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A Necessary and Sufficient Condition for Convergence of Steepest Descent Approximation to Accretive Operator Equations

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A necessary and sufficient condition is established which ensures the strong convergence of the steepest descent approximation to a solution of equations involving quasi-accretive operators defined on a uniformly smooth Banach space. © 1992 Academic Press, Inc.

1. INTRODUCTION

The aim of the present paper is to characterise conditions for the convergence of the steepest descent approximation process

$$x_{n+1} = x_n - t_n A x_n \text{ with } x_0 \text{ given and}$$

$$t_n \in (0, \infty), \quad \sum_{n=0}^{\infty} t_n = +\infty, \; t_n \to 0 \; (n \to \infty), \; \forall n \geqslant 0 \qquad (*)$$

to a solution of the accretive operator equations $Ax = 0$ in Banach spaces.

Let X be a Banach space, X^* its dual space, and $J: X \to 2^{X^*}$ the normalized duality mapping defined by

$$Jx = \{x^* \in X^*: \langle x^*, x\rangle = \|x^*\| \, \|x\|, \|x^*\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. An operator A with

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domain $D(A)$ and kernal $N(A)$ is said to be accretive if, for every $x, y \in D(A)$, there exist $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geqslant 0. \tag{1.1}$$

It is said to be strongly accretive if, in addition, there is a strictly increasing function ϕ: $[0, \infty) = R^+ \to R^+$, $\phi(0) = 0$, such that

$$\langle Ax - Ay, j(x-y) \rangle \geqslant \phi(\|x-y\|)\|x-y\|. \tag{1.2}$$

The operator is uniformly accretive if there is a positive constant $\alpha > 0$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geqslant \alpha \|x-y\|^2. \tag{1.3}$$

Furthermore, if $N(A) \neq \varnothing$ and the inequalities (1.1), (1.2), and (1.3) hold for any $x \in D(A)$ but $y \in N(A)$, then the corresponding operator A is said to be quasi-accretive, strongly quasi-accretive, and uniformly quasi-accretive, respectively. Such operators have been extensively studied and used by various authors (see, e.g., [1, 2, 4, 8, 13]). The interest and importance of these operators stems mainly from the fact that many physically significant problems can be modelled in terms of an initial value problem of the form

$$\begin{aligned} \frac{dx}{dt} &= -Ax \\ x(0) &= x_0, \end{aligned} \tag{1.4}$$

where A is either an accretive or strongly or uniformly accretive operator in an appropriate Banach space. In this case, the solutions of the equation $Ax = 0$ are just the equilibrium points of the system (1.4).

In the special case when A is a uniformly accretive operator and of the form $I - T$ with T being a nonexpansive mapping (namely, $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in D(T)$), many authors have used the iterative process $(*)$ to determine solutions of the equation $Ax = 0$. See, for instance, Zarantonello [24]; Vainberg [22, 23]; Bruck [4]; Crandall and Pazy [5]; Reich [17–19]; Browder and Petryshn [3]; Dotson [9] and Mann [14]. Basically, they have established the following typical results:

(i) If the Banach space X is uniformly smooth (see next section for the definition) and A is bounded, uniformly accretive, then there exists a positive real number $T(x_0) > 0$ such that the iterative scheme $(*)$, with $t_n \leqslant T(x_0)$ for each n, converges strongly to the unique solution of $Ax = 0$ (if it exists) [26, 27];

(ii) If $A = I - T$, where, for a closed convex subset D of X, $T: D \to D$ is a nonexpansive mapping with a nonempty fixed point set $F(T)$, then the iterative scheme $(*)$, with $t_n < 1$ for each n, or equivalently, the Mann type iterative process

$$x_0 \in D$$

$$x_{n+1} = (1 - t_n) x_n + t_n T x_n, \ n \geq 0, \ \{t_n\} \subset (0, 1), \sum_{n=0}^{\infty} t_n = +\infty, \tag{1.5}$$

strongly converges to a fixed point x^* of T provided T satisfies the condition

$$\|x - Tx\| \geq f(d(x, F(T))), \tag{1.6}$$

where $f: R^+ \to R^+$, $f(0) = 0$, is a strinctly increasing function, and $d(x, F(T)) = \inf\{\|x - p\|: p \in F(T)\}$ [10, 21].

