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Projective dimension is a lattice invariant

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This paper is dedicated to the memory of Richard Pierce

Abstract

We show that, for a free abelian group G and prime power p^v, every direct sum decomposition of the group G/p^vG lifts to a direct sum decomposition of G. This is the key result we use to show that, for R a commutative von Neumann regular ring, and $\mathscr{E}$ a set of idempotents in R, then the projective dimension of the ideal $\mathscr{E}R$ as an R-module the same as the projective dimension of the ideal $\mathscr{E}\mathscr{B}$ as a $\mathscr{B}$-module, where $\mathscr{B}$ is the boolean algebra generated by $\mathscr{E} \cup \{1\}$. This answers a 30 year old open question of R. Wiegand. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Back in the late 1960s, Roger Wiegand asked the following question in [10]:

Let R be a commutative [von Neumann] regular ring and J an ideal of R generated by a set $\mathscr{E}$ of idempotents. Let $\mathscr{B}$ be the Boolean algebra of all idempotents of R. Then is the projective dimension of $J = \mathscr{E}R$ as an R-module the same as the projective dimension of $\mathscr{E}\mathscr{B}$ as a $\mathscr{B}$-module?

In this paper we show that the answer to this question is 'yes'.

Richard Pierce popularized this problem, and did some of the early work on it. It is not difficult to see that the answer is 'yes' if J is projective. In [8] Pierce showed

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that projective dimension of an ideal generated by an independent set of idempotents in a boolean algebra was κ where the independent set had cardinality $\aleph_\kappa$ (here $\kappa \geq \omega$ is replaced by ∞ for projective dimension). Osofsky [4] proved the same result for arbitrary commuting idempotents in any ring, so in the case of ideals generated by independent idempotents the answer to the Wiegand question is 'yes'. Then Pierce [9] showed that it is 'yes' in case either the projective dimension of $\mathscr{E}R$ or the projective dimension of $\mathscr{E}\mathscr{B}$ is one. Since then, the problem has been solved in some special cases with extra hypotheses on the idempotents forcing projective dimension to be the subscript of the minimal aleph of a generating set, although the general problem remained open.

The essence of the problem is that the additive order of some of the idempotents in $\mathscr{E}$ might be one prime (for example the prime 2 in case $R = \mathscr{B}$) and a different prime in another ring R', or perhaps even infinite in a third ring R''. Here we conquer the problem of different primes by working in a (not regular) ring R of characteristic 0. We show that the answer to Wiegand's question is 'yes' in all cases.

In Section 2, we prove a subtle but elementary result about free abelian groups, namely for any free abelian group G and any direct sum decomposition of G/p^vG, this decomposition lifts to a direct sum decomposition of G. In Section 3, we apply this result to any commutative von Neumann regular ring R containing a lattice of idempotents isomorphic to $\mathscr{B}$. Unlike Pierce's paper concerning the case of projective dimension 1 [19], we do not give an internal characterization of projective dimension of ideals in a commutative von Neumann regular ring. However, there is a candidate for such a characterization in a series of papers by the author [5–7].

2. A theorem on abelian groups

The aim in this section is to prove the following:

Theorem A. *Let G be a free abelian group and $\{\overline{b_\alpha}: \alpha \in \mathfrak{I}\}$ a (free) basis for G/p^vG with p a prime. Then there exists a family of integers $\{u_\alpha: \alpha \in \mathfrak{I}\}$, relatively prime to p, and a free basis of G, $\{y_\alpha: \alpha \in \mathfrak{I}\}$, such that $\overline{y_\alpha} = u_\alpha \overline{b_\alpha}$ in G/p^vG for all $\alpha \in \mathfrak{I}$.*

A way of restating this theorem is that the direct sum decomposition $G/p^vG = \bigoplus_\alpha \overline{b_\alpha}\,\mathbb{Z}/p^v\mathbb{Z}$ lifts to a direct sum decomposition $G = \bigoplus_\alpha y_\alpha \mathbb{Z}$. In fact, any direct sum decomposition of G/p^vG will lift to a direct sum decomposition of G by taking bases of each of the summands and lifting them. We use the fact that the ring $\mathbb{Z}/p^v\mathbb{Z}$ is local, that is, has a unique maximal ideal. If p^v is replaced by an arbitrary integer which has at least two distinct prime factors, the result is false since $\mathbb{Z}$ is indecomposable whereas $\mathbb{Z}/n\mathbb{Z}$ decomposes if n is a product of two relatively prime factors > 1.

Basic notation. We fix a prime power p^v. For any abelian group G, we denote the natural map from G to G/p^vG by an overline. If $\bar{x}$ is an element of $\bar{G} = G/p^vG$ we will assume from the notation that $x \in G$ is some preimage of $\bar{x}$. If G is some free abelian group, we will denote some free basis for G by

$$\mathfrak{X} = \{x_\sigma \colon \sigma \in \mathfrak{K}\}$$

and we will denote a basis of $\bar{G}$ as a (free) $\bar{\mathbb{Z}}$-module by

$$\mathfrak{B} = \{\overline{b_\alpha} \colon \alpha \in \mathfrak{I}\}.$$

Reduction to the countable case. Much of this paper relies heavily on a beautiful paper by Kaplansky [2] for both technique and results. Here we adapt the basic technique of Kaplansky's paper to get a specialized result on free abelian groups. We have the same objective as Kaplansky did, namely to reduce the question under study to the countable case.

Lemma 2.1. *Let G be a nonzero free abelian group with free basis $\mathfrak{X}$, and let $\mathfrak{B}$ be a basis of $\bar{G}$ as a (free) $\bar{\mathbb{Z}}$-module. Let $\mathfrak{c}$ be any countable subset of $\mathfrak{B}$. Then there exists a nonzero countably generated direct summand H of G such that*

$$\bar{H} = \sum_{i=0}^{\infty} \overline{b_{\alpha_i}}\, \bar{\mathbb{Z}}$$

for $\{\overline{b_{\alpha_i}} \colon i \in \omega\}$ some countable subset of $\mathfrak{B}$ containing $\mathfrak{c}$. Moreover, H itself is generated by a countable subset of $\mathfrak{X}$.

