J. Math. Anal. Appl. 373 (2011) 464–474

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Dirichlet series of Rankin–Cohen brackets

YoungJu Choie [a,∗,1], Min Ho Lee [b,2]

[a] Department of Mathematics and PMI (Pohang Mathematical Institute), POSTECH, Pohang 790-784, Republic of Korea
[b] Department of Mathematics, University of Northern Iowa, Cedar Falls, IA 50614, USA

ARTICLE INFO

Article history:
Received 4 January 2010
Available online 11 August 2010
Submitted by Richard M. Aron

Keywords:
Quasimodular forms
Modular forms
Dirichlet series

ABSTRACT

Given modular forms f and g of weights k and ℓ, respectively, their Rankin–Cohen bracket $[f,g]_n^{(k,\ell)}$ corresponding to a nonnegative integer n is a modular form of weight $k+\ell+2n$, and it is given as a linear combination of the products of the form $f^{(r)}g^{(n-r)}$ for $0\leqslant r\leqslant n$. We use a correspondence between quasimodular forms and sequences of modular forms to express the Dirichlet series of a product of derivatives of modular forms as a linear combination of the Dirichlet series of Rankin–Cohen brackets.

1. Introduction and statement of main results

Given modular forms f and g of weights k and ℓ, respectively, their Rankin–Cohen bracket $[f,g]_n^{(k,\ell)}$ corresponding to a nonnegative integer n is a modular form of weight $k+\ell+2n$, and it is given as a linear combination of the products of the form $f^{(r)}g^{(n-r)}$ for $0\leqslant r\leqslant n$ (see e.g. [3]). Although such products are not modular forms, they are quasimodular forms.

Quasimodular forms generalize classical modular forms and first introduced by Kaneko and Zagier in [6]. It appears naturally in various places (see [4,5,7], for instance). One of the useful features of quasimodular forms is that their derivatives are also quasimodular forms (see [1,9]). In particular, derivatives of modular forms are quasimodular forms. Since products of quasimodular forms are quasimodular forms, it follows that the products $f^{(r)}g^{(n-r)}$ considered above are quasimodular forms. As in the case of modular forms, we can consider the Dirichlet series associated to quasimodular forms by using their Fourier coefficients. From the formula for Rankin–Cohen brackets it follows that the Dirichlet series of the modular forms $[f,g]_n^{(k,\ell)}$ can be written as the Dirichlet series of the quasimodular forms $f^{(r)}g^{(n-r)}$. The goal of this paper is to express the Dirichlet series of a product of derivatives of modular forms in terms of the Dirichlet series of Rankin–Cohen brackets. More precisely, we prove the following theorem:

Theorem 1.1. *Given a discrete subgroup Γ, containing translations, of $SL(2,\mathbb{R})$, let ϕ and ψ be modular forms for Γ with width h and weights μ and ν, respectively. Then the Dirichlet series of the quasimodular form $\phi^{(m)}\psi^{(n)}$ can be written in the form*

$$L\big(\phi^{(m)}\psi^{(n)},s\big)=\sum_{\ell=0}^{m+n}a_{\mu,\nu}^{m,n}(\ell)L\big([\phi,\psi]_{m+n-\ell}^{(\mu,\nu)},s-\ell\big),\tag{1.1}$$

where

∗ Corresponding author.
E-mail addresses: yjc@postech.ac.kr (Y.J. Choie), lee@math.uni.edu (M.H. Lee).

1 Supported in part by NRF20090083909 and NRF2009-0094069.
2 Supported in part by a summer fellowship from the University of Northern Iowa.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.07.055

$$a_{\mu,\nu}^{m,n}(\ell) = \frac{(2\pi i)^{\ell} n!(\mu+m-1)!(\nu+n-1)!(\mu+\nu+2\ell-1)!}{(\mu+\ell-1)!(\mu+\nu+2m+2n-\ell-1)!h^{\ell}} \times \sum_{j=0}^{\ell}(-1)^j \frac{(m+n-\ell+j)!(2\ell+\mu+\nu-j-2)!}{j!(\ell-j)!(n-\ell+j)!(\nu+\ell-j-1)!} \tag{1.2}$$

for $0 \leqslant \ell \leqslant m+n$.

The proof of this theorem is carried out by using a correspondence between quasimodular forms and sequences of modular forms discussed in [8]. For example, each quasimodular form can be written as a linear combination derivatives of a finite number of modular forms.

2. Quasimodular and modular polynomials

In this section we describe $SL(2,\mathbb{R})$-equivariant automorphisms of spaces of polynomials introduced in [8] which determine correspondences between quasimodular polynomials and modular polynomials.

Let $\mathcal{H}$ be the Poincaré upper half plane, and let $\mathcal{F}$ be the ring of holomorphic functions $f:\mathcal{H}\to\mathbb{C}$ satisfying the following growth condition

$$\left|f(z)\right| \ll \left(\frac{\operatorname{Im} z}{1+|z|^2}\right)^{-\nu} \tag{2.1}$$

for some $\nu > 0$ (see e.g. [9, Section 17.1] for a more precise description of this condition). We fix a nonnegative integer m and denote by $\mathcal{F}_m[X]$ the complex vector space of polynomials in X over $\mathcal{F}$ of degree at most m. Thus $\mathcal{F}_m[X]$ consists of polynomials of the form

$$\Phi(z,X) = \sum_{r=0}^{m} \phi_r(z) X^r \tag{2.2}$$

with $\phi \in \mathcal{F}$ for $0 \leqslant r \leqslant m$. If $\Phi(z,X) \in \mathcal{F}_m[X]$ is as in (2.2) and if λ is an integer with $\lambda > 2m$, we set