From these studies two questions arise quite naturally: (i) Can the iterative process $(*)$ be used for general quasi-accretive operators rather than uniformly accretive ones? and, (ii) is the condition (1.6) necessary for strong convergence of (1.5) when dealing with nonexpansive mappings? We shall show in the present paper that the answer to both questions is in the affirmative. More precisely, by establishing a necessary and sufficient condition for strong convergence of the iterative process $(*)$, we show that the process $(*)$ can be applied to any equation involving a strongly quasi-accretive operator which is not necessarily uniformly accretive. In the case when $A = I - T$ with T a nonexpansive mapping, we show that, in a certain sense, the assumption (1.6) is indeed a necessary and sufficient condition for strong convergence of the iterative process (1.5).

2. Main Results

We establish our main results by using certain special geometrical aspects of Banach spaces. Recall that a Banach space X is said to be uniformly convex if $\delta_X(\varepsilon)$, the modulus of convexity of X, which is defined by

$$\delta_X(\varepsilon) = \inf\{1 - \tfrac{1}{2}\|x + y\|: \|x\| = 1, \|y\| = 1, \|x - y\| \geq \varepsilon\},$$

satisfies $\delta_X(0) = 0$ and $\delta_X(\varepsilon) > 0$ for any $0 < \varepsilon \leq 2$. A Banach space X is said to be uniformly smooth if the modulus of smoothness of X, defined by

$$\rho_X(\tau) = \sup\{\tfrac{1}{2}\|x + y\| + \tfrac{1}{2}\|x - y\| - 1: \|x\| = 1, \|y\| \leq \tau\},$$

satisfies

$$\lim_{\tau \to 0} \rho_X(\tau)/\tau = 0. \tag{2.1}$$

It is known [12] that any Hilbert space H, the Lebesgue spaces L^p $(1 < P < \infty)$, and Sobolev spaces W_m^p $(1 < p < \infty)$ all are uniformly convex and uniformly smooth and, furthermore that,

$$\delta_H(\varepsilon) = 1 - \left(1 - \frac{1}{4}\varepsilon^2\right)^{1/2}$$

$$\delta_{L^p}(\varepsilon) \geqslant \delta_{W_m^p}(\varepsilon) \geqslant \begin{cases} \dfrac{p-1}{8}\varepsilon^2, & 1 < p < 2 \\ 1 - \left[1 - \left(\dfrac{\varepsilon}{2}\right)^p\right]^{1/p}, & p \geqslant 2 \end{cases}$$

$$\rho_H(\tau) = (1 + \tau^2)^{1/2} - 1$$

$$\rho_{L^p}(\tau) \leqslant \rho_{W_m^p}(\tau) \leqslant \begin{cases} (1 + \tau^p)^{1/p} - 1, & 1 < p \leqslant 2 \\ \dfrac{p-1}{2}\tau^2, & p > 2. \end{cases}$$

DEFINITION 1. A quasi-accretive operator A in Banach space X is said to satisfy the condition (I) if, for any $x \in D(A)$, $p \in N(A)$, and any $j(x-p) \in J(x-p)$ the equality $\langle Ax, j(x-p) \rangle = 0$ holds if and only if $Ax = Ap = 0$.

From the definition, it follows that any strongly quasi-accretive operator satisfies the condition (I). In Theorem 2 below we will show that, for any nonexpansive mapping T in a uniformly convex Banach space, the operator $A = I - T$ also satisfies the condition (I).

We now prove our first main result.

THEOREM 1. *Let X be a uniformly smooth Banach space and let $A\colon D(A) = X \to X$ be a quasi-accretive, bounded operator which satisfies the condition* (I). *Then, for any initial value $x_0 \in D(A)$ there are positive real numbers $T(x_0)$ such that the steepest descent approximation method* $(*)$, *with $t_n \leqslant T(x_0)$ for any n, converges strongly to a solution x^* of the equation $Ax = 0$ if and only if there is a strictly incresing function $\phi\colon R^+ \to R^+$, $\phi(0) = 0$, such that*

$$\langle Ax_n - Ax^*, J(x_n - x^*) \rangle \geqslant \phi(\|x_n - x^*\|)\,\|x_n - x^*\|. \tag{2.2}$$

Proof. Necessity. Let $M=\sup_{n\geqslant 0}\{\|x_n-x^*\|\}$ where $\{x_n\}$ is the sequence defined by the process $(*)$ with $t_n\leqslant T(x_0)$. If $M=0$ then $\{x_n\}=\{x^*\}$ and the condition (2.2) follows trivially. Suppose $M>0$ and for $0<t<M$ define

$$C_t=\{n\in N: \|x_n-x^*\|\geqslant t\}$$

and

$$f(t)=\inf\left\{\frac{\langle Ax_n, J(x_n-x^*)\rangle}{\|x_n-x^*\|}: n\in C_t\right\}. \tag{2.3}$$