Proof. We are given that $\mathfrak{X} = \{x_\sigma \colon \sigma \in \mathfrak{K}\}$ is a free basis for G. Fix a lifting $\{b_\alpha\}$ of $\mathscr{B}$. For any countable subset $\mathfrak{c} \subseteq \mathfrak{I}$, let $X_\mathfrak{c} \subseteq \mathfrak{K}$ be the smallest (necessarily countable) subset of $\mathfrak{K}$ such that $\sum_{\alpha \in \mathfrak{c}} b_\alpha \mathbb{Z} \subseteq \sum_{\sigma \in X_\mathfrak{c}} x_\sigma \mathbb{Z}$. Similarly, for any countable subset $\mathfrak{c}' \subseteq \mathfrak{K}$, let $B_{\mathfrak{c}'} \subseteq \mathfrak{I}$ be the smallest (necessarily countable) subset of $\mathfrak{I}$ such that $\sum_{\sigma \in \mathfrak{c}'} x_\sigma \mathbb{Z} \subseteq \sum_{\alpha \in B_{\mathfrak{c}'}} b_\alpha \mathbb{Z}$.

Now start with any nonempty countable set $\mathfrak{c}_0$ such that $\mathfrak{c} \subseteq \mathfrak{c}_0 \subseteq \mathfrak{I}$. We use finite induction to define two sequences $\{\mathfrak{c}_i, \mathfrak{c}_i' \colon i < \omega\}$ of countable sets by

$$\mathfrak{c}_n' = X_{\mathfrak{c}_n},$$
$$\mathfrak{c}_{n+1} = B_{\mathfrak{c}_n'}.$$

In words, think of $\mathfrak{B}$ as images of $\{b_\alpha \colon \alpha \in \mathfrak{I}\}$. Starting with a countable subset $\mathfrak{c}_0$ of the basis $\mathfrak{B}$ of $\bar{G}$, use our lifting of $\mathscr{B}$ to get an inverse image $c_0 \subseteq G$ and take the smallest countable subset $\mathfrak{c}_0'$ of the basis $\mathfrak{X}$ of G whose span contains c_0. Now take images of $\mathfrak{c}_0'$ modulo p^ν and find the smallest countable subset $\mathfrak{c}_1 \supseteq \mathfrak{c}_0$ of the basis $\mathfrak{B}$ which span a group containing all of the elements of $\overline{\mathfrak{c}_i}$. Iterate a countable number of times.

We then have for all i, $\mathfrak{c}_i' \subseteq \mathfrak{c}_{i+1}'$ and

$$\overline{F_n} = \sum_{\alpha \in \mathfrak{c}_n} \overline{b_\alpha}\, \bar{\mathbb{Z}} \subseteq \overline{G_n} = \sum_{\sigma \in \mathfrak{c}_n'} \overline{x_\sigma}\, \bar{\mathbb{Z}} \subseteq \overline{F_{n+1}} = \sum_{\alpha \in \mathfrak{c}_{n+1}} \overline{b_\alpha}\, \bar{\mathbb{Z}}. \qquad (*)$$

Set $H = \sum_{\sigma\in\bigcup_{n=0}^{\infty} \mathfrak{C}'_n} x_\sigma \mathbb{Z}$. Clearly H is a direct summand of G. Moreover, H is countably generated since the indexing set is a countable union of countable sets. Eq. $(*)$ forces

$$\bar{H} = \sum_{\alpha\in\bigcup_{i=0}^{\infty} F_i} \overline{b_\alpha}\,\bar{\mathbb{Z}}. \qquad \square$$

Lemma 2.2. *Let G be a nonzero free abelian group, and let $\mathfrak{B}$ be a basis of $\bar{G}$ as a (free) $\bar{\mathbb{Z}}$-module. Then G is the union of a well-ordered (by inclusion) family $\{H_\mu\colon \mu < \Omega\}$ of subgroups such that H_μ and $\bigcup_{\kappa<\mu} H_\kappa$ are direct summands of G for every μ in the ordinal Ω; for each μ, $H_\mu/\bigcup_{\kappa<\mu} H_\kappa$ is countable; and each $\overline{H_\mu}$ is generated by some subset of the $\{\overline{b_\alpha}\colon \alpha\in\mathfrak{I}\}$.*

Proof. Fix a basis $\mathfrak{X}$ of G. Well order $\mathfrak{B}$. Assume we have H_κ for all $\kappa<\mu$ such that

(i) Each H_κ is generated by a subset of $\mathfrak{X}$;
(ii) $\overline{H_\kappa}$ is generated by some subset of $\mathfrak{B}$; and
(iii) $H_\kappa \supset H_{\kappa'}$ if $\kappa > \kappa'$.
(iv) $H_\kappa/\bigcup_{\kappa'<\kappa} H_{\kappa'}$ is countably generated.

H_κ and $\bigcup_{\mu<\kappa} H_\mu$ are direct summands of G since they are generated by subsets of our fixed basis. If $\bigcup_{\kappa<\mu} H_\kappa \neq G$, that union cannot map onto $\bar{G}$. Let $\overline{b_\beta}$ be the smallest element of $\mathfrak{B}$ (under the well ordering of $\mathfrak{B}$) not in $\overline{\bigcup_{\kappa<\mu} H_\kappa}$. Apply Lemma 2.1 to get a countably generated subgroup K_μ generated by elements of $\mathfrak{X}$ with $\overline{b_\beta}\in\overline{K_\mu}$ and $\overline{K_\mu}$ generated by a subset of $\mathfrak{B}$. Set $H_\mu = K_\mu + \bigcup_{\nu<\mu} H_\nu$. Since H_μ clearly has the required properties and this process must eventually give all of G (at least by the order type of $\mathfrak{B}$), by transfinite induction we are done. $\square$

Corollary 2.3. *Assume that, for any countably generated free abelian group G with $\mathfrak{B}$ a basis for $\bar{G}$, there is a direct decomposition lifting of*

$$\bar{G} = \bigoplus_{\alpha\in\mathfrak{I}} \overline{b_\alpha}\,\bar{\mathbb{Z}}$$

to the direct decomposition

$$G = \bigoplus_{\alpha\in\mathfrak{I}} y_\alpha \mathbb{Z}.$$

Then, Theorem A is true for any free abelian group G.

