1) to complete the Hamiltonian path.

Case 2: A cycle has two vertices.

(3a.2) Remove $(1, 3)$ and $(4t + 1, 4a + 4b + 3)$, where $4t + 1$ is adjacent to $4a + 4b + 4$, from the cycle generated by $(1, 3)$. Add the edge $(3, 4a + 2)$ and the path $4t + 1, 4a + 4b + 4, 4a + 4b + 3$ to obtain a path with end vertices 1 and $4a + 2$. Then attach the rest of the degenerate cycles $(y_i, z_i)$ by removing the edges $(x_i, x_i + 2), x_i \in L_1, y_i \in L_4, z_i \in L_2$, and adding the edges $(x_i, y_i)$ and $(z_i, x_i + 2)$ to obtain the required Hamiltonian path.

(4) If $(1, 4k) \notin E(G)$, partition $L_2 \cup L_4$ into cycles generated by $(2, 4b + 4)$.

Case 1: A cycle has more than two vertices.

(4a.1) Obtain a path with end vertices 1 and $4k$ from the cycle generated by $(1, 3)$ and the cycle containing $4k$ by removing $(1, 4k - 1), (4k - 4b - 3, x)$ and $(4k - 4b - 2, 4k, 4b + 2)$, where $(x, 4b + 2) \in E(G)$ and $x \in L_3$, and then adding the edges $(4k - 1, 4k), (x, 4b + 2)$ and $(4k - 4b - 3, 4k - 4b - 2)$.

(4b) Complete the Hamiltonian path by following the procedure in (1b.1).

Case 2: A Cycle has only two vertices.
(4a.2) Partition instead $L_2$ and $L_4$ into cycles generated by the edge $(2, 4c + 2)$. Then obtain a path with end vertices 1 and $4k$ from the cycle generated by $(1, 3)$ and the cycles containing $4k$ and $4k - 4b - 2$ by removing $(4k - 4b - 2, 4c - 4b - 2), (4k, 4c)$ and $(1, 3)$ and then adding the edges $(4c - 4b - 2, 4c) = (2, 4b + 4) a^{4c - 4b - 4}$ and $(3, 2k)$. The last edge exists because $2k \neq 2$.

(4b.2) Complete the Hamiltonian path by following the procedure in (1b.1).

(5) If $(2, 4a + 2) \notin E(G)$, partition $L_2 \cup L_4$ into cycles generated by $(2, 4b + 4)$.

Case 1: A cycle has more than two vertices.

Case 1.1: Vertices 2 and $4a + 2$ are in the same cycle.

(5a.1.1) Obtain a path with end vertices 2 and $4a + 2$ by removing the edges $(2, 4b + 4)$ and $(4a + 2, 4a + 4b + 4)$ from the cycle containing 2 and the edge $(4a + 4b + 1, 4a + 4b + 3)$ from the cycle generated by $(1, 3)$ and then adding the edges $(4b + 4, 4a + 4b + 1)$ and $(4a + 4b + 3, 4a + 4b + 4)$. If $(4b + 4, 4a + 4b + 1) \notin E(G)$, replace $4a + 4b + 1$ by $4a + 4b + 5$. Complete the Hamiltonian path by following (1b.1).

Case 1.2: Vertices 2 and $4a + 2$ are in different cycles.

(5a.1.2) Obtain a path with end vertices 2 and $4a + 2$ by removing $(2, 4b + 4)$ and $(4a + 2, 4a + 4b + 4)$ from the two cycles containing 2 and $4a + 2$ and the edge $(4a + 4b + 1, 4a + 4b + 3)$ from the cycle generated by $(1, 3)$ and then adding the edges $(4b + 4, 4a + 4b + 1)$ and $(4a + 4b + 3, 4a + 4b + 4)$. If $(4b + 4, 4a + 4b + 1) \notin E(G)$, then replace $4a + 4b + 1$ by $4a + 4b + 5$. Finally, complete the Hamiltonian path following the procedure in (1b.1).

Case 2: A cycle has only two vertices.

Treat this case as in (5a.1.1), except that no edge is removed since each partition is already a path.

(6) If $(2, 4a + 4) \notin E(G)$, partition $L_2 \cup L_4$ into cycles generated by $(2, 4b + 4)$.

Case 1: A cycle has more than two vertices.

Case 1.1: Vertices 2 and $4a + 4$ are in the same cycle.

(6a.1.1) Remove the edges $(4a - 4b + 2, 4a + 4)$ and $(2, 4k - 4b)$ from the cycle containing 2 and the edge $(1, 3)$ from the cycle generated by $(1, 3)$ and then add the edges $(1, 4k - 4b)$ and $(4a - 4b + 2) + (3, 4a - 4b + 2)$ to obtain a path with end vertices 2 and $4a + 4$. The last edge exists since $a \neq b$ and $b < k$. Complete the Hamiltonian path as in (1b.1).

Case 1.2: Vertices 2 and $4a + 4$ are in different cycles.

(6a.1.2) Remove $(2, 4b + 4)$ and $(4a + 4, 4a - 4b + 2)$ from the cycles containing 2 and $4a + 4$ and the edge $(1, 3)$ from the cycle generated by $(1, 3)$ and then add $(1, 4b + 4)$ and $(3, 4a - 4b + 2)$ to obtain a path with end vertices 2 and $4a + 4$. Complete the Hamiltonian path as in the procedure in (1b.1).

Case 2: There are two vertices in each cycle.

In this case obtain a Hamiltonian path with end vertices 2 and $4a + 4$ as in (5a.2). This completes the proof. □

**Theorem 3.4.** Let $G$ be an $SC$ graph having a CP $\sigma = [1, 2, \ldots , 4k]$, $k \geq 2$, and having the edges $(1, 2)$ and $(1, 3)$. If it is an SHSC graph such that $N_{L_2}(1) = \{2\}$, then it is Hamiltonian connected.
4. Summary and recommendations

Strongly Hamiltonian self-complementary graphs (when \( k \geq 3 \)) having properties (P1) and (P2) where \( N_{L_2}(1) = L_2 \) or \( N_{L_2}(1) = \{2\} \) are also Hamiltonian connected. If the Hamiltonian connectedness of the classes \( G_{n}(4k), G_{n}^{\prime}(4k) \) and \( G_{n}^{\prime \prime}(4k) \), where \( n \) is neither 1 or 2, is decided, then the question as to which strongly Hamiltonian self-complementary graphs with properties (P1) and (P2) are also Hamiltonian connected will have been settled.

References