Clearly, $f(t)$ is nonnegative and nondecreasing. We now prove that $f(t)>0$ for any $0<t<M$. Assume this is not the case. Consequently there is a $t_0\in(0, M)$ such that $f(t_0)=0$. Hence, by (2.3) there exist $n_k\in C_{t_0}\neq\varnothing$ such that

$$0\leqslant\delta_{n_k}=\frac{\langle Ax_{n_k}, J(x_{n_k}-x^*)\rangle}{\|x_{n_k}-x^*\|}\leqslant f(t_0)+\frac{1}{k}=\frac{1}{k}$$

for every integer k. Thus we find subsequences $\{\|x_{n_k}-x^*\|\}$ and $\{\delta_{n_k}\}$ such that

$$\|x_{n_k}-x^*\|\geqslant t_0>0 \qquad \text{and} \qquad \delta_{n_k}\to 0 \qquad (k\to\infty). \tag{2.4}$$

Since, by hypothesis, $\|x_n-x^*\|\to 0$ $(n\to\infty)$, the sequence $\{\|x_{n_k}-x^*\|\}$ must be finitely circulative, that is, there must be an $x_{n0}\in\{x_n\}$ and a subsequence $\{\|x_{n_{k_i}}-x^*\|\}$ such that

$$\|x_{n_{k_i}}-x^*\|=\|x_{n_0}-x^*\|>t_0>0, \qquad i=1, 2, \ldots.$$

This implies by (2.4) that

$$\delta_{n_0}=\frac{\langle Ax_{n_0}-Ax^*, J(x_{n_0}-x^*)\rangle}{\|x_{n_0}-x^*\|}=0.$$

Because the operator A satisfies the condition (I), we therefore have $Ax_{n_0}=Ax^*=0$, that is to say, x_{n_0} is a solution of the equation $Ax=0$. However, from the definition of the iterative process $(*)$, we then have $x_n=x_{n_0}$ for every $n\geqslant n_0$, which obviously contradicts the fact that $x_n\to x^*$ $(n\to\infty)$. Thus $f(t)>0$ for any $0<t<M$.

We extend the domain of f to R^+ by defining $f(0)=0$ and $f(t)=\sup\{f(s): s<M\}$ for $t\geqslant M$. It is easy to verify that f so defined is

nondecreasing and fulfills the inequality (2.2) with $\phi = f$. In consequence, let

$$\phi(t) = \frac{tf(t)}{1+t}, \qquad \forall t \in R^+.$$

Then $\phi: R^+ \to R^+$, $\phi(0) = 0$, is strictly increasing and, with this function, the condition (2.2) is established. This completes the proof of the necessity.

Sufficiency. Suppose that the condition (2.2) is satisfied. We prove that there is $T(x_0) > 0$ such that the iterative process $(*)$ with $t_n \leqslant T(x_0)$ for every n converges strongly to x^*, a solution of the equation $Ax = 0$. To do this, we will make repeated use of the inequality

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle Jx, y\rangle + K_1 \max\{\|x\| + \|y\|, \tfrac{1}{2}C\}\, \rho_X(\|y\|), \qquad \forall x, y \in X \tag{2.5}$$

which is valid for uniformly smooth Banach spaces as shown in [25, 26]. Here K_1 and C are both fixed positive constants. Let

$$M(x_0) = \sup\{\|Ay\|: \|y - x_0\| \leqslant 3\phi^{-1}(\|Ax_0\|)\} \tag{2.6}$$

and let β be the largest positive real number such that

$$\beta^{-1}\rho_X(\beta M(x_0)) \leqslant \frac{2\phi^{-1}(\|Ax_0\|)\,\|Ax_0\|}{K_1[3\phi^{-1}(\|Ax_0\|) + (1/2)C]}. \tag{2.7}$$

(This is possible because $\lim_{\tau \to 0} \rho_X(\tau)/\tau = 0$.) We define

$$T(x_0) = \min\{\beta, \phi^{-1}(\|Ax_0\|)/M(x_0)\} \tag{2.8}$$

and proceed by the following two steps:

Step I. We prove the boundedness of the sequence $\{x_n\}$, which is generated by the process $(*)$ with $t_n \leqslant T(x_0)$. Assume that $\{x_n\}$ is not bounded. Then we distinguish two possible cases:

Case I. There exist integers $n_0 > 0$ such that

$$\|x_n - x^*\| > \phi^{-1}(\|Ax_0\|), \qquad \forall n \geqslant n_0. \tag{2.9}$$