Proof. Using the notation of Lemma 2.2, we let $G=\bigcup_{\mu<\Omega} H_\mu$ where for all $\mu<\Omega$, $H_\mu = K_\mu + \bigcup_{\kappa<\mu} H_\kappa$ with K_μ countably generated. For each b_{α_i} in $K_\mu \setminus \bigcup_{\kappa<\mu} H_\kappa$, set $b_{\alpha_i} = c_i + b'_i$, where c_i is the projection of b_{α_i} to $\bigcup_{\kappa<\mu} H_\kappa$. If $b'_i = 0$, ignore it and renumber. By assumption, we can lift the direct sum decomposition of the quotient

$$\overline{H_\mu \Big/ \bigcup_{\kappa<\mu} H_\kappa} \approx \overline{K_\mu \Big/ K_\mu \cap \bigcup_{\kappa<\mu} H_\kappa} = \bigoplus_{i=0}^{\infty} \overline{b'_i}\,\bar{\mathbb{Z}}$$

to a direct sum decomposition

$$K_\mu \Big/ K_\mu \cap \bigcup_{\kappa<\mu} H_\kappa = \bigoplus_{b_{\alpha_i} \in K_\mu \setminus \cup_{\kappa<\mu} H_\kappa} y'_i \mathbb{Z}$$

with units $\{u_i\}$ such that $y'_i - u_i b'_i \in p^\nu G$. Now set $y_{\alpha_i} = y'_i + u_i c_i$ so that y_{α_i} lifts b_{α_i}.

Assume for all $\mu < \lambda$, $H_\mu = \bigoplus_{\kappa \le \mu} L_\kappa$, where L_κ is the free group generated by a lifting of the decomposition of $H_\kappa / \bigcup_{\kappa' < \kappa} H_{\kappa'}$ generated by the appropriate subset of $\mathfrak{B}$. Then we have $\bigcup_{\mu<\lambda} H_\mu = \bigoplus_{\mu<\lambda} L_\mu$ and by the above, $H_\lambda = L_\lambda \oplus \bigoplus_{\mu<\lambda} L_\mu$. By transfinite induction we get $G = \bigoplus_{\lambda<\Omega} L_\lambda$. □

Infinite Gaussian elimination modulo p^ν. The reader is assumed thoroughly familiar with the details of Gaussian elimination as developed in an introductory linear algebra course. Infinite Gaussian elimination on a row finite $\omega \times \omega$ matrix can proceed very much like the algorithm on a finite matrix. As in [1], one looks for a pivot in a row rather than a column as in many texts and standard implementations of finite Gaussian elimination. That insures that only a finite number of entries need to be examined to either obtain a unit pivot or to know that no such pivot exists. Subtracting multiples of a pivot row from all other rows to make entries in the pivot column equal to 0 will, in general, involve an infinite number of operations before the algorithm is complete. To avoid this, in the infinite case, rows are included with previously obtained pivot rows one at a time, and one clears the previously obtained pivot columns in a row at the time that the row is included, and then finds a pivot if possible and clears above the pivot in the new pivot column. In the infinite case there is no LU decomposition or forward pass and back substitution because these might lead to rows changing infinitely often, and there are no row permutations because some row might conceivably be permuted to a higher numbered position an infinite number of times and thus never examined for a pivot. However, it is still the case that a row finite $\omega \times \omega$ matrix is invertible if and only if with these modifications of standard Gaussian elimination, infinite Gaussian elimination will now reduce the matrix to a matrix whose columns are a permutation of the columns of the identity matrix.

We now modify infinite Gaussian elimination to produce an algorithm which we call infinite Gaussian elimination modulo p^ν.[1] This algorithm clearly also works if we have a finite matrix A. We indicate the variables needed in the algorithm with a little information about them, then give the steps of the algorithm, and then add a step-by-step explanation of what unusual steps do. We start with a row finite $\omega \times \omega$ matrix $\boldsymbol{A}$ with entries in $\mathbb{Z}$. In our proof of Theorem A, the rows of $\boldsymbol{A}$ will be some lifting of a given basis for $\bar{\bar{\mathbb{Z}}}^{(\omega)}$ to elements of $\mathbb{Z}^{(\omega)}$.

By the expression 'principal submatrix' of an infinite matrix, we will mean the submatrix obtained by taking the first n rows and first k columns of the matrix, where

[1] The author has a working Maple V implementation of this algorithm. See the appendix in a copy of this paper archived on http://arXiv.org as math.RA/0007091, and there is a link to the Maple V worksheet at http://www.math.rutgers.edu/pub/∼osofsky/index.html.

n and k are both finite. A 'principal minor' will be the determinant of a square principal submatrix.

