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ADS modules

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ABSTRACT

We study the class of ADS rings and modules introduced by Fuchs (1970) [F]. We give some connections between this notion and classical notions such as injectivity and quasi-continuity. A simple ring R such that R_R is ADS must be either right self-injective or indecomposable as a right R -module. Under certain conditions we can construct a unique ADS hull up to isomorphism. We introduce the concept of completely ADS modules and characterize completely ADS semiperfect right modules as direct sum of semisimple and local modules.

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1. Introduction

The purpose of this note is to study the class of ADS rings and modules. Fuchs [F] calls a right module M right ADS if for every decomposition $M = S \oplus T$ of M and every complement T' of S we have $M = S \oplus T'$. Clearly any ring in which idempotents are central (in particular commutative rings or reduced rings) has the property that R_R is ADS. Moreover, if R is commutative then every cyclic R -module is ADS. We note that every right quasi-continuous module (also known as π -injective module) is right ADS, but not conversely. However, a right ADS module which is also CS is quasi-continuous. We provide equivalent conditions for a module to be ADS. A module need not have an ADS hull in the usual sense but we show that, under some hypotheses, every nonsingular right module possesses a right ADS hull which is unique up to isomorphism. We call a right module M completely ADS if each of its subfactors is ADS. We characterize completely ADS semiperfect right

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modules as direct sums of semisimple and local modules. In particular we give an alternative proof of the characterizations of semiperfect πc -rings (rings whose cyclics are quasi-continuous).

2. Definitions and notations

Throughout every module will be a right module unless otherwise stated. All rings have identity and all modules are unital. A module M is called *continuous* if it satisfies (C1): each complement in M is a direct summand, and (C2): if a submodule N of M is isomorphic to a direct summand of M then N itself is a direct summand of M . A module M is called *quasi-continuous* (π -injective) if it satisfies (C1) and (C3): the sum of two direct summands of M with zero intersection is again a direct summand of M . Equivalently a module M is quasi-continuous if and only if every projection $\pi_i : N_1 \oplus N_2 \rightarrow N_i$, where N_i ($i = 1, 2$) are submodules of M , can be extended to M .

For two modules A and B , we say that A is *B-injective* if any homomorphism from a submodule C of B to A can be extended to a homomorphism from B to A . We note that if A is B -injective and A is contained in B then A is a direct summand of B . A module M is called *semiperfect* if each of its homomorphic images has a projective cover. A submodule N of a module M is *small* in M if for any proper submodule P of M , $P + N \neq M$. We will write $N \ll M$. Let A and P be submodules of a module M . Then P is called a *supplement* of A if it is minimal with the property $A + P = M$.

A module M is *discrete* if it satisfies (D₁): for every submodule A of M there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subset A$ and $M_2 \cap A$ is small in M , and (D₂): if A is a submodule of M such that M/A is isomorphic to a direct summand of M , then A is a direct summand of M . A module M is called *quasi-discrete* if it satisfies (D₁) and (D₃): if M_1 and M_2 are summands of M and $M = M_1 + M_2$ then $M_1 \cap M_2$ is a summand of M .

For any module M , $E(M)$ denotes the injective hull of M . We recall a useful result of Azumaya that for any two modules M and N , if M is N -injective then for any R -homomorphism $\sigma : E(N) \rightarrow E(M)$, $\sigma(N) \subseteq M$.

3. Properties of ADS modules

We begin with a lemma which is useful in checking the ADS property of a module. This was proved by Burgess and Raphael [BR], however, for the sake of completeness, we provide the proof.

Lemma 3.1. *An R -module M is ADS if and only if for each decomposition $M = A \oplus B$, A and B are mutually injective.*

Proof. Suppose M is ADS. We prove A is B -injective. Let C be a submodule of B and let $f : C \rightarrow A$ be an R -homomorphism. Set $X = \{c + f(c) \mid c \in C\}$. Then $X \cap A = 0$. So X is contained in a complement, say K , of A . Then by hypothesis, $M = A \oplus K$. The trick is to define an R -homomorphism $g : B \rightarrow A$ which is a composition of the projection $\pi_K : M \rightarrow K$ along A followed by the projection $\pi_A : M \rightarrow A$ along B and restricting to B . By writing an element $c \in C$ as $c = (c + f(c)) - f(c)$, we see that $\pi_A \pi_K = f$ on C and hence $\pi_A \pi_K$ is an extension of f .