Without loss of generality we assume that n_0 is the smallest integer n such that (2.9) holds (hence $\|x_{n_0-1}-x^*\| \leqslant \phi^{-1}(\|Ax_0\|)$). From (2.8), (2.6) we have

$$\begin{aligned}\|x_{n_0}-x^*\| &\leqslant \|x_{n_0-1}-x^*\| + t_{n_0-1}\|Ax_{n_0-1}\| \\ &\leqslant \phi^{-1}(\|Ax_0\|) + t_{n_0-1}M(x_0) \\ &\leqslant \phi^{-1}(\|Ax_0\|) + T(x_0)\,M(x_0) \leqslant 2\phi^{-1}(\|Ax_0\|).\end{aligned}$$

Therefore $\|x_{n_0}-x_0\| \leqslant \|x_{n_0}-x^*\| + \|x_0-x^*\| \leqslant 3\phi^{-1}(\|Ax_0\|)$. (Notice that the condition (2.3) implies $\|x_0-x^*\| \leqslant \phi^{-1}(\|Ax_0\|)$.) It then follows from (2.2), (2.4)–(2.9) that

$$\begin{aligned}\|x_{n_0+1}-x^*\|^2 &= \|x_{n_0}-x^*-t_{n_0}Ax_{n_0}\|^2 \\ &\leqslant \|x_{n_0}-x^*\|^2 - 2t_{n_0}\langle J(x_{n_0}-x^*), Ax_{n_0}\rangle \\ &\quad + K_1[\|x_{n_0}-x^*\| + t_{n_0}\|Ax_{n_0}\| + \tfrac{1}{2}C]\cdot\rho_X(t_{n_0}\|Ax_{n_0}\|) \\ &\leqslant \|x_{n_0}-x^*\|^2 - 2t_{n_0}\phi(\|x_{n_0}-x^*\|)\,\|x_{n_0}-x^*\| \\ &\quad + K_1[2\phi^{-1}(\|Ax_0\|) + T(x_0)\,M(x_0) + \tfrac{1}{2}C]\;\rho_X(t_{n_0}M(x_0)) \\ &\leqslant \|x_{n_0}-x^*\|^2 - 2t_{n_0}\phi^{-1}(\|Ax_0\|)\,\|Ax_0\| + K_1 \\ &\quad \cdot[[3\phi^{-1}(\|Ax_0\|) + \tfrac{1}{2}C]\;\rho_X(t_{n_0}M(x_0))].\end{aligned}$$

It is known [12] that $\rho_X(\tau)/\tau$ is nondecreasing. Hence, we obtain from (2.7)–(2.8) that

$$\begin{aligned}\|x_{n_0+1}-x^*\|^2 \leqslant \|x_{n_0}-x^*\|^2 - t_{n_0}\{&2\phi^{-1}(\|Ax_0\|)\,\|Ax_0\| \\ &- K_1[3\phi^{-1}(\|Ax_0\|) + \tfrac{1}{2}C] \\ &\cdot\rho_X(\beta M(x_0))/\beta\} \leqslant \|x_{n_0}-x^*\|^2.\end{aligned}$$

In the same way, it is easy to show that

$$\|x_{n+1}-x^*\|^2 \leqslant \|x_n-x^*\|^2 \leqslant \cdots \leqslant \|x_{n_0}-x^*\|^2, \qquad \forall n \geqslant n_0.$$

This clearly contradicts the assumption that $\{x_n\}$ is not bounded.

Case II. The interval $[0, \phi^{-1}(\|Ax_0\|)]$ contains infinitely many $\|x_n-x^*\|$ of the sequence $\{\|x_n-x^*\|\}$. In this case there is some real $\delta \geqslant 0$ and a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\|x_{n_k}-x^*\| \to \delta$ $(k \to \infty)$. Since $\{\|x_n-x^*\|\}$ is unbounded by assumption, then $R^+\backslash[0, 2\phi^{-1}(\|Ax_0\|)]$ also contains infinitely many $\|x_n-x^*\|$, and hence, the sequence $\{\|x_n-x^*\|\}$ must pass through the interval $[\delta+\frac{1}{3}\phi^{-1}(\|Ax_0\|), \delta+\frac{2}{3}\phi^{-1}(\|Ax_0\|)]$

infinitely many times. Thus, we can find two subsequences of $\{x_n\}$, say $\{x_{n_j}\}$ and $\{x_{m_j}\}$, in such a manner that

(a) $n_j < m_j$,

(b) $\|x_{n_j-1} - x^*\| \leqslant \delta + \frac{1}{3}\phi^{-1}(\|Ax_0\|)$ and $\|x_{n_j} - x^*\| > \delta + \frac{1}{3}\phi^{-1}(\|Ax_0\|)$,

(c) $\|x_{m_j-1} - x^*\| \leqslant \delta + \frac{2}{3}\phi^{-1}(\|Ax_0\|)$ and $\|x_{m_j} - x^*\| > \delta + \frac{2}{3}\phi^{-1}(\|Ax_0\|)$.