Additional variables are needed to perform the algorithm. We use a diagonal matrix $\boldsymbol{U}$ (or a countable row vector) to hold units modulo p^v. Multiplying row i of $\boldsymbol{A}$ by an appropriate unit $\boldsymbol{U}_{i,i}$ enables us to make a crucial determinant 1. The actual row reduction is done in arbitrarily large but finite principal submatrices of an $\omega \times \omega$ matrix $\boldsymbol{R}$. Another $\omega \times \omega$ matrix $\boldsymbol{C}$ (for candidates) holds, in a finite principal submatrix, the current candidates for lifting basis elements times units. These candidates change during the elimination but each row only changes a finite number of times. As the algorithm progresses, we multiply (an initial segment of) row i of $\boldsymbol{A}$ by the appropriate unit $\boldsymbol{U}_{i,i}$ (integer relatively prime to p^v) and then insert it into both $\boldsymbol{R}$ and $\boldsymbol{C}$. All changes to $\boldsymbol{C}$ other than the concatenation of rows from $\boldsymbol{UA}$ consist of adding multiples of p^v to entries so nothing changes modulo p^v. In addition, we use a finite square matrix $\boldsymbol{M}$ which is generated from a submatrix of $\boldsymbol{C}$ and has determinant 1.

At the end of each loop of this algorithm, the matrix $\boldsymbol{R}$ will be a row reduction of $\boldsymbol{C}$ with row operations captured by $\boldsymbol{M}$. Also, any entry of $\boldsymbol{R}$ which is a multiple of p^v is 0; it is set to 0 before any arithmetic is done using it. At any given stage of the algorithm we work with finite matrices large enough to hold all nonzero entries in a finite number of rows. Moreover, the results of each loop of the algorithm applied to $\bar{\boldsymbol{A}}$ are identical with the results of applying normal infinite Gaussian elimination to $\boldsymbol{A}$.

Algorithm 1 (Infinite Gaussian elimination modulo p^v)**.** We start with an $\omega \times \omega$ integer valued row finite matrix $\boldsymbol{A}$.

Step 1: Initialize. Let your row index I be set to 0. Set up the matrix variables $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{R}$ and $\boldsymbol{U}$. Set up a row vector J to hold pivot columns. Read the 0^{th} row of $\boldsymbol{A}$ into $\boldsymbol{R}$, replacing any element divisible by p^v with 0.

Step 2: For K going from 0 to $I-1$, subtract $\boldsymbol{R}_{I,J(K)}$ times row K of $\boldsymbol{R}$ from row I of $\boldsymbol{R}$.

Step 3: Search row I of $\boldsymbol{R}$ for the first entry which is relatively prime to p. If no such element is found then STOP. The rows of $\boldsymbol{A}$ do not form a basis modulo p^v. Otherwise, let the first entry relatively prime to p be in column $J(I)$, and call column $J(I)$ the Ith pivot column.

Step 4: Set $\boldsymbol{U}_{I,I}$ equal to an integer u such that $u\boldsymbol{R}_{I,J(I)} \equiv 1 \bmod p^v$. Multiply row I of $\boldsymbol{A}$ by u. If some entry in the resulting row is a multiple of p^v, set that entry to 0. Insert the result as row I in both $\boldsymbol{C}$ and $\boldsymbol{R}$.

Step 5: For K going from 0 to $I-1$, subtract $\boldsymbol{R}_{I,J(K)}$ times row K of $\boldsymbol{R}$ from row I of $\boldsymbol{R}$.

Step 6: The pivot in row I of $\boldsymbol{R}$ is now congruent to 1 modulo p^v. Subtract a multiple of p^v from it to make the pivot 1. Subtract the same multiple of p^v from the $(I,J(I))$ entry of $\boldsymbol{C}$.

Step 7: If any entry in row I of $\boldsymbol{R}$ is a multiple of p^v, subtract that multiple of p^v from the corresponding entry in $\boldsymbol{C}$ and set the entry in $\boldsymbol{R}$ equal 0.

Step 8: For K going from 0 to $I-1$, subtract $\boldsymbol{R}_{K,J(I)}$ times row I of $\boldsymbol{R}$ from row K of $\boldsymbol{R}$ to clear every entry in column $J(I)$ above the Ith row.

Step 9: If any entry in $\boldsymbol{R}$ is a multiple of p^v, then set that entry equal to 0.

Step 10: Set $\boldsymbol{M}$ equal to the matrix $[\boldsymbol{C}_{K,J(K)}]_{0\le K\le I}$. Set $\boldsymbol{C}=\boldsymbol{MR}$.

Step 11: For each nonpivot column ℓ of $\boldsymbol{C}$, check to see if the first nonzero entry $\boldsymbol{C}_{k,\ell}$ is devisible by p^v. If so, form the set $\mathfrak{S}_\ell$ consisting of all l_i such that column l_i is a pivot column, $\boldsymbol{C}_{k,l_i}$ is the first nonzero entry in column l_i, and $\boldsymbol{R}_{l_i,\ell}\neq 0$. If $\mathfrak{S}_\ell\neq\emptyset$, check if p^v times the gcd, d, of $\mathfrak{S}_\ell$ divides $\boldsymbol{C}_{k,\ell}$. If so, express this gcd d as a sum $\sum_{\mathfrak{S}_\ell}\boldsymbol{C}_{k,l_i b_{l_i}}$. Form a column vector with zeros everywhere except for $b_{l_i}\cdot\boldsymbol{C}_{k,\ell}/d$ in row l_i, and add this to column ℓ of $\boldsymbol{R}$. Premultiply by $\boldsymbol{M}$, and use the result as the new column ℓ of $\boldsymbol{C}$. The new $\boldsymbol{C}_{k,\ell}$ will be 0.

Step 12: Read row $I+1$ of $\boldsymbol{A}$ into $\boldsymbol{R}$, replacing multiples of p^v by 0.

Step 13: Increment I by 1 and GOTO Step 2.

END

That is the end of the algorithm. To get a picture of what is happening, at the end of the $(n-1)$th loop at Step 12 the column permuted matrix $\boldsymbol{R}$ (picturing $j(i)$ as though it were i) looks like

$$R=\left[\begin{array}{cccc|c|c}
1 & 0 & \cdots & 0 & r_{0,n} & \\
0 & 1 & \cdots & 0 & r_{1,n} & \\
\vdots & \vdots & \ddots & \vdots & \vdots & B\ \cdots \\
0 & 0 & \cdots & 1 & r_{n-1,n} & \\
\hline
r_{n,0} & r_{n,1} & \cdots & r_{n,n-1} & r_{n,n} & \boldsymbol{D}\ \cdots
\end{array}\right]$$

for an appropriate finite matrix $\boldsymbol{B}$ and finite row $\boldsymbol{D}$, and all entries in $\boldsymbol{R}$ which are divisible by p^v are 0.