Conversely, suppose for each decomposition $M = A \oplus B$, A and B are mutually injective. Let C be a complement of A . Set $U = B \cap (A \oplus C)$ which is nonzero because $A \oplus C$ is essential in M . Let π_A be the projection of $A \oplus C$ on to A and $f : U \rightarrow A$ be the restriction of π_A to U . This can be extended to $g : B \rightarrow A$, by assumption. Let $b \in B$ and let $D = (b - g(b))R + C$. We claim $D \cap A = 0$. Let $a \in A$ and let $a = br - g(b)r + c$ for some $c \in C$. This gives $br = a + g(b)r - c \in U$ and so $f(br) = a + g(b)r$ because f is the identity on A and 0 on C . This yields $a = 0$, proving our claim. Thus $D = C$ and hence $b - g(b) \in C$ for all $b \in B$. Therefore, $b = (b - g(b)) + g(b) \in C \oplus A$ and so $M = A \oplus B \subseteq C \oplus A$, proving that $M = C \oplus A$. \square

Our next proposition gives equivalent statements as to when a module is ADS analogous to characterization of quasi-continuous modules (cf. [GJ]).

Proposition 3.2. For an R -module M the following are equivalent:

- (i) M is ADS.
- (ii) For any direct summand S_1 and a submodule S_2 having zero intersection with S_1 , the projection map $\pi_i : S_1 \oplus S_2 \rightarrow S_i$ ($i = 1, 2$) can be extended to an endomorphism (indeed a projection) of M .
- (iii) If $M = M_1 \oplus M_2$ then M_1 and M_2 are mutually injective.
- (iv) For any decomposition $M = A \oplus B$, the projection $\pi_B : M \rightarrow B$ is an isomorphism when it is restricted to any complement C of A in M .
- (v) For any decomposition $M = A \oplus B$ and any $b \in B$, A is bR -injective.
- (vi) For any direct summand $A \subseteq^{\oplus} M$ and any $c \in M$ such that $A \cap cR = 0$, A is cR -injective.

Proof. (i) \Rightarrow (ii) Let \hat{S}_2 be a complement of S_1 containing S_2 . Then by definition of ADS module, $M = S_1 \oplus \hat{S}_2$. Hence the canonical projections $\hat{\pi}_1 : S_1 \oplus \hat{S}_2 \rightarrow S_1$ and $\hat{\pi}_2 : S_1 \oplus \hat{S}_2 \rightarrow \hat{S}_2$ are clearly extensions of π_1 and π_2 .

(ii) \Rightarrow (i) Let $M = A \oplus B$ and let C be a complement of A in M . We must show that $M = A \oplus C$. By hypothesis, the projection $\pi : A \oplus C \rightarrow C$ can be extended to an endomorphism $f : M \rightarrow M$. We claim $f(M) \subset C$. Since $A \oplus C$ is essential in M , for any $0 \neq m \in M$, there exists an essential right ideal E of R such that $0 \neq mE \subset A \oplus C$. This gives $f(m)E = \pi(mE) \subset C$. Since C is closed in M , this yields $f(m) \in C$, proving our claim. We also remark that $f^2 = f$, $M = \text{Ker}(f) \oplus \text{im}(f)$ and $\text{Ker}(f) = \{m - f(m) \mid m \in M\}$. We now show that $\text{Ker}(f) = A$. For any $a \in A$, clearly $a = a - f(a) \in \text{Ker}(f)$, hence $A \subset \text{Ker}(f)$. Now let $0 \neq m - f(m) \in \text{Ker}(f)$. There exists $r \in R$ such that $0 \neq (m - f(m))r \in A \oplus C$. This implies $f[(m - f(m))r] = f(mr) - f(f(m)r) = f(mr) - f(mr) = 0$. Since f extends π , this means that $0 \neq (m - f(m))r \in \text{Ker}(\pi) = A$. But A being closed in M , we conclude $A = \text{Ker}(f)$, completing the proof.