$$\left(\Lambda_\lambda^m \Phi\right)(z,X) = \sum_{r=0}^{m} \phi_r^{\Lambda}(z) X^r, \qquad \left(\Xi_\lambda^m \Phi\right)(z,X) = \sum_{r=0}^{m} \phi_r^{\Xi}(z) X^r \tag{2.3}$$

where

$$\phi_r^{\Lambda} = \frac{1}{r!} \sum_{\ell=0}^{m-r} \frac{1}{\ell!(\lambda-2r-\ell-1)!} \phi_{m-r-\ell}^{(\ell)}, \tag{2.4}$$

$$\phi_r^{\Xi} = (\lambda+2r-2m-1) \sum_{\ell=0}^{r} \frac{(-1)^{\ell}}{\ell!} (m-r+\ell)!(2r+\lambda-2m-\ell-2)! \phi_{m-r+\ell}^{(\ell)}, \tag{2.5}$$

for each $r \in \{0,1,\dots,m\}$. Then it can be shown that the resulting maps

$$\Lambda_\lambda^m, \Xi_\lambda^m : \mathcal{F}_m[X] \to \mathcal{F}_m[X]$$

are complex linear isomorphisms with

$$\left(\Xi_\lambda^m\right)^{-1} = \Lambda_\lambda^m$$

(see [8]).

The group $SL(2,\mathbb{R})$ acts on the Poincaré upper half plane $\mathcal{H}$ as usual by linear fractional transformations, so that

$$\gamma z = \frac{az+b}{cz+d}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. For the same z and γ, by setting

$$\mathfrak{J}(\gamma,z) = cz+d, \qquad \mathfrak{K}(\gamma,z) = \frac{c}{cz+d}, \tag{2.6}$$

we obtain the maps $\mathfrak{J}, \mathfrak{K}: SL(2,\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ which satisfy

$$\mathfrak{J}(\gamma\gamma',z) = \mathfrak{J}(\gamma,\gamma' z)\mathfrak{J}(\gamma',z), \qquad \mathfrak{K}(\gamma,\gamma' z) = \mathfrak{J}(\gamma',z)^2\left(\mathfrak{K}(\gamma\gamma',z) - \mathfrak{K}(\gamma',z)\right)$$

for all $z \in \mathcal{H}$ and $\gamma, \gamma' \in SL(2,\mathbb{R})$.

Given $\Phi(z, X) \in \mathcal{F}_m[X]$ as in (2.2) and elements $f \in \mathcal{F}$, $\gamma \in SL(2, \mathbb{R})$ and $\lambda \in \mathbb{Z}$, we set

$$(f \mid_\lambda \gamma)(z) = \mathfrak{J}(\gamma, z)^{-\lambda} f(\gamma z), \tag{2.7}$$

$$\big(\Phi \mid_\lambda^X \gamma\big)(z, X) = \sum_{r=0}^{m} (\phi_r \mid_{\lambda+2r} \gamma)(z) X^r, \tag{2.8}$$

$$(\Phi \parallel_\lambda \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} \Phi\big(\gamma z, \mathfrak{J}(\gamma, z)^2\big(X - \mathfrak{K}(\gamma, z)\big)\big) \tag{2.9}$$

for all $z \in \mathcal{H}$. Then $|_\lambda$ determines an action of $SL(2, \mathbb{R})$ on $\mathcal{H}$ as usual, and the other two operations $|_\lambda^X$ and $\parallel_\lambda$ determine actions of the same group on $\mathcal{F}_m[X]$. If Λ_λ^m and Ξ_λ^m are the linear automorphisms of $\mathcal{F}_m[X]$ given by (2.3), then it is known that

$$\big(\big(\Lambda_\lambda^m \Phi\big) \parallel_\lambda \gamma\big)(z, X) = \Lambda_\lambda^m \big(\Phi \mid_{\lambda-2m}^X \gamma\big)(z, X), \tag{2.10}$$

$$\big(\big(\Xi_\lambda^m \Phi\big) \mid_{\lambda-2m}^X \gamma\big)(z, X) = \Xi_\lambda^m (\Phi \parallel_\lambda \gamma)(z, X) \tag{2.11}$$

for all $\gamma \in SL(2, \mathbb{R})$ (cf. [8]).

We now consider a discrete subgroup Γ of $SL(2, \mathbb{R})$. Then an element $f \in \mathcal{F}$ is a *modular form for Γ of weight λ* if it satisfies

$$f \mid_\lambda \gamma = f$$

for all $\gamma \in \Gamma$, where the operation $|_\lambda$ is given by (2.7).

Definition 2.1. Let $|_\lambda^X$ and $\parallel_\lambda$ with $\lambda \in \mathbb{Z}$ be the operations in (2.8) and (2.9).

(i) An element $F(z, X) \in \mathcal{F}_m[X]$ is a *modular polynomial for Γ of weight λ and degree at most m* if it satisfies

$$F \mid_\lambda^X \gamma = F$$

for all $\gamma \in \Gamma$.

(ii) An element $\Phi(z, X) \in \mathcal{F}_m[X]$ is a *quasimodular polynomial for Γ of weight λ and degree at most m* if it satisfies

$$\Phi \parallel_\lambda \gamma = \Phi$$

for all $\gamma \in \Gamma$.