Now for a more detailed explanation of how this algorithm works. In the permuted matrix used in the discussion, $j(i)$ will be treated as though it were i to aid in visualization of the progress of the algorithm. That is, we will pretend that we have permuted the columns of the matrix.

Step 2 is the first pass at clearing already obtained pivot columns (which have pivot 1) in row i. It is used to get the unit mod p^v we must multiply the ith row of $\boldsymbol{A}$ by to make sure that we can make the pivot in row i equal to 1. It is not performed when $i=0$.

Step 6 relies on the claim that the pivot is congruent to 1 modulo p^v. Why is that claim true? Adding one row of a matrix to another corresponds to premultiplication by a matrix of determinant 1. After Step 3, if we look at the principal minor of the column permuted matrix $\boldsymbol{R}$, it has determinant the (i,i) entry of the permuted $\boldsymbol{R}$ because it is upper triangular with all other diagonal entries 1. When we multiply what was the last row before Step 3 by u, we make that determinant congruent to 1 modulo p^v. Now we redo the elementary row operations of determinant 1 to get an upper triangular matrix with element in the $(i,j(i))$ slot equal to the determinant.

In Step 6, subtracting multiples of p^ν from the same entries in both $\boldsymbol{R}$ and $\boldsymbol{C}$ does not change $\bar{\boldsymbol{C}}$ and does insure that the elementary row operations we have done so far will reduce the new $\boldsymbol{C}$ to the new $\boldsymbol{R}$.

Since we want entries in $\boldsymbol{R}$ congruent to $0 \bmod p^\nu$ to be 0, we set them to 0 in Step 9. This can only affect entries in nonpivot columns. Now, we must make sure that our $\boldsymbol{C}$ row reduces to the new $\boldsymbol{R}$. This is done in Step 10. At this stage, the appropriate principal submatrix of the column permuted matrix $\boldsymbol{R}$ is the identity matrix. So the row operations we have done have reduced the corresponding principal submatrix of the column permuted matrix $\boldsymbol{C}$ to the identity. By standard linear algebra, the matrix $\boldsymbol{M}$ is the inverse of the product of the elementary matrices which produce this elimination by premultiplication. Thus from $\boldsymbol{R} = \boldsymbol{M}^{-1}\boldsymbol{M}\boldsymbol{R}$ we see that setting $\boldsymbol{C} = \boldsymbol{M}\boldsymbol{R}$ gives us a matrix which row reduces to the new $\boldsymbol{R}$, and since $\boldsymbol{R}$ did not change modulo p^ν, neither did $\boldsymbol{M}\boldsymbol{R}$.

In Step 11, the algorithm bounds the power of p^ν that can divide entries of $\boldsymbol{C}$ after the corresponding row of $\boldsymbol{A}$ becomes all zeros. This step may change $\boldsymbol{C}$ and nonzero entries in $\boldsymbol{R}$ modulo p^ν. The several imposed conditions on $\mathfrak{S}_\ell$ insure that no zero entry in column ℓ above row k of $\boldsymbol{R}$ becomes nonzero, and the divisibility property makes the added vector a multiple of p^ν. If a nonzero entry appears in $\boldsymbol{C}$ after all the nonzero mod p^ν entries in its row occur in pivot columns, that multiple of p^ν may propagate, but the propagation eventually leads to entries in the row divisible by higher powers of p^ν, and eventually Step 11 will make all of these entries zero. Thus Step 11 makes sure that no row has an infinite number of entries congruent to 0 modulo p^ν.

New row operations are only done to the rows above the pivot row when their entries in the current pivot column is nonzero. Hence once the finite set of rows from 0 to i have zero entries except for a pivot of 1, and there are no more nonzero multiples of p^ν in these rows of $\boldsymbol{C}$, those rows will no longer be affected by the elimination process.

The last steps of the algorithm just set up for the next loop.

Proof of Theorem A. By Corollary 2.3, it is enough to show that, for a countably generated free abelian group G with $\mathfrak{B}$ a basis for $\bar{G}$, there is a direct decomposition lifting of

$$\bar{G} = \bigoplus_{\alpha \in \mathfrak{B}} \overline{b_\alpha}\, \bar{\mathbb{Z}}$$

to the direct decomposition

$$G = \bigoplus_{\alpha \in \mathfrak{B}} y_\alpha \mathbb{Z}$$

Form a matrix $\boldsymbol{A}$ whose rows are some lifting of $\mathfrak{B}$. Do infinite Gaussian elimination modulo p^ν on $\boldsymbol{A}$. Since the rows of $\bar{\boldsymbol{A}}$ form a basis for $\bar{\mathbb{Z}}^{(\omega)}$ and modulo p^ν this algorithm agrees with infinite Gaussian elimination, after a finite number of steps, the top $i+1$ rows of $\boldsymbol{R}$ will be rows of the identity and all rows of the identity will eventually arise as rows of $\boldsymbol{R}$. Since every entry of R which is zero modulo p^ν is actually 0, $\boldsymbol{C}$ is row reduced to the identity provided every row at some point stops changing in taking the product $\boldsymbol{MR}$. Since all of the entries of row n of $\boldsymbol{C}$ which are not congruent to $0 \bmod p^{v}$ are contained in a finite number of columns, any row of $\boldsymbol{C}$ ceases to change when all the rows of the identity with 1 in those columns have been obtained in the matrix $\boldsymbol{R}$. Hence after an infinite number of steps each row of $\boldsymbol{C}$ will have stabilized and the stabilized rows of $\boldsymbol{C}$ will form a basis for $\boldsymbol{Z}^{(\omega)}$ which lifts the direct sum decomposition. $\square$

3. Lattices of commuting idempotents

Definitions and notation. The following notation will be used, usually without comment, in the rest of this paper.