Let

$$M_0 = \sup\{\|Ay\| : \|y - x^*\| \leqslant \delta + \phi^{-1}(\|Ax_0\|)\}.$$

Then we have

$$\|x_{n_j} - x_{n_j-1}\| \leqslant t_{n_j-1}\|Ax_{n_j-1}\| \leqslant t_{n_j-1} M_0 \to 0 \qquad (j \to \infty).$$

Therefore

$$\|x_{n_j} - x^*\| \to \delta + \tfrac{1}{3}\phi^{-1}(\|Ax_0\|) \qquad (j \to \infty). \tag{2.10}$$

On the other hand, by inequality (2.5) and the definition of $(*)$, we find for every $n \in [n_j, m_j - 1]$ that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|x_n - t_n A x_n - x^*\|^2 \\ &\leqslant \|x_n - x^*\|^2 - 2t_n\phi(\|x_n - x^*\|)\|x_n - x^*\| + K_1 \\ &\quad \cdot [\|x_n - x^*\| + t_n\|Ax_n\| + \tfrac{1}{2}C]\,\rho_X(t_n\|Ax_n\|) \qquad (2.11) \\ &\leqslant \|x_n - x^*\|^2 - 2t_n\phi(\delta + \tfrac{1}{3}\phi^{-1}(\|Ax_0\|))(\delta + \tfrac{1}{3}\phi^{-1}(\|Ax_0\|)) \\ &\quad + K_1[\delta + \tfrac{2}{3}\phi^{-1}(\|Ax_0\|) + T(x_0)\,M_0 + \tfrac{1}{2}C]\,\rho_X(t_n M_0). \end{aligned}$$

Since $\rho_X(\tau)/\tau \to 0$ $(\tau \to 0)$ and $t_n \to 0$ there is an $N_0 > 0$ such that

$$\begin{aligned} &2\phi(\delta + \tfrac{1}{3}\phi^{-1}(\|Ax_0\|))(\delta + \tfrac{1}{3}\phi^{-1}(\|Ax_0\|)) \\ &\quad \geqslant K_1[\delta + \tfrac{2}{3}\phi^{-1}(\|Ax_0\|) + T(x_0)\,M_0 + \tfrac{1}{2}C] \cdot \rho_X(t_n M_0)/t_n \end{aligned}$$

for all $n \geqslant N_0$. Hence, (2.11) implies that

$$\|x_{n+1} - x^*\|^2 \leqslant \|x_n - x^*\|^2, \qquad \forall n \in [n_j, m_j - 1],\ n_j \geqslant N_0.$$

In particular,

$$\|x_{m_j} - x^*\| \leqslant \|x_{m_j-1} - x^*\| \leqslant \cdots \leqslant \|x_{n_j} - x^*\|, \qquad \forall n_j \geqslant N_0.$$

From (2.10) we therefore have

$$\delta + \tfrac{2}{3}\phi^{-1}(\|Ax_0\|) \leqslant \limsup_{(j)} \|x_{m_j} - x^*\| \leqslant \lim_{j\to\infty} \|x_{n_j} - x^*\|$$
$$= \delta + \tfrac{1}{3}\phi^{-1}(\|Ax_0\|)$$

which is a contradiction. This, together with Case I, justifies the boundedness of the sequence $\{x_n\}$.

Step II. Show that the sequence $\{x_n\}$ converges strongly to x^* as $n \to \infty$. First we prove that the sequence $\{\|x_n - x^*\|\}$ is convergent. Assume it is not. Then the boundedness of $\{x_n\}$ implies that there exist at least two real numbers δ_1 and δ_2 (say, $\delta_1 > \delta_2$) and subsequences $\{x_{n_i}\}$, $\{x_{n_j}\}$ such that

$$\|x_{n_i} - x^*\| \to \delta_1\ (i \to \infty); \qquad \|x_{n_j} - x^*\| \to \delta_2\ (j \to \infty).$$