(i) \Leftrightarrow (iii) This is Lemma 3.1 above.

(i) \Leftrightarrow (iv) Let C be a complement of A . Then $\text{ker}(\pi_B|_C) = 0$. Since $A \oplus C = (A \oplus C) \cap (A \oplus B) = ((A \oplus C) \cap B) + A$, we have $\pi_B(C) = \pi_B(A \oplus C) = \pi_B((A \oplus C) \cap B) = (A \oplus C) \cap B$. This gives $\pi_B(C) = B$ when M is ADS. On the other hand if $\pi_B(C) = B$ then $M = A \oplus C$, hence M is ADS.

(i) \Leftrightarrow (v) This is classical (cf. Proposition 1.4 in [MM]).

(i) \Rightarrow (vi) Consider C a complement of A containing cR . Since M is ADS we have $M = A \oplus C$. Using (v), this leads to A being cR -injective.

(vi) \Rightarrow (i) This is clear since if $M = A \oplus B$, (vi) implies that A is bR -injective for all $b \in B$ and Proposition 1.4 in [MM] yields that A is B -injective. \square

Let us mention the following necessary condition for a module to be ADS.

Corollary 3.3. Let M_R be an ADS module. For any direct summand $A \subseteq^{\oplus} M$ and any $(a, c, r) \in A \times M \times R$ such that $cR \cap A = 0$ and $\text{ann}(cr) \subseteq r.\text{ann}(a)$ there exists $a' \in A$ such that $a = a'$. If R is a right PID the converse is true.

Proof. By Proposition 3.2(vi), we know that A is cR -injective. Consider $\varphi \in \text{Hom}_R(crR, A)$ defined by $\varphi(cr) = a$. The condition on annihilators guarantees that φ is well defined. By relative injectivity, this map can be extended to $\bar{\varphi} : cR \rightarrow A$, and hence we get $a = \varphi(cr) = \bar{\varphi}(c)r$. We obtain the desired result by defining $a' = \bar{\varphi}(c)$.

If R is a principal ideal domain then the submodules of cR are of the form crR for some $r \in R$. The condition mentioned in the statement of the corollary makes it possible to extend any map in $\text{Hom}_R(crR, A)$ to a map in $\text{Hom}_R(cR, A)$ for any direct summand $A \subseteq^{\oplus} M$. Invoking Proposition 3.2(vi), we can thus conclude that M is ADS. \square

It is known that the sum of two closed submodules of a quasi-continuous module is closed [GJ]. We prove that the direct sum of two closed submodules of an ADS module is again closed when one of them is a summand.

Proposition 3.4. *Let A and B be two closed submodules of an ADS module M such that A is a summand and $A \cap B = 0$. Then $A \oplus B$ is a closed submodule of M .*

Proof. Let C be a complement of A containing B . Since M is ADS, we have $M = A \oplus C$. Let $x = a + c$ be in the closure of $A \oplus B$ in M , where $a \in A$ and $c \in C$. Since $a \in A \subseteq cl(A \oplus B)$, we have that $a \in cl(A \oplus B)$. Hence there exists an essential right ideal E of R such that $cE \subseteq (A \oplus B) \cap C = B$. The fact that B is closed implies $c \in B$. Hence $x \in A \oplus B$, as desired. \square

Remark 3.5. Let A, B be closed submodules of an ADS module M such that A is a direct summand of M . If $A \cap B$ is a direct summand of M , then $A + B$ is closed. Indeed let K be a complement of $A \cap B$. Since M is ADS we have $M = (A \cap B) \oplus K$. Hence $A + B = A \oplus (K \cap B)$. The above proposition then yields the result.

The proposition that follows gives an interesting property of an ADS module. The original statement is due to Gratzner and Schmidt (cf. Theorem 9.6 in [F]). We first prove the following lemma.