Let $\mathscr{E}$ be a lattice of commuting idempotents in a ring R with 1, that is, $\mathscr{E}$ is closed under multiplication and addition of orthogonal idempotents. The idempotents in $\mathscr{E}$ together with the identity generate a boolean algebra $\mathscr{B}$ under multiplication as in R but in addition the symmetric difference $e +_{\mathscr{B}} f = e(1-f) + f(1-e)$. Let $\mathbb{Z}[\mathscr{B}]$ be the semigroup algebra of $\langle \mathscr{B}, \cdot \rangle$, that is, the free abelian group with basis the elements of $\mathscr{B}$ and multiplication the multiplication as in $\mathscr{B}$. Let

$$\mathscr{S} = \mathbb{Z}[\mathscr{B}] / \langle (e+f) - e - f \colon ef = 0 \rangle,$$

where $\mathscr{S}$ is a free lattice ring in the sense that it can be formed for any modular, complemented lattice and has appropriate universal properties with respect to embedding such lattices in rings.

For convenience, we will assume that $\mathscr{E}$ is a Boolean ideal, that is, if $f = f^2 \in \mathscr{E}R$, then $f \in \mathscr{E}$. This does not change $\mathscr{E}R$.

Elementary properties of $\mathscr{S}$. Much of the known material assumed in this subsection can be found in graduate level text books such as [3].

The next proposition is essentially a sequence of remarks, included with short proofs.

Proposition 3.1. *The following hold for the free lattice ring $\mathscr{S}$:*
(a) *The additive group of $\mathscr{S}$ is torsionfree.*
(b) *The lattice of idempotent generated ideals of $\mathscr{S}$ is isomorphic to $\mathscr{B}$.*
(c) *Any finitely generated ideal of $\mathscr{S}$ is cyclic and isomorphic to a sum $\sum_{i=1}^{n} f_i\mathscr{S}$ for some set of orthogonal idempotents $\{f_i\} \subseteq \mathscr{B}$.*
(d) *R is an $\mathscr{S}$-module under the map induced by the inclusion of $\mathscr{B}$ in R.*
(e) *The projective dimension of an idempotent generated ideal I of $\mathscr{S}$ is greater than or equal to the projective dimension over R of the module $I \otimes_{\mathscr{S}} R$.*

Proof. (a) The kernel of the ring map from $\mathbb{Z}$ to $\mathscr{S}$ is generated by idempotents and so pure.

(b) Any element of $\mathscr{S}$ is of the form $\sum_{i=1}^{n} e_i n_i$ where $\{e_i\} \subseteq \mathscr{B}$ are pairwise orthogonal and $n_i \in \mathbb{Z}$. Assume such an element is idempotent. By the torsionfree property of $\langle \mathscr{S}, + \rangle$, the n_i must be all 1, and $e = \sum_{i=1}^{n} e_i \in \mathscr{E}$. But then the symmetric difference of e and f is the same as in $\mathscr{B}$.

(c) Given a finite set of idempotents $\{e_i\colon 1 \le i \le n\} \subseteq \mathscr{B}$, the minimal nonzero idempotents in the lattice they generate will be pairwise orthogonal and generate the same lattice. Since $\mathscr{S}$ is a quotient of the ring $\mathbb{Z}[\mathscr{B}]$, any element of $\mathscr{S}$ is of the form $\sum_{j=1}^{m} e_j n_j$.

Moreover, if the $\{e_j\}$ happen to be orthogonal, $(\sum_{j=1}^{m} e_j n_j)\mathscr{S} = \sum_{j=1}^{m} (e_j n_j \mathscr{S})$.

Now, let I be the finitely generated ideal

$$I = \sum_{i=1}^{k} \left(\sum_{j=1}^{l_j} e_{i,j} n_{i,j} \mathscr{S} \right) \subseteq \mathscr{S}.$$

Split each $e_{i,j}$ into an orthogonal sum of the nonzero minimal elements in the lattice generated by $\{e_{i,j}\colon 1 \le j \le l_j, 1 \le i \le k\}$. Collecting multiples of each of these minimal elements, we get a generator for I of the form $\sum_{i=1}^{k'} f_i m_i$ where the $\{f_i\}$ are pairwise orthogonal idempotents in $\mathscr{E}$. But $(\sum_{i=1}^{k''} f_i m_i)\mathscr{S} \approx \bigoplus_{i=1}^{k'} f_i \mathscr{S}$ if we ignore terms with $m_i = 0$.

(d) The obvious map $\mathbb{Z}[\mathscr{B}] \to R$ is a ring homomorphism whose kernel contains

$$\langle (e+f) - e - f : ef = 0 \rangle.$$

(e) I is a direct limit of idempotent generated cyclics and so flat. A projective resolution

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_0 \to I \to 0$$

is therefore pure exact. Moreover, since P_i is a projective $\mathscr{S}$-module, $P_i \otimes_{\mathscr{S}} R$ is a projective R-module. Thus,

$$\cdots \to P_i \otimes_{\mathscr{S}} R \to P_{i-1} \otimes_{\mathscr{S}} R \to \cdots \to P_0 \otimes_{\mathscr{S}} R \to I \otimes_{\mathscr{S}} R \to 0$$

is a projective resolution of $I \otimes_{\mathscr{S}} R$. If the kernel of a map $P_i \to P_{i-1}$ is $\mathscr{S}$-projective, by pure exactness and the fact that tensoring preserves projectivity we see that the kernel of $P_i \otimes_{\mathscr{S}} R \to P_{i-1} \otimes_{\mathscr{S}} R$ is R-projective. Thus, the $\mathscr{S}$-projective dimension of I is at most i implies that the R-projective dimension of $I \otimes_{\mathscr{S}} R$ is also at most i. □