We denote by $MP_\lambda^m(\Gamma)$ and $QP_\lambda^m(\Gamma)$ the spaces of modular and quasimodular, respectively, polynomials for Γ of weight λ and degree at most m. If a polynomial $F(z, X) \in \mathcal{F}_m[X]$ of the form

$$F(z, X) = \sum_{r=0}^{m} f_r(z) X^r$$

belongs to $MP_\lambda^m(\Gamma)$, from (2.8) and Definition 2.1(i) we see that

$$f_r \in M_{\lambda+2r}(\Gamma) \tag{2.12}$$

for $0 \leqslant r \leqslant m$. From (2.10), (2.11) and Definition 2.1 it follows that the automorphisms Λ_λ^m and Ξ_λ^m of $\mathcal{F}_m[X]$ induce the isomorphisms

$$\Lambda_\lambda^m : MP_{\lambda-2m}^m(\Gamma) \to QP_\lambda^m(\Gamma), \qquad \Xi_\lambda^m : QP_\lambda^m(\Gamma) \to MP_{\lambda-2m}^m(\Gamma) \tag{2.13}$$

for each $\lambda \geqslant 2m$.

Example 2.2. Given an integer $\lambda \geqslant 2m$, we consider two modular forms

$$\xi \in M_{\lambda-2m}(\Gamma), \qquad \eta \in M_{\lambda-2m+2}(\Gamma)$$

and the associated modular polynomial

$$F(z, X) = \sum_{r=0}^{m} f_r(z) X^r \in MP_{\lambda-2m}(\Gamma)$$

with

$$f_r = \begin{cases} \xi & \text{if } r = 0; \\ \eta & \text{if } r = 1; \\ 0 & \text{if } 2 \leqslant r \leqslant m. \end{cases}$$

If $\Lambda_\lambda^m F(z,X) = \sum_{r=0}^m f_k^\Lambda(z) X^r \in QP_\lambda^m(\Gamma)$ is the corresponding quasimodular polynomial, from (2.4) we obtain

$$\begin{aligned} f_k^\Lambda &= \frac{f_0^{(m-k)}}{k!(m-k)!(\lambda-k-m-1)!} + \frac{f_1^{(m-k-1)}}{k!(m-k-1)!(\lambda-k-m)!} \\ &= \frac{(\lambda-k-m)\xi^{(m-k)} + (m-k)\eta^{(m-k-1)}}{k!(m-k)!(\lambda-k-m)!}. \end{aligned}$$

Thus we have

$$\Lambda_\lambda^m F(z,X) = \sum_{k=0}^m \frac{(\lambda-k-m)\xi^{(m-k)}(z) + (m-k)\eta^{(m-k-1)}(z)}{k!(m-k)!(\lambda-k-m)!} X^k. \tag{2.14}$$

3. Quasimodular forms

In this section we discuss the correspondence between quasimodular polynomials and quasimodular forms. We also express the Dirichlet series of a quasimodular form in terms of Dirichlet series of the modular forms associated to the corresponding quasimodular polynomial.

Let $\mathcal{F}$ be the ring of holomorphic functions on $\mathcal{H}$ satisfying (2.1) as in Section 2, and let Γ be a discrete subgroup of $SL(2,\mathbb{R})$ containing translations.

Definition 3.1. Given integers m and λ with $m \geqslant 0$, an element $f \in \mathcal{F}$ is a *quasimodular form for Γ of weight λ and depth at most m* if there are functions $f_0, \dots, f_m \in \mathcal{F}$ such that

$$(f \mid_\lambda \gamma)(z) = \sum_{r=0}^m f_r(z) \mathfrak{K}(\gamma, z)^r \tag{3.1}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$, where $\mathfrak{K}(\gamma, z)$ is as in (2.6) and $|_\lambda$ is the operation in (2.7). We denote by $QM_\lambda^m(\Gamma)$ the space of quasimodular forms for Γ of weight λ and depth at most m.

Quasimodular forms correspond to quasimodular polynomials as described below. If $0 \leqslant \ell \leqslant m$, we consider the complex linear map

$$\mathfrak{S}_\ell : \mathcal{F}_m[X] \to \mathcal{F}$$

defined by

$$\mathfrak{S}_\ell\left(\sum_{r=0}^m \phi_r(z) X^r\right) = \phi_\ell(z)$$

for all $z \in \mathcal{H}$. Then it can be shown (cf. [2]) that

$$\mathfrak{S}_\ell\big(QP_\lambda^m(\Gamma)\big) \subset QM_{\lambda-2\ell}^{m-\ell}(\Gamma);$$

hence we obtain the map

$$\mathfrak{S}_\ell : QP_\lambda^m(\Gamma) \to QM_{\lambda-2\ell}^{m-\ell}(\Gamma) \tag{3.2}$$

for each ℓ. For $\ell = 0$ it is known that the map

$$\mathfrak{S}_0 : QP_\lambda^m(\Gamma) \to QM_\lambda^m(\Gamma)$$

is an isomorphism whose inverse is the map

$$\mathcal{Q}_\lambda^m : QM_\lambda^m(\Gamma) \to QP_\lambda^m(\Gamma) \tag{3.3}$$

defined by

$$\big(\mathcal{Q}_\lambda^m f\big)(z,X) = \sum_{r=0}^m f_r(z) X^r$$

for a quasimodular form $f \in QM_\lambda^m(\Gamma)$ and functions $f_0, \dots, f_m \in \mathcal{F}$ as in (3.1).