Lemma 3.6. *Let $M = B \oplus C$ be a decomposition of M with projections $\beta : M \rightarrow B, \gamma : M \rightarrow C$. Then $M = B \oplus C_1$ if and only if there exists $\theta \in End(M)$ such that $C_1 = (\gamma - \beta\theta\gamma)(M)$.*

Proof. Suppose that $M = B \oplus C_1$ with projections β_1 on B and γ_1 on C_1 . We will show that $\beta_1 = \beta + \beta\theta\gamma$ and $\gamma_1 = \gamma - \beta\theta\gamma$ with $\theta = \gamma - \gamma_1$. We have $B < \ker(\theta)$, so $\theta = \theta\beta + \theta\gamma = \theta\gamma$.

If $m = b + c = b_1 + c_1$, where $b, b_1 \in B, c \in C, c_1 \in C_1$. Then $\theta(m) = c - c_1 = b_1 - b \in B$. Thus $\beta\theta = \theta$. Hence $\gamma_1 = \gamma - \theta = \gamma - \beta\theta\gamma$. Also $\beta_1 = 1_A - \gamma_1 = \beta + \gamma - \gamma_1 = \beta + \beta\theta\gamma$.

Conversely, if β_1, γ_1 are defined as above, that is $\beta_1 = \beta + \beta\theta\gamma$ and $\gamma_1 = \gamma - \beta\theta\gamma$ for any $\theta \in End(M)$, then $\beta_1 + \gamma_1 = 1_A, \beta_1^2 = \beta_1, \gamma_1^2 = \gamma_1, \beta_1\gamma_1 = \gamma_1\beta_1 = 0$. Therefore, $M = \beta_1 M \oplus \gamma_1 M$. Since $\beta_1(M) \subset B$ and $\beta_1(b) = \beta(b) = b$ for $b \in B$, we have $M = B \oplus (\gamma - \beta\theta\gamma)(M)$, as required. \square

Using the same notations as in the previous lemma we state the following corollary.

Corollary 3.7. *A module M is ADS if and only if for any decomposition $M = B \oplus C$ the complements of B in M are all of the form $(\gamma - \beta\theta\gamma)(M)$ for some $\theta \in End(M)$.*

Proposition 3.8. *Let $M = B \oplus C$ be a decomposition of an ADS R -module M . Let β and γ be projections on B and C respectively. Then the intersection D of all the complements of B is the maximal fully invariant submodule of M which has zero intersection with B .*

Proof. Let $\theta \in End(M)$. Then $C_1 = (\gamma - \beta\theta\gamma)(M)$ is again a complement of B . For $c \in D$ we have $(\gamma - \beta\theta\gamma)(c) = c$ and $\gamma c = c$, because $c \in C_1 \cap C$. Hence $\beta\theta c = 0$ and $\theta c \in C$. This holds for all complements C , so $\theta c \in D$, so D is fully invariant in M with $D \cap B = 0$. On the other hand, assume X is fully invariant with $X \cap B = 0$. Since $M = B \oplus C$, and $\pi_B(X) \subseteq X$ and $\pi_C(X) \subseteq X$, this leads to $X = (X \cap B) \oplus (X \cap C) = X \cap C$. Hence $X < C$. Since M is ADS this holds for any complement of B in M , and hence $X \subseteq D$. \square

It is known that an indecomposable regular ring which is right continuous is right self-injective (cf. Corollary 13.20 in [G]). The following theorem is a generalization of this result for simple rings without the assumption of regularity. We may add that an indecomposable two-sided continuous regular ring is simple (cf. [G, Corollary 13.26]).

Theorem 3.9. *Let R be an ADS simple ring. Then either R_R is indecomposable or R is a right self-injective regular ring.*

Proof. Let Q be the right maximal quotient of R which is regular right self-injective. Since R is right (left) nonsingular $E(R) = Q$. Suppose R is not right indecomposable and let e be a nontrivial

idempotent. Then since R is ADS eR is $(1 - e)R$ -injective (cf. Lemma 3.1). Furthermore, since $\text{Hom}((1 - e)Q, eQ) \cong eQ(1 - e)$, $(eQ(1 - e)(1 - e)R \subseteq eR$. Because R is simple, so $R = R(1 - e)R \subseteq Q(1 - e)R$. This yields, $1 \in Q(1 - e)R$. Therefore $Q = Q(1 - e)R$, and so $eQ = eR$. Similarly $(1 - e)Q = (1 - e)R$ hence $R = Q$, i.e. R is a right self-injective regular ring. \square