Proposition 3.2. *The additive group of $\mathscr{S}$ is a free abelian group.*

Proof. Let $\mathfrak{X}$ be the family of all subsets X of $\mathscr{B} \setminus \{0\}$ such that whenever $\{e_i\}$ is a set of orthogonal idempotents in X, if $\{f_j\}$ is any set of orthogonal idempotents such that $\{e_i\} \neq \{f_j\}$ and $\sum_i e_i = \sum_j f_j$, then at least one $f_j \notin X$. $\mathfrak{X}$ is an inductive poset under $\subseteq$, so by Zorn's lemma there is a maximal element B in $\mathfrak{X}$. B is $\mathbb{Z}$-linearly independent in $\mathscr{S}$ because the only relations on the $\mathbb{Z}$-linearly independent idempotents in $\mathbb{Z}[\mathscr{B}]$ set an idempotent equal to an orthogonal sum of other idempotents. B will be a vector space basis for $\mathscr{B}$ over the field of 2 elements. Let $f \in \mathscr{B}\setminus\{0\}$. If $f \notin B$, then $B \cup \{f\} \notin \mathfrak{X}$. Hence there must be a set $\{e_i\}$ of orthogonal idempotents in $B \cup \{f\}$ and a different set $\{f_i\} \subseteq B \cup \{f\}$ of orthogonal idempotents with $\sum_{i=1}^{n} e_i = \sum_{j=1}^{m} f_j$. If $f \in \{e_i\} \cap \{f_j\}$ then we get $\sum_{e_i \neq f} e_i = \sum_{f_j \neq f} f_j$ with all summands in B, a contradiction. Similarly, if $f \notin \{e_i\} \cup \{f_j\}$ we get a contradiction. Hence f is in precisely one of the two sets, say $f = e_1$. Then $f = \sum_j f_j - \sum_{i=2}^{n} e_i$ is in the span of B. $\square$

Proposition 3.2 strongly reinforces the observation that $\mathscr{S}$ is a free object. The basis found for its additive group will be a basis for $\mathscr{S} \otimes_{\mathscr{S}} F$ over F for any field F.

In his proof of the affirmative answer to the Wiegand question in the case $n = 1$, R.S. Pierce proved the next lemma with completely different terminology. See [9, Lemma 2.7].

Proposition 3.3. *Let $\{\kappa_\alpha\}$ be a set of elements in a submodule of a free $\mathscr{S}$-module K, where the $\{\kappa_\alpha \otimes 1\}$ are all nonzero. Then if $\{\kappa_\alpha \otimes_{\mathscr{S}} R\}$ is R-independent in $K \otimes_{\mathscr{S}} R$, then $\{\kappa_\alpha\}$ is $\mathscr{S}$-independent in K.*

Proof. Assume not. Then there is a shortest sum $\sum_{i=1}^{n} \kappa_{\alpha_i} s_i = 0$ where the summands are all nonzero in (S). Considering elements of the free module K as consisting of sums of idempotents times basis elements, we see that the annihilator of each $\kappa_{\alpha_i} s_i$ is generated by an idempotent $(1 - \varepsilon_i)$. Since n is the smallest number of summands that can give you a zero and $\sum_{i=1}^{n} \kappa_{\alpha_i} s_i \varepsilon_1 = 0$, we have $\kappa_{\alpha_i} s_i \varepsilon_1 \neq 0$ for all i. Similarly $\kappa_{\alpha_i} s_i \varepsilon_1 \varepsilon_2 \neq 0$ for all i. Continuing in this manner we get $\kappa_{\alpha_i} s_i \prod_{j=1}^{n} \varepsilon_j \neq 0$ for all i. Then $\sum_i \kappa_i s_i \prod_{j=1}^{n} \varepsilon_j$ has all summand nonzero and there is an integer m such that $\sum_{i=1}^{n} \kappa_{\alpha_i} s_i m^{-1} \prod_{i=1}^{n} \varepsilon_i$ is an element not divisible by any integers other than ± 1 in the free abelian additive group of K. But then $\sum_{i=1}^{n} \kappa_{\alpha_i} s_i m^{-1} \prod_{i=1}^{n} \varepsilon_i \otimes 1$ is nonzero in $K \otimes_{\mathscr{S}} R$ and each of the summands is nonzero. $\square$

We quote a proposition due to Kaplansky that is basic to almost all studies of infinitely generated projective modules, with two consequences giving rise to the same result for von Neumann regular rings.

Proposition 3.4 (Kaplansky)**.** *A projective module over any ring is a direct sum of countably generated submodules. From this we obtain*

(a) *Any projective right module over a von Neumann regular ring is isomorphic to a direct sum of cyclic* (*idempotent generated*) *right ideals.*
(b) *Any projective module over a commutative semihereditary ring is isomorphic to a direct sum of finitely generated right ideals.*

See [2] for a proof. The proof of this theorem is the template on which the preliminary proofs in Section 2 are based.

The proof of an affirmative answer to the Wiegand question. We now complete our work on the Wiegand question.

Proposition 3.5. *Let R be a commutative von Neumann regular ring. Let F be a projective $\mathscr{S}$-module and let K be any pure submodule of F. Then if $K \otimes_{\mathscr{S}} R$ is projective as an R-module, then K is projective as an $\mathscr{S}$-module.*

Proof. Since $K \otimes_{\mathscr{S}} R$ is a projective R-module, it is a direct sum of the form $K \otimes_{\mathscr{S}} R = \bigoplus_{\alpha} x_\alpha R$ where for each α there is an e_α such that $x_\alpha R \approx e_\alpha R$. If any e_α is of finite but composite order, express it as an orthogonal sum of idempotents of prime power order by the Chinese Remainder Theorem. In the von Neumann regular case where there are no nilpotent elements, the prime power must be the prime itself. We can then divide the indexing set into a family of subsets

$$\mathfrak{F}_p = \{\alpha\colon char(e_\alpha \otimes_{\mathscr{S}} R) = p\}$$

for p a prime or 0.