Thus the sequence $\{\|x_n - x^*\|\}$ must pass through the interval $[\delta_2 + \tfrac{1}{3}\delta_1, \delta_2 + \tfrac{2}{3}\delta_1\}$ infinitely many times. Consequently, a similar argument to that used in Case II of Step I shows that this is impossible. That is, the sequence $\{\|x_n - x^*\|\}$ must be convergent. Consequently we can write

$$l = \lim_{n\to\infty} \|x_n - x^*\|.$$

Now we show that $l = 0$ and, in turn, that $\{x_n\}$ converges strongly to x^*. To this end we assume it is not the case, namely $l > 0$. Let n_0 be the integer such that

$$\tfrac{3}{2}l \geqslant \|x_n - x^*\| \geqslant \tfrac{1}{2}l, \qquad \forall n \geqslant n_0.$$

Then we have that for all $n \geqslant n_0$

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leqslant \|x_n - x^*\|^2 - 2t_n\phi(\|x_n - x^*\|)\,\|x_n - x^*\| \\ &\quad + K_1[\|x_n - x^*\| + t_n\|Ax_n\| + \tfrac{1}{2}C\}\,\rho_X(t_n\|Ax_n\|) \\ &\leqslant \|x_n - x^*\|^2 - t_n\phi(\|x_n - x^*\|)\,\|x_n - x^*\| \\ &\quad - t_n\{\phi(\tfrac{1}{2}l)\,\tfrac{1}{2}l - K_1[\tfrac{3}{2}l + T(x_0)M + \tfrac{1}{2}C]\,\rho_X(t_nM)/t_n\}, \end{aligned} \tag{2.12}$$

where $M = \sup\{\|Ax_n\|\} < \infty$. Let $N \geqslant n_0$ be integer such that

$$\phi(\tfrac{1}{2}l)(\tfrac{1}{2}l) \geqslant K_1[\tfrac{3}{2}l + T(x_0)M + \tfrac{1}{2}C]\,\rho_X(t_nM)/t_n, \qquad \forall n \geqslant N.$$

Thus, (2.12) implies

$$\|x_{n+1}-x^*\|^2 \leqslant \|x_n-x^*\|^2 - t_n\phi(\|x_n-x^*\|)\,\|x_n-x^*\|, \qquad \forall n \geqslant N.$$

Summarising this inequality then gives

$$\sum_{n=N}^{\infty} t_n\phi(\|x_n-x^*\|)\,\|x_n-x^*\| \leqslant \|x_N-x^*\|^2 - l^2 < \infty.$$

Since $\sum_{n=0}^{\infty} t_n = +\infty$ it follows that $\liminf_{(n)} \phi(\|x_n-x^*\|)\,\|x_n-x^*\| = 0$. Therefore, the conclusion $l=0$ now follows directly from the strict increasing monotonicity of ϕ and the convergence of $\{\|x_n-x^*\|\}$. With this, the proof of Theorem 1 is completed.

Remark 1. In Theorem 1 the assumption $D(A)=X$ can naturally be relaxed by requiring simply that the sequence $\{x_n\}$, defined by $(*)$ with $t_n \leqslant T(x_0)$ for any n, remains in $D(A)$. This is the case, for instance, if $D(A)$ is a convex subset and there is a positive real number $r \geqslant T(x_0)$ such that $(I-rA)\,D(A) \subset D(A)$. (Indeed, in this case the process $(*)$ can be rewritten as $x_{n+1} = (1-r^{-1}t_n)\,x_n + r^{-1}t_n(I-rA)\,x_n$ and, hence, $\{x_n\} \subset D(A)$ as long as $x_0 \in D(A)$.)

Remark 2. From the proof of Theorem 1, one sees that the assumptions of uniform smoothness of X and $t_n \to 0$ $(n \to \infty)$ are in fact not required for the necessity of the theorem and, also, the assumption that A satisfies condition (I) is not required for the sufficiency of the theorem.

With these observations in mind, we prove

THEOREM 2. *Let X be a uniformly convex Banach space, let $D \subset X$ be a nonempty closed convex subset of X, and let $T\colon D \to D$ be a quasi-nonexpansive mapping* (*that is, $F(T) \neq \varnothing$ and $\|Tx-Tp\| \leqslant x-p\|$ for all $x \in D$ and $p \in F(T)$*). *Then, for any initial value $x_0 \in D$, the Mann type iterative process* (1.5) *converges strongly to a fixed point x^* of T if and only if there is a strictly increasing function $f\colon R^+ \to R^+$, $f(0)=0$, such that*