Corollary 3.10. *A simple regular right continuous ring is right self-injective.*

4. ADS hulls

We now proceed to construct an ADS hull of a nonsingular module. Burgess and Raphael (cf. [BR]) claimed that an example can be constructed of a finite dimensional module over a finite dimensional algebra which has no ADS hull. We show that, under some circumstances, such an ADS hull does exist.

Lemma 4.1. *Suppose M is nonsingular. Then M is ADS iff for every decomposition $E(M) = E_1 \oplus E_2$ where $E_1 \cap M$ is a direct summand of M , then $M = (E_1 \cap M) \oplus (E_2 \cap M)$.*

Proof. Suppose M is ADS. We may write $M = (E_1 \cap M) \oplus K$ where K is a complement of $E_1 \cap M$. Let $e_i : (E_1 \cap M) \oplus (E_2 \cap M) \rightarrow E_i \cap M$ be the projection map. Then by Proposition 3.2(ii) there exists $e_i^* : M \rightarrow M$ that extends e_i . Let $\pi_i : E_1 \oplus E_2 \rightarrow E_i$ be the natural projection. Since $E(M)$ is injective we can further extend e_i^* to $e_i^{**} \in \text{End}(E(M))$. We claim e_i^{**} is an idempotent in $\text{End}(E(M))$. Indeed let $x \neq 0$ be any element in $E(M)$ and A an essential right ideal of R such that $0 \neq xA \subseteq M$. We have $(e_i^{**})^2(x)A = (e_i^*)^2(xA) = (e_i^*)^2(xA) = e_i^*(xA) = e_i^{**}(xA) = e_i^{**}(x)A$. This yields the claim, since M is nonsingular. Thus $e_i^{**}(E(M)) = \pi_i(E(M)) = E_i$. Now $M \subseteq_e E(M) = E_1 \oplus E_2$ implies $E_1 \cap M \subseteq_e (E_1 \oplus E_2) \cap M$. Similarly $E_2 \cap M \subseteq_e E_2$ and so $e_i^{**} = \pi_i$ on $E_1 \cap M \oplus E_2 \cap M \subseteq_e M \subseteq_e E(M)$. Since M is nonsingular $e_i^{**} = \pi_i$ on $E(M)$. In particular, $\pi_i(M) \subseteq M$ and so $M = (\pi_1 + \pi_2)(M) \subseteq \pi_1(M) \oplus \pi_2(M) \subseteq (E_1 \cap M) \oplus (E_2 \cap M)$.

Conversely, let $M = A \oplus B$ and C be a complement of A . We must show that $M = A \oplus C$. Since $A \oplus C \subseteq_e M$, we get $E(M) = E(A) \oplus E(C)$. Since both A and C are closed in M , we have $E(A) \cap M = A$ and $E(C) \cap M = C$. Since A is a direct summand of M we have, thanks to the hypothesis, $M = (E(A) \cap M) \oplus (E(C) \cap M) = A \oplus C$, as desired. \square

Theorem 4.2. *Let M be a right R -module. Then M is ADS if and only if for every $e = e^2, f = f^2 \in \text{End}(E(M))$ with $eM \subseteq M$ and $f(E(M)) = eE(M)$, we have $fM \subseteq M$.*

Proof. Let us prove necessity: $(1 - f)(E(M)) \cap M \subseteq_e (1 - f)(E(M))$ and $f(E(M)) \cap M \subseteq_e f(E(M))$. Thus $((1 - f)(E(M)) \cap M) \oplus (f(E(M)) \cap M) \subseteq_e M$. We claim $f(E(M)) \cap M = eM$. Note first that $e(E(M)) \cap M = f(E(M)) \cap M$. Clearly $eE(M) \cap M \subseteq_e eM$. Let $C = (1 - f)(E(M)) \cap M$. Then $C \oplus eM \subseteq_e M$. Because eM is closed C is a complement of eM in M (cf. Lemma 6.32 in Lam's book). Because M is ADS we have $M = e(M) \oplus C$. Let g be the projection of eM along C , so that $g(M) = e(M)$. Now $g(M) = e(M) \subseteq f(E(M))$. This gives $eM = (M) = fg(M) = fe(M)$. Since C is contained in $(1 - f)(E(M))$, $f(C) = 0$. Then $fM = f(C \oplus eM) = eM \subseteq M$.