Let $\psi \in QM_\lambda^m(\Gamma)$ be a quasimodular form whose Fourier expansion is of the form

$$\psi(z) = \sum_{k=0}^\infty a_k e^{2\pi i k z/h} \tag{3.4}$$

with $h \in \mathbb{R}$, so that the corresponding Dirichlet series is given by

$$L(\psi, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{3.5}$$

where it converges when $\operatorname{Re} s \gg 0$. For $0 \leqslant r \leqslant m$, using (2.12) and the isomorphisms in (2.13) and (3.3), we see that the function $(\mathfrak{S}_r \circ \Xi_\lambda^m \circ \mathcal{Q}_\lambda^m)\psi$ is a modular form belonging to $M_{\lambda-2m+2r}(\Gamma)$. We now set

$$f_r^\psi = (\mathfrak{S}_r \circ \Xi_\lambda^m \circ \mathcal{Q}_\lambda^m)\psi \in M_{\lambda-2m+2r}(\Gamma), \tag{3.6}$$

and assume that its Fourier expansion is given by

$$f_r^\psi(z) = \sum_{k=0}^{\infty} c_{r,k} e^{2\pi ikz/h}. \tag{3.7}$$

Thus the corresponding Dirichlet series can be written as

$$L(f_r^\psi, s) = \sum_{n=1}^{\infty} \frac{c_{r,n}}{n^s}, \tag{3.8}$$

where it converges for $\operatorname{Re} s \gg 0$.

Proposition 3.2. *The Dirichlet series in* (3.5) *and* (3.8) *satisfy the relation*

$$L(\psi, s) = \sum_{\ell=0}^{m} \frac{(2\pi i)^\ell}{\ell!(\lambda-\ell-1)!h^\ell} L(f_{m-\ell}^\psi, s-\ell), \tag{3.9}$$

where it converges for $\operatorname{Re} s \gg 0$.

Proof. From (3.6) we see that

$$((\Xi_\lambda^m \circ \mathcal{Q}_\lambda^m)\psi)(z, X) = \sum_{r=0}^{m} f_r^\psi(z) X^r \in MP_{\lambda-2m}^m(\Gamma).$$

Applying the isomorphism Λ_λ^m in (2.13) to this relation and using (2.3) and (2.4), we have

$$(\mathcal{Q}_\lambda^m \psi)(z, X) = ((\Lambda_\lambda^m \circ \Xi_\lambda^m \circ \mathcal{Q}_\lambda^m)\psi)(z, X) = \sum_{r=0}^{m} \psi_r(z) X^r \in QP_\lambda^m(\Gamma),$$

where

$$\psi_r = \frac{1}{r!} \sum_{\ell=0}^{m-r} \frac{1}{\ell!(\lambda-2r-\ell-1)!} (f_{m-r-\ell}^\psi)^{(\ell)}$$

for $0 \leqslant r \leqslant m$. From this and the fact that $(\mathcal{Q}_\lambda^m)^{-1} = \mathfrak{S}_0$ we obtain

$$\psi = \mathfrak{S}_0(\mathcal{Q}_\lambda^m \psi) = \psi_0 = \sum_{\ell=0}^{m} \frac{1}{\ell!(\lambda-\ell-1)!} (f_{m-\ell}^\psi)^{(\ell)}.$$

Using (3.7), we have

$$(f_{m-\ell}^\psi)^{(\ell)}(z) = \sum_{k=0}^{\infty} c_{m-\ell,k} \left(\frac{2\pi ik}{h}\right)^\ell e^{2\pi ikz/h};$$

hence we obtain

$$\psi(z) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{m} \frac{(2\pi ik)^\ell c_{m-\ell,k}}{\ell!(\lambda-\ell-1)!h^\ell} e^{2\pi ikz/h}.$$

Comparing this with (3.4), we have

$$a_k = \sum_{\ell=0}^{m} \frac{(2\pi ik)^\ell c_{m-\ell,k}}{\ell!(\lambda-\ell-1)!h^\ell}$$

for $k \geqslant 0$. Thus from (3.5) and (3.8) we see that

$$\begin{aligned} L(\psi, s) &= \sum_{n=1}^{\infty} \sum_{\ell=0}^{m} \frac{(2\pi i n)^\ell c_{m-\ell,n}}{\ell!(\lambda-\ell-1)! h^\ell n^s} \\ &= \sum_{\ell=0}^{m} \sum_{n=1}^{\infty} \frac{(2\pi i)^\ell c_{m-\ell,n}}{\ell!(\lambda-\ell-1)! h^\ell n^{s-\ell}} \\ &= \sum_{\ell=0}^{m} \frac{(2\pi i)^\ell}{\ell!(\lambda-\ell-1)! h^\ell} L\big(f^{\psi}_{m-\ell}, s-\ell\big); \end{aligned}$$

hence the proposition follows. □

4. Proof of Theorem 1.1

We first recall that the Rankin–Cohen brackets are the bilinear maps

$$[\,,\,]_w^{(k,\ell)} : M_k(\Gamma) \times M_\ell(\Gamma) \to M_{k+\ell+2w}(\Gamma)$$

defined by

$$[f, g]_w^{(k,\ell)} = \sum_{r=0}^{w} (-1)^r \binom{k+w-1}{w-r} \binom{\ell+w-1}{r} f^{(r)} g^{(w-r)} \tag{4.1}$$

for $k, \ell \in \mathbb{Z}$, $w \geqslant 0$, $f \in M_k(\Gamma)$ and $g \in M_\ell(\Gamma)$ (see e.g. [10,3]). It is known that the Rankin–Cohen brackets are unique up to constant. More precisely, if

$$B_w : M_k(\Gamma) \times M_\ell(\Gamma) \to M_{k+\ell+2w}(\Gamma)$$

is a bilinear differential operator, there is a constant $c \in \mathbb{C}$ such that

$$B_w(f, g) = c[f, g]_w^{(k,\ell)}$$

for all $(f, g) \in M_k(\Gamma) \times M_\ell(\Gamma)$.