$$\|x_n - Tx_n\| \geqslant f(d(x_n, F(T))), \qquad n \geqslant 0. \tag{2.13}$$

Proof. Let $A = I - T$. Then A is a quasi-accretive operator [2]. We now check that A also satisfies the condition (I). By Theorem 1 of [25], for all $x, y \in X$ we have

$$\|x+y\|^2 \geqslant \|x\|^2 + 2\langle j(x), y\rangle + \sigma(x, y),$$

where $j(x) \in Jx$ arbitrarily and

$$\sigma(x, y) = c \int_0^1 \frac{(\|x+ty\| \vee \|x\|)^2}{t} \delta_X \left(\frac{t\,\|y\|}{2(\|x+ty\| \vee \|x\|)} \right) dt \qquad (2.14)$$

with a positive constant $c > 0$. Employing this inequality for $x := x - y$ and $y := -(Ax - Ay)$ we obtain

$$\begin{aligned} \|Tx - Ty\|^2 &= \|(x-y) - (Ax - Ay)\|^2 \\ &\geqslant \|x-y\|^2 - 2\langle j(x-y), Ax - Ay \rangle + \sigma(x-y, Ay - Ax). \end{aligned}$$

Since T is quasi-nonexpansive, this implies that

$$\langle j(x-y), Ax - Ay \rangle \geqslant \tfrac{1}{2}\sigma(x-y, Ay-Ax), \qquad \forall x \in D(T),\ y \in F(T). \qquad (2.15)$$

We observe that for any $x \in D(T)$, $y \in F(T)$, and $t \in (0, 1)$ we have

$$\begin{aligned} \|x - y + t(Ay - Ax)\| &= \|(1-t)(x-y) + t(Tx - Ty)\| \\ &\leqslant (1-t)\,\|x-y\| + t\,\|Tx - Ty\| \leqslant \|x-y\|. \end{aligned}$$

Hence, from (2.14) and (2.15), we obtain

$$\begin{aligned} \langle j(x-y), Ax - Ay \rangle &\geqslant \frac{c}{2} \int_0^1 \frac{\|x-y\|^2}{t} \delta_X \left(\frac{t\,\|Ax - Ay\|}{2\,\|x-y\|} \right) dt \\ &= \frac{c}{2}\,\|x-y\|^2 \int_0^{\|Ax - Ay\|/2\,\|x-y\|} \delta_X(\varepsilon)/\varepsilon\, d\varepsilon. \end{aligned}$$

It is known [12] that the function $\delta_X(\varepsilon)/\varepsilon$ is nondecreasing and positive. Therefore, $\langle j(x-y), Ax - Ay \rangle = 0$ if and only if $x = y$ or $Ax = Ay$. That is, A satisfies condition (I). Thus from Theorem 1 and Remarks 1 and 2, there must be a strictly increasing function ϕ: $R^+ \to R^+$, $\phi(0) = 0$, such that when the iterative process (1.5) converges strongly to a fixed point x^* of T one has

$$\langle x_n - Tx_n, j(x_n - x^*) \rangle \geqslant \phi(\|x_n - x^*\|)\,\|x_n - x^*\|$$

or,

$$\|x_n - Tx_n\| \geqslant \phi(\|x_n - x^*\|) \geqslant \phi(d(x_n F(T))).$$

This obviously completes the proof of necessity of the theorem.

Conversely, when assumption (2.13) is fulfilled, Theorem 2 of [21] implies directly that the sequence $\{x_n\}$ converges strongly to a fixed point x^* of T. (We notice that condition (1.6) in this case can obviously be replaced by (2.13) in this theorem.) This completes the proof of Theorem 2.

3. Consequences and Concluding Remarks

From Theorem 1, Remarks 1 and 2, and Theorem 2 we can conclude

(i) For any strongly quasi-accretive and bounded operator A which is defined on a uniformly smooth Banach space X, the steepest descent approximation method (*) can be used to find the unique solution x^* of the equation $Ax=0$. In particular, from the proof of Theorem 1 and Remark 1 of [26], we know that in this case the iterative process

$$x_{n+1}=x_n-t_nAx_n, \qquad n\geqslant 0,$$

with

$$\{t_n\}\subset(0,T(x_0)),\ \sum_{n=0}^{\infty} t_n=+\infty,\ t_n\to 0\ (n\to\infty)$$

converges strongly to the unique solution x^*, where

$$T(x_0)=\min\{\beta,\phi^{-1}(\|Ax_0\|)/M(x_0)\},$$

ϕ is the function given in (1.2), β is the largest positive real number such that

$$\beta^{-1}\rho_X(\beta M(x_0))\leqslant\frac{2\phi^{-1}(\|Ax_0\|)\,\|Ax_0\|}{K_1[3\phi^{-1}(\|Ax_0\|)+\frac{1}{2}C]},$$

and

$$K_1=\max\{8\sqrt{128}(\sqrt{3}-1)C\}$$

$$C=\frac{4\tau_0}{(1+\tau_0^2)^{1/2}-1}\sum_{j=1}^{\infty}\left(1+\frac{15}{2^{j+2}}\tau_0\right)\quad \text{with}\quad \tau_0=\frac{\sqrt{339}-18}{30}.$$