Conversely, let $M = eM \oplus (1 - e)(M)$ and C be a complement of $e(M)$ in M . We want to show $M = e(M) \oplus C$. Now, $C \oplus e(M) \subseteq_e M$ and so $E(C) \oplus E(eM) = E(M)$. Hence $E(C) \oplus eE(M) = E(M)$. Let f be the projection on $eE(M)$ along $E(C)$. We have $f(E(M)) = e(E(M))$ and $E(C) = (1 - f)(E(M))$. By hypothesis we have $f(M) \subseteq M$. Let m be in M . Then $m \in M = E(C) \oplus f(E(M))$, say $m = c + f(m)$, where $c \in E(C)$. $c = m - f(m) \in E(C) \cap M = C$, because C is closed. We conclude that $M = C \oplus e(M)$. \square

We may recall that any endomorphism $f \in \text{End}_R(M)$ of a nonsingular module M can be uniquely extended to an endomorphism f^* of its injective hull $E(M)$. Let us mention moreover that if $f = f^2$ then $f^* = (f^*)^2$. Under these notations we obtain the following corollary.

Corollary 4.3. *Let M be a right nonsingular R -module. M is ADS if and only if for every $e = e^2 \in \text{End}(M)$ and $f = f^2 \in \text{End}(E(M))$ with $fE(M) = e^*E(M)$, we have $fM \subset M$.*

We are now ready to show, that under some circumstances, an ADS hull can be constructed for a nonsingular module. For a nonsingular right R -module M , we continue to let e^* denote the unique extension of $e^2 = e \in \text{End}(M)$ to the injective hull $E(M)$ of M .

Theorem 4.4. *Let M_R be a nonsingular right R -module. Let \bar{M} denote the intersection of all the ADS submodules of $E(M)$ containing M . Suppose that for any $e^2 = e \in \text{End}(\bar{M})$ and for any ADS submodule N of $E(M)$ containing M we have $e^*(N) \subset N$. Then, \bar{M} is, up to isomorphism, the unique ADS hull of M .*

Proof. Let Ω be the set of ADS submodules N such that $M < N < E(M)$. Then $\bar{M} = \bigcap_{N \in \Omega} N$. We claim that \bar{M} is ADS. Clearly $E(\bar{M}) = E(M)$. Let $e = e^2 \in \text{End}_R(M)$, $f^2 = f \in \text{End}(E(M))$ such that $e(\bar{M}) \subseteq \bar{M}$ and $f(E(M)) = e^*(E(M))$. Since M is nonsingular and $e(\bar{M}) \subseteq \bar{M}$, we have $e(N) \subseteq N$ for every $N \in \Omega$. So, for every $N \in \Omega$, $f(N) \subset N$ because N is ADS. Let $x \in \bar{M}$. Then $x \in N$ for every $N \in \Omega$. Hence $f(x) \in N$ for every $N \in \Omega$. Therefore $f(x) \in \bigcap_{N \in \Omega} N = \bar{M}$, that is $f(\bar{M}) \subseteq \bar{M}$, proving our claim. \square

Remarks 4.5. Let us remark that the condition stated in the above theorem is in particular fulfilled if we consider the ADS hull of a nonsingular ring. Indeed in this case we consider the ADS rings between R and $Q := E(R)$ and projections are identified with idempotents of the rings. Of course, these idempotents remain idempotents in overrings.