Proof of Theorem 1.1. Given nonnegative integers μ and ν, we consider modular forms $\phi \in M_\mu(\Gamma)$ and $\psi \in M_\nu(\Gamma)$. Then their derivatives $\phi^{(m)}$ and $\psi^{(n)}$ with $m, n \geqslant 0$ are quasimodular forms with

$$\phi^{(m)} \in QM^m_{\mu+2m}, \qquad \psi^{(n)} \in QM^n_{\nu+2n};$$

hence we see that $\phi^{(m)}\psi^{(n)}$ is a quasimodular form belonging to $QM^{m+n}_{\mu+\nu+2m+2n}(\Gamma)$. By setting $\lambda = \mu + 2m$, $\xi = \phi$ and $\eta = 0$ in (2.14), we obtain

$$\Lambda^m_{\mu+2m} \Phi(z, X) = \sum_{k=0}^{m} \frac{\phi^{(m-k)}}{k!(m-k)!(\mu+m-k-1)!} X^k \in QP^m_{\mu+2m}(\Gamma), \tag{4.2}$$

$$\big(\big(\mathfrak{S}_0 \circ \Lambda^m_{\mu+2m}\big)\Phi\big)(z) = \frac{\phi^{(m)}}{m!(\mu+m-1)!} \in QM^m_{\mu+2m}(\Gamma).$$

Similarly, if $\psi \in M_\nu(\Gamma)$, we have

$$\Lambda^n_{\nu+2n} \Psi(z, X) = \sum_{\ell=0}^{n} \frac{\psi^{(n-\ell)}(z)}{\ell!(n-\ell)!(\nu+n-\ell-1)!} X^\ell \in QP^n_{\nu+2n}(\Gamma),$$

$$\big(\big(\mathfrak{S}_0 \circ \Lambda^n_{\nu+2n}\big)\Psi\big)(z) = \frac{\psi^{(n)}}{n!(\nu+n-1)!} \in QM^n_{\nu+2n}(\Gamma).$$

Thus, if we set

$$F(z, X) = \big(\Lambda^m_{\mu+2m} \Phi(z, X)\big) \cdot \big(\Lambda^n_{\nu+2m} \Psi(z, X)\big),$$

it is a quasimodular polynomial belonging to $QP^{m+n}_{\mu+\nu+2m+2n}(\Gamma)$, and we obtain

$$\begin{aligned} F(z,X) &= \sum_{k=0}^{m}\sum_{\ell=0}^{n} \frac{\phi^{(m-k)}(z)\psi^{(n-\ell)}(z)}{k!\ell!(m-k)!(n-\ell)!(\mu+m-k-1)!(\nu+n-\ell-1)!} X^{k+\ell} \\ &= \sum_{r=0}^{m+n}\sum_{k=0}^{r} K_{k,r}^{m,n;\mu,\nu} \phi^{(m-k)}(z)\psi^{(n-r+k)}(z)X^r \in QP_{\mu+\nu+2m+2n}^{m+n}(\Gamma), \end{aligned}$$

where

$$K_{k,r}^{m,n;\mu,\nu} = \frac{1}{k!(r-k)!(m-k)!(n-r+k)!(\mu+m-k-1)!(\nu+n-r+k-1)!}$$

for $0 \leqslant k \leqslant r \leqslant m+n$; here we assume that $\phi^{(a)} = 0$ and $\psi^{(b)} = 0$ for $a, b < 0$. Using (2.3) and (2.5), we have

$$\Xi_{\mu+\nu+2m+2n}^{m+n} F(z,X) = \sum_{j=0}^{m+n} \phi_j^{\Xi}(z)X^j \in MP_{\mu+\nu}^{m+n}(\Gamma),$$

where

$$\phi_\ell^{\Xi} = (\mu+\nu+2\ell-1)\sum_{j=0}^{\ell} \frac{(-1)^j}{j!}(m+n-\ell+j)!(2\ell+\mu+\nu-j-2)!\phi_{m+n-\ell+j}^{(j)}$$

with

$$\phi_r = \sum_{k=0}^{r} K_{k,r}^{m,n;\mu,\nu}\phi^{(m-k)}\psi^{(n-r+k)}$$

for $0 \leqslant \ell, r \leqslant m+n$. However, we have

$$\begin{aligned} \phi_{m+n-\ell+j}^{(j)} &= \sum_{k=0}^{m+n-\ell+j} K_{k,m+n-\ell+j}^{m,n;\mu,\nu}\big(\phi^{(m-k)}\psi^{(\ell+k-m-j)}\big)^{(j)} \\ &= \sum_{k=0}^{m+n-\ell+j} K_{k,m+n-\ell+j}^{m,n;\mu,\nu}\sum_{p=0}^{j}\binom{j}{p}\phi^{(m-k+j-p)}\psi^{(\ell+k+p-m-j)}; \end{aligned}$$

hence it follows that

$$\begin{aligned} \phi_\ell^{\Xi} = (\mu+\nu+2\ell-1)\sum_{j=0}^{\ell}\sum_{k=0}^{m+n-\ell+j}\sum_{p=0}^{j}\frac{(-1)^j}{j!}\binom{j}{p}(m+n-\ell+j)!(2\ell+\mu+\nu-j-2)!K_{k,m+n-\ell+j}^{m,n;\mu,\nu} \\ \times\phi^{(m-k+j-p)}\psi^{(\ell+k+p-m-j)} \end{aligned} \tag{4.3}$$

for $0 \leqslant \ell \leqslant m+n$. Since $\phi_\ell^{\Xi} \in M_{\mu+\nu+2\ell}(\Gamma)$, we obtain the bilinear differential operator

$$(\phi,\psi) \mapsto \phi_\ell^{\Xi} : M_\mu(\Gamma)\times M_\nu(\Gamma) \to M_{\mu+\nu+2\ell}(\Gamma)$$

on $M_\mu(\Gamma)\times M_\nu(\Gamma)$ for each ℓ. Thus, using the uniqueness of Rankin–Cohen brackets, we see that

$$\phi_\ell^{\Xi} = b_\ell[\phi,\psi]_\ell^{(\mu,\nu)} \tag{4.4}$$

for some $b_\ell \in \mathbb{C}$. Using $p = j$ and $k = m$, the coefficient of $\phi\psi^{(\ell)}$ in (4.3) is given by

$$(\mu+\nu+2\ell-1)\sum_{j=0}^{\ell}\frac{(-1)^j}{j!}(m+n-\ell+j)!(2\ell+\mu+\nu-j-2)!K_{m,m+n-\ell+j}^{m,n;\mu,\nu}, \tag{4.5}$$

where

$$K_{m,m+n-\ell+j}^{m,n;\mu,\nu} = \frac{1}{m!(n-\ell+j)!(\ell-j)!(\mu-1)!(\nu+\ell-j-1)!}.$$