Consider the map $K \xrightarrow{I_k \otimes 1} K \otimes_{\mathscr{S}} R \to \bigoplus_{\alpha \in \mathfrak{F}_0} x_\alpha R$. Its image is a projective $\mathscr{S}$-module, so it splits. Hence, without loss of generality we can work with the kernel of this map in place of K and assume that $K \otimes_{\mathscr{S}} R$ is torsion. But then it is the orthogonal sum of its p-primary components so we need only look at sums of the form $\bigoplus_{\alpha \in \mathfrak{F}_p} x_\alpha R$ for a fixed prime p. That is, without loss of generality, $K \otimes_{\mathscr{S}} R$ is p-primary. Since the additive group of $\mathscr{S}$ is free, the additive group of F is free and hence K is a subgroup of a free abelian group and so free. By Theorem A, there is a basis $\{b_\lambda\}$ of K which lifts the direct sum decomposition $G_p/pG_p = \bigoplus_{\alpha \in \mathfrak{F}_p} x_\alpha R$ to a direct sum decomposition of K.

For every α, let $\mathfrak{B}_\alpha = \{b_\lambda\colon b_\lambda \otimes 1 \in x_\alpha R\}$. Let H_α be the $\mathscr{S}$-submodule of K generated by $\mathfrak{B}_\alpha$. Since the generators of H_α all map to $x_\alpha R$ under $Id_K \otimes 1_R$, so must H_α. Since H_α contains $\mathfrak{B}_\alpha$ and $\bigcup_\alpha \mathfrak{B}_\alpha$ is a basis for K, $K = \sum_\alpha H_\alpha$. By Proposition 3.3, that sum is direct.

Select any element y in H_α which maps to x_α. This y is an element lying in a finitely generated free submodule of F. Hence it is of the form $y = \sum_{i=1}^{m} \sum_{j=1}^{k_i} c_{i,j} e_{i,j} n_{i,j}$ where the $c_{i,j}$ are basis elements of F, and we can use our little trick of decomposing into the minimal idempotents in a finite lattice to get that $e_{i,j}$ and $e_{k,l}$ are either the same idempotent or orthogonal. Because of the $\mathbb{Z}$-purity of K, we may find a $y_\alpha \in H_\alpha$ such that each sum of the form $\sum_{e_{i,j} = e_{k,l}} c_{i,j} e_{i,j} n_{i,j}$ is of content 1 and hence this y_α generates a direct summand of F. But then $y_\alpha \mathscr{S}$ is a direct summand of H_α which maps to the same submodule of $K \otimes_{\mathscr{S}} R$. We conclude that $H_\alpha = y_\alpha S$ for all α. Thus $K = \bigoplus_\alpha y_\alpha \mathscr{S}$ so K is projective. $\square$

Corollary 3.6. *Let F be a projective $\mathscr{S}$-module of the form*

$$F = \bigoplus_{\alpha \in \mathfrak{A}} e_\alpha \mathscr{S},$$

where each $e_\alpha \mathscr{S}$ is isomorphic to an ideal of $\mathscr{S}$ contained in $\mathscr{E}\mathscr{S}$. Then for any pure submodule K of F, $\mathrm{pd}_R(K \otimes_{\mathscr{S}} R) = \mathrm{pd}_{\mathscr{S}}(K)$.

Proof. We can take a short projective resolution of K over $\mathscr{S}$, say

$$0 \to L \to P \to K \to 0$$

is exact with P projective and, like F, a direct sum of cyclic projectives of the form $e\mathscr{S}$ for some $e \in \mathscr{E}$. Then if we let $\infty - 1 = \infty$, $\mathrm{pd}_{\mathscr{S}}(L) = \mathrm{pd}_{\mathscr{S}}(K) - 1$. This short exact sequence is pure, so tensoring with R over $\mathscr{S}$ gives a short projective resolution of $K \otimes_{\mathscr{S}} R$

$$0 \to L \otimes_{\mathscr{S}} R \to P \otimes_{\mathscr{S}} R \to K \otimes_{\mathscr{S}} R \to 0$$

with $\mathrm{pd}_R(L \otimes_{\mathscr{S}} R) = \mathrm{pd}_R(K \otimes_{\mathscr{S}} R) - 1$. Induction on $\mathrm{pd}_{\mathscr{S}}(K)$ completes the proof. □

Theorem B (The answer to the Wiegand question)**.** *For any commutative von Neumann regular ring R with a commuting set of idempotents $\mathscr{E}$*, $\mathrm{pd}_R(\mathscr{E}R) = \mathrm{pd}_{\mathscr{S}}(\mathscr{E}\mathscr{S}) = \mathrm{pd}_{\mathscr{B}}(\mathscr{E}\mathscr{B})$.

Proof. $\mathscr{E}\mathscr{S}$ has a projective resolution of the form required in Corollary 3.6. Then Corollary 3.6 gives the desired conclusion. □

One way to summarize this answer to the Wiegand question is to say that, when working in a submodule of a free module over a commutative regular ring, the lattice of direct summands carries all of the information about the module, and the coefficients essentially none. For example, note that in Theorem B, the lattices of direct summands in the three ideals $\mathscr{E}R$, $\mathscr{E}\mathscr{S}$, and $\mathscr{E}\mathscr{B}$ are isomorphic, as they correspond to the idempotents themselves. However, as soon as one gets to free modules on more than one generator, that property fails. Since the number of one-dimensional subspaces of a two-dimensional vector space depends on the cardinality of the field, if $R = \mathscr{S}/3\mathscr{S}$ then the number of direct summands of $eR \oplus eR$ isomorphic to eR and the number of direct summands of $e\mathscr{B} \oplus e\mathscr{B}$ isomorphic to $e\mathscr{B}$ will always be different for any idempotent e.

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