This consequence generalizes and improves various results concerned with constructive techniques for the solution of uniformly accretive operator equations [5, 6, 18, 26, 27], in the sense that here global convergence is obtained for general bounded strongly quasi-accretive operators rather than for uniformly accretive operators and, moreover, without any continuity requirement on the operator.

(ii) Although it is now clear that the method $(*)$ can be used for quasi-accretive operators which are not necessarily uniformly accretive (such an example, for instance, is given by $Ax = x - [1 - \exp(-x)]x \sin(1/x)$, when $x \neq 0$, $Ax = 0$ when $x = 0$, where $X = R^1$ and $N(A) = \{0\}$), Theorem 1 says that, in a certain sense, the strongly quasi-accretive operator is the largest subclass of quasi-accretive operators for which the method $(*)$ is valid. This conclusion provides a practically suitable criterion for choosing an iterative algorithm to solve accretive operator equations so that it should have minimal computational complexity. We notice that other iterative methods than $(*)$ have been developed for use with accretive operator equations (see, e.g., [16, 19, 20, 28, 15]), but it is easy to see that the steepest descent approximation method $(*)$ really possesses the least computational complexity among all these available iterative algorithms.

(iii) For the strong convergence of the Mann type iteration to a fixed point of quasi-nonexpansive mapping, Theorem 2 says that condition (1.6) is indeed not only sufficient but also necessary in the sense of (2.19). This extends the known results (see, e.g., [21, Theorem 2]).

Remark. We notice that, if $\{t_n\}$ in $(*)$ is further restricted to be such that $\sum_{(n)} \rho_x(t_n) < \infty$, various other sufficient conditions are known for strong convergence of $(*)$; e.g., see O. Nevanlinna and S. Reich [29], R. E. Bruck and S. Reich [30], and A. Pazy [31].

Theorem 1 is related to these convergence results but they cannot be deduced from Theorem 1 of the present paper. However, our approach is capable of yielding those results. Namely, let $N_0(A)$ be a proximinal and convex subset of $N(A)$, let P_0 be an arbitrary selection of the nearest point mapping from X onto $N_0(A)$, and let $J_0(x - P_0x)$ be an element in $J(x - P_0x)$ that satisfies $\langle J_0(x - P_0x), P_0x - y \rangle \geqslant 0$ for all $y \in N_0(A)$. If we say, corresponding to Definition 1, that the quasi-accretive operator A with $N(A) \neq \varnothing$ satisfies condition (J) whenever $\langle Ax, J_0(x - P_0x) \rangle = 0$ implies $x \in N(A)$, and that a sequence $\{y_n\}$ in X is pseudo-monotonically convergent whenever it is convergent and there exists an integer $K_0 > 0$ such that $\|x_n - P_0x_n\| \geqslant \|x_{n+1} - P_0x_{n+1}\|$ for any $n \geqslant K_0$, then we have recently proved the following

THEOREM. *Let X be a uniformly smooth Banach space and let $A: X \to X$ be a quasi-accretive, bounded, and semi-closed operator which satisfies condition* (J). *Then, for any $x_0 \in X$ there is a positive real number $T(x_0)$ such that the sequence $\{x_n\}$ defined by $(*)$ with*

$$t_n \leqslant T(x_0),\ n = 0, 1, 2, \ldots, \qquad \text{and} \qquad \sum_{(n)} \rho_X(t_n) < \infty$$

converges pseudo-monotonically and strongly to an element x^* in $N(A)$ if and only if there is a strictly increasing function ψ: $R^+ \to R^+$, $\psi(0)=0$, such that

$$\langle Ax_n, J_0(x_n - P_0 x_n)\rangle \geqslant \psi(\|x_n - P_0 x_n\|)\,\|Ax_n\|, \qquad n=0, 1, 2, \ldots.$$

Since any operator either being semipositive [30] or satisfying the convergence condition introduced in [29] obviously satisfies condition (J), this theorem unifies and generalizes all of the corresponding results in [29–31]. The detailed proof of this theorem will be given in a subsequent paper.

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