On the other hand, by (4.1) the coefficient of $\phi\psi^{(\ell)}$ in (4.4) is equal to

$$\binom{\mu+\ell-1}{\ell}b_\ell = \frac{(\mu+\ell-1)!b_\ell}{\ell!(\mu-1)!}.$$

Comparing this with (4.5), we obtain

$$b_\ell = \frac{(\mu+\nu+2\ell-1)\ell!}{(\mu+\ell-1)!m!} \sum_{j=0}^{\ell} (-1)^j \frac{(m+n-\ell+j)!(2\ell+\mu+\nu-j-2)!}{j!(n-\ell+j)!(\ell-j)!(\nu+\ell-j-1)!} \tag{4.6}$$

for $0 \leqslant \ell \leqslant m+n$. Since we have

$$\mathfrak{S}_0 F = \frac{\phi^{(m)}\psi^{(n)}}{m!n!(\mu+m-1)!(\nu+n-1)!} \in QM^{m+n}_{\mu+\nu+2m+2n}(\Gamma),$$

it follows that

$$\left(\mathfrak{S}_r \circ \Xi^{m+n}_{\mu+\nu+2m+2n} \circ \mathcal{Q}^{m+n}_{\mu+\nu+2m+2n}\right)(\mathfrak{S}_0 F) = \left(\mathfrak{S}_r \circ \Xi^{m+n}_{\mu+\nu+2m+2n}\right) F = \phi^{\Xi}_r$$

for $0 \leqslant r \leqslant m+n$. Thus, using Proposition 3.2, we obtain

$$\begin{aligned} L\left(\phi^{(m)}\psi^{(n)}, s\right) &= m!n!(\mu+m-1)!(\nu+n-1)! \sum_{\ell=0}^{m+n} \frac{(2\pi i)^\ell}{\ell!(\mu+\nu+2m+2n-\ell-1)!h^\ell} L\left(\phi^{\Xi}_{m+n-\ell}, s-\ell\right) \\ &= m!n!(\mu+m-1)!(\nu+n-1)! \sum_{\ell=0}^{m+n} \frac{(2\pi i)^\ell b_\ell}{\ell!(\mu+\nu+2m+2n-\ell-1)!h^\ell} L\left([\phi,\psi]^{(\mu,\nu)}_{m+n-\ell}, s-\ell\right); \end{aligned}$$

hence the theorem follows from this and (4.6). □

5. Examples

In this section we consider two modular forms

$$\phi \in M_\mu(\Gamma), \qquad \psi \in M_\nu(\Gamma),$$

and provide examples of the formula (1.1) for $(m,n) = (1,1), (2,0), (0,2)$.

We first consider the case where $m = n = 1$ by regarding the given modular forms as modular polynomials

$$\Phi(z,X) = \phi(z) \in MP^1_\mu(\Gamma), \qquad \Psi(z,X) = \psi(z) \in MP^1_\nu(\Gamma).$$

Then from (4.2) we see that

$$\left(\Lambda^1_{\mu+2}\Phi(z,X)\right)\cdot\left(\Lambda^1_{\nu+2}\Psi(z,X)\right) = \frac{1}{\mu!\nu!}\left[\phi'(z)\psi'(z) + \left(\nu\phi'(z)\psi(z) + \mu\phi(z)\psi'(z)\right)X + \mu\nu\phi(z)\psi(z)X^2\right],$$

which is a quasimodular form belonging to $QP^2_{\mu+\nu+4}(\Gamma)$. Thus, if we set

$$F(z,X) = \mu!\nu!\left(\Lambda^1_{\mu+2}\Phi(z,X)\right)\cdot\left(\Lambda^1_{\nu+2}\Psi(z,X)\right),$$

we have

$$F(z,X) = f_0(z) + f_1(z)X + f_2(z)X^2 \in QP^2_{\mu+\nu+4}(\Gamma),$$

where

$$f_0 = \phi'\psi' = \mathfrak{S}_0 F \in QM^2_{\mu+\nu+4}(\Gamma), \qquad f_1 = \nu\phi'\psi + \mu\phi\psi', \qquad f_2 = \mu\nu\phi\psi.$$

Using (2.5), we have

$$\left(\Xi^2_{\mu+\nu+4}F\right)(z,X) = \sum_{r=0}^{2} f^{\Xi}_r(z)X^r,$$

where

$$\begin{aligned} f^{\Xi}_0 &= 2\mu\nu(\mu+\nu-1)!\phi\psi, \\ f^{\Xi}_1 &= (\mu+\nu+1)(\mu+\nu-1)!(\mu-\nu)\left(\mu\phi\psi' - \nu\phi'\psi\right), \\ f^{\Xi}_2 &= -(\mu+\nu+3)(\mu+\nu)! \times \left(\mu(\mu+1)\phi\psi'' - 2(\mu+1)(\nu+1)\phi'\psi' + \nu(\nu+1)\phi''\psi\right). \end{aligned}$$

However, using (4.1), we have

$$\begin{aligned}
&[\phi,\psi]_0^{(\mu,\nu)} = \phi\psi,\\
&[\phi,\psi]_1^{(\mu,\nu)} = \mu\phi\psi' - \nu\phi'\psi,\\
&[\phi,\psi]_2^{(\mu,\nu)} = \frac{1}{2}\big(\mu(\mu+1)\phi\psi'' - 2(\mu+1)(\nu+1)\phi'\psi' + \nu(\nu+1)\phi''\psi\big).
\end{aligned}\tag{5.1}$$

Thus we see that

$$\begin{aligned}
&f_0^{\Xi} = 2\mu\nu(\mu+\nu-1)![\phi,\psi]_0^{(\mu,\nu)},\\
&f_1^{\Xi} = (\mu-\nu)(\mu+\nu+1)(\mu+\nu-1)![\phi,\psi]_1^{(\mu,\nu)},\\
&f_2^{\Xi} = -2(\mu+\nu+3)(\mu+\nu)![\phi,\psi]_2^{(\mu,\nu)}.
\end{aligned}$$

From this and (3.9) with $\lambda = \mu+\nu+4$ we obtain

$$\begin{aligned}
L(\phi'\psi', s) &= \sum_{j=0}^{2} \frac{(2\pi i)^j}{j!(\mu+\nu+3-j)!h^j} L\big(f_{2-j}^{\Xi}, s-j\big)\\
&= -\frac{2}{(\mu+\nu+2)(\mu+\nu+1)} L\big([\phi,\psi]_2^{(\mu,\nu)}, s\big)\\
&\quad + \frac{2\pi i(\mu-\nu)}{(\mu+\nu+2)(\mu+\nu)h} L\big([\phi,\psi]_1^{(\mu,\nu)}, s-1\big)\\
&\quad + \frac{(2\pi i)^2\mu\nu}{(\mu+\nu+1)(\mu+\nu)h^2} L\big([\phi,\psi]_0^{(\mu,\nu)}, s-2\big).
\end{aligned}\tag{5.2}$$

We now consider the case where $m=2$ and $n=0$ by regarding the given modular forms as the modular polynomials

$$\Phi(z,X) = \phi(z) \in MP_\mu^2(\Gamma), \qquad \Psi(z,X) = \psi(z) \in MP_\nu^0(\Gamma),$$

so that from (4.2) we obtain

$$\big(\Lambda_{\mu+4}^2\Phi(z,X)\big)\cdot\big(\Lambda_\nu^0\Psi(z,X)\big) = \frac{1}{(\nu-1)!}\left(\frac{\phi''(z)\psi(z)}{2(\mu+1)!} + \frac{\phi'(z)\psi(z)}{\mu!}X + \frac{\phi(z)\psi(z)}{2(\mu-1)!}X^2\right),$$

which is a quasimodular polynomial belonging to $QP_{\mu+\nu+4}^2(\Gamma)$. Thus, if we set

$$G(z,X) = 2(\nu-1)!(\mu+1)!\big(\Lambda_{\mu+4}^2\Phi(z,X)\big),$$

then we have

$$G(z,X) = g_0(z) + g_1(z)X + g_2(z)X^2 \in QP_{\mu+\nu+4}^2(\Gamma),$$

where

$$g_0 = \phi''\psi = \mathfrak{S}_0 G \in QM_{\mu+\nu+4}^2(\Gamma), \qquad g_1 = 2(\mu+1)\phi'\psi, \qquad g_2 = \mu(\mu+1)\phi\psi.$$

Using this, (2.5) and (5.1), we see that

$$\big(\Xi_{\mu+\nu+4}^2 G\big)(z,X) = \sum_{r=0}^{2} g_r^{\Xi}(z)X^r,$$

where

$$\begin{aligned}
g_0^{\Xi} &= 2(\mu+\nu-1)!g_2 = 2\mu(\mu+1)(\mu+\nu-1)![\phi,\psi]_0^{(\mu,\nu)},\\
g_1^{\Xi} &= (\mu+\nu+1)\big((\mu+\nu)!g_1 - 2(\mu+\nu-1)!g_2'\big)\\
&= 2(\mu+\nu+1)(\mu+\nu-1)!\big(\nu\phi'\psi - \mu\phi\psi'\big)\\
&= -2(\mu+\nu+1)(\mu+\nu-1)![\phi,\psi]_1^{(\mu,\nu)},\\
g_2^{\Xi} &= (\mu+\nu+3)\big((\mu+\nu+2)!g_0 - (\mu+\nu+1)!g_1' + (\mu+\nu)!g_2''\big)\\
&= (\mu+\nu+3)(\mu+\nu)!\big((\mu+\nu+2)(\mu+\nu+1)\phi''\psi - 2(\mu+1)(\mu+\nu+1)\big(\phi''\psi+\phi'\psi'\big)\\
&\quad + \mu(\mu+1)\big(\phi''\psi + 2\phi'\psi' + \phi\psi''\big)\big)\\
&= (\mu+\nu+3)(\mu+\nu)!\big(\nu(\nu+1)\phi''\psi + 2(\mu+1)(\nu+1)\phi'\psi' + \mu(\mu+1)\phi\psi''\big)\\
&= 2(\mu+\nu+3)(\mu+\nu)![\phi,\psi]_2^{(\mu,\nu)}.
\end{aligned}$$

From this and (3.9), we have

$$
\begin{aligned}
L(\phi''\psi, s) &= \sum_{j=0}^{2} \frac{(2\pi i)^j}{j!(\mu+\nu+3-j)!h^j} L\big(g_{2-j}^{\Xi}, s-j\big) \\
&= \frac{2}{(\mu+\nu+2)(\mu+\nu+1)} L\big([\phi,\psi]_2^{(\mu,\nu)}, s\big) \\
&\quad - \frac{2(2\pi i)(\mu+1)}{(\mu+\nu+2)(\mu+\nu)h} L\big([\phi,\psi]_1^{(\mu,\nu)}, s-1\big) \\
&\quad + \frac{(2\pi i)^2\mu(\mu+1)}{(\mu+\nu+1)(\mu+\nu)h^2} L\big([\phi,\psi]_0^{(\mu,\nu)}, s-2\big).
\end{aligned} \tag{5.3}
$$

The case where $m=0$ and $n=2$ can be obtained from (5.3) and the fact that Rankin–Cohen brackets $[\phi,\psi]_w^{(\mu,\nu)}$ are $(-1)^w$-symmetric. Thus we see that

$$
\begin{aligned}
L(\phi\psi'', s) &= \frac{2}{(\mu+\nu+2)(\mu+\nu+1)} L\big([\psi,\phi]_2^{(\mu,\nu)}, s\big) \\
&\quad - \frac{2(2\pi i)(\nu+1)}{(\mu+\nu+2)(\mu+\nu)h} L\big([\psi,\phi]_1^{(\mu,\nu)}, s-1\big) \\
&\quad + \frac{(2\pi i)^2\nu(\nu+1)}{(\mu+\nu+1)(\mu+\nu)h^2} L\big([\psi,\phi]_0^{(\mu,\nu)}, s-2\big) \\
&= \frac{2}{(\mu+\nu+2)(\mu+\nu+1)} L\big([\phi,\psi]_2^{(\mu,\nu)}, s\big) \\
&\quad + \frac{2(2\pi i)(\nu+1)}{(\mu+\nu+2)(\mu+\nu)h} L\big([\phi,\psi]_1^{(\mu,\nu)}, s-1\big) \\
&\quad + \frac{(2\pi i)^2\nu(\nu+1)}{(\mu+\nu+1)(\mu+\nu)h^2} L\big([\phi,\psi]_0^{(\mu,\nu)}, s-2\big).
\end{aligned}
$$

6. Concluding remarks

Given two modular forms $\phi \in M_\mu(\Gamma)$ and $\psi \in M_\nu(\Gamma)$, from (4.1) we see that

$$
L\big([\phi,\psi]_w^{(k,\ell)}, s\big) = \sum_{r=0}^{w} (-1)^r \binom{k+w-1}{w-r}\binom{\ell+w-1}{r} L\big(\phi^{(r)}\psi^{(w-r)}, s\big)
$$

for each nonnegative integer w. Using this and (1.1), we have

$$
L\big([\phi,\psi]_w^{(\mu,\nu)}, s\big) = \sum_{\ell=0}^{w}\sum_{r=0}^{w} (-1)^r \binom{\mu+w-1}{w-r}\binom{\nu+w-1}{r} a_{\mu,\nu}^{r,w-r}(\ell) L\big([\phi,\psi]_{w-\ell}^{(\mu,\nu)}, s-\ell\big),
$$

where the constants $a_{\mu,\nu}^{r,w-r}(\ell) \in \mathbb{C}$ are as in (1.2). Hence we obtain the identities

$$
\begin{aligned}
&\sum_{r=0}^{w} (-1)^r \binom{\mu+w-1}{w-r}\binom{\nu+w-1}{r} a_{\mu,\nu}^{r,w-r}(0) = 1, \\
&\sum_{r=0}^{w} (-1)^r \binom{\mu+w-1}{w-r}\binom{\nu+w-1}{r} a_{\mu,\nu}^{r,w-r}(\ell) = 0
\end{aligned}
$$

for $1 \leqslant \ell \leqslant w$.

Acknowledgment

We would like to thank the referee for various helpful comments and suggestions.

References

[1] J. Bruinier, G. van der Geer, G. Harder, D. Zagier, The 1-2-3 of modular forms, in: Kristian Ranestad (Ed.), Lectures from the Summer School on Modular Forms and Their Applications Held in Nordfjordeid, June 2004, in: Universitext, Springer-Verlag, Berlin, 2008, x+266 pp.

[2] Y. Choie, M.H. Lee, Quasimodular forms, Jacobi-like forms, and pseudodifferential operators, preprint.

[3] P.B. Cohen, Y. Manin, D. Zagier, Automorphic pseudodifferential operators, in: Algebraic Aspects of Nonlinear Systems, Birkhäuser, Boston, 1997, pp. 17–47.

[4] A. Eskin, A. Okounkov, Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials, Invent. Math. 145 (1) (2001) 59–103.

[5] A. Eskin, A. Okounkov, Andrei Pillowcases and quasimodular forms, in: Algebraic Geometry and Number Theory, in: Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 1–25.

[6] M. Kaneko, D. Zagier, A generalized Jacobi theta function and quasimodular forms, in: Progr. Math., vol. 129, Birkhäuser, Boston, 1995, pp. 165–172.

[7] S. Lelievre, E. Royer, Orbitwise countings in $H(2)$ and quasimodular forms, Int. Math. Res. Not. IMRN (2006), Art. ID 42151, 30 pp.

[8] M.H. Lee, Quasimodular forms and Poincaré series, Acta Arith. 137 (2009) 155–169.

[9] F. Martin, E. Royer, Formes modulaires et transcendance, in: S. Fischler, E. Gaudron, S. Khémira (Eds.), Formes modulaires et périodes, Soc. Math. de France, 2005, pp. 1–117.

[10] D. Zagier, Modular forms and differential operators, in: K.G. Ramanathan Memorial Issue, Proc. Indian Acad. Sci. Math. Sci. 104 (1) (1994) 57–75.