5. Completely ADS modules

Theorem 5.1. *Let $M = \bigoplus_{i \in I} M_i$ be a decomposition of a module M into a direct sum of indecomposable modules M_i . Suppose M is completely ADS. Then*

- (i) *For every $(i, j) \in I^2, i \neq j, M_i$ is M_j -injective.*
- (ii) *If $(i, j) \in I^2, i \neq j$ are such that $\text{Hom}_R(M_i, M_j) \neq 0$, then M_j is simple.*
- (iii) *$M = S \oplus T$ where S is semisimple and $T = \bigoplus_{j \in J \subset I} M_j$ is a direct sum of indecomposable modules. Moreover, for any $\theta \in \text{End}(M)$ we have $\theta(S) \subset S$ and for $j \in J, \theta(M_j) \subseteq M_j \oplus S$.*

Proof. Since the ADS property is inherited by direct summands, statement (i) is an obvious consequence of Lemma 3.1.

(ii) For convenience, let us write $i = 1, j = 2$ and suppose that $0 \neq \sigma \in \text{Hom}_R(M_1, M_2)$. We have $\sigma(M_1) \oplus M_2 \oplus \dots \cong M_1 / \ker(\sigma) \oplus M_2 \oplus \dots = M / \ker(\sigma)$ is ADS, by assumption. Hence $\sigma(M_1)$ is M_2 -injective and, since $\sigma(M_1) \subseteq M_2$, we get that $\sigma(M_1)$ is a direct summand of M_2 . But M_2 is indecomposable, hence $\sigma(M_1) = M_2$. We conclude that $M_2 \oplus M_2 = \sigma(M_1) \oplus M_2$ is ADS. This means that M_2 is M_2 -injective i.e. M_2 is quasi-injective.

Let us now show that for any $0 \neq m_2 \in M_2, m_2R = M_2$. Since $\sigma(M_1) = M_2$, there exists $m_1 \in M_1$ such that $\sigma(m_1) = m_2$. We remark that $\sigma(m_1R) \oplus M_2 = \frac{m_1R}{\ker \sigma \cap m_1R} \oplus M_2 = \frac{m_1R \oplus M_2}{\ker \sigma \cap m_1R}$ is a submodule of $\frac{M}{\ker \sigma \cap m_1R}$. Since M is completely ADS, we conclude that $\sigma(m_1R) \oplus M_2$ is ADS. As earlier in this proof, relative injectivity and indecomposability lead to $\sigma(m_1R) = M_2$. Hence $m_2R = M_2$, as desired.

(iii) Let I_1 consist of those $i \in I$ such that there exists $j \in I, j \neq i$ with $\text{Hom}_R(M_j, M_i) \neq 0$. We define $S := \bigoplus_{i \in I_1} M_i$ and $T := \bigoplus_{j \in J} M_j$ where $J := I \setminus I_1$. Statement (ii) above implies that $M = S \oplus T$ where S is semisimple and T is a sum of indecomposable modules. Moreover if $j \in J$, then for any $i \in I, i \neq j$, we have $\text{Hom}_R(M_i, M_j) = 0$. It is clear that, for any $\theta \in \text{End}(M)$ we must have $\theta(S) \subset S$. For $j \in J$ and $x \in M_j$ let us write $\theta(x) = y + z$, where $z \in S$ and $y \in T$. Since, for $l \in J, l \neq j, \text{Hom}_R(M_j, M_l) = 0$, we have $\pi_l \theta(x) = 0$, where $\pi_l : M \rightarrow M_l$ is the natural epimorphism. Thus $\pi_l(y) = 0$. This shows that $y \in M_j$, as required. \square

Oshiro’s theorem states that any quasi-discrete module is a direct sum of indecomposable modules (cf. [MM, Theorem 4.15]). Hence the above Theorem 5.1 applies to completely ADS quasi-discrete modules. In general for a quasi-discrete module we have the following theorem:

Theorem 5.2. *Let M be a completely ADS quasi-discrete module. Then M can be written as $M = S \oplus M_1 \oplus M_2$, where S is semisimple, M_1 is a direct sum of local modules and M_2 is equal to its own radical.*

Proof. Corollary 4.18 and Proposition 4.17 in [MM] imply that $M = N \oplus M_2$ where N has a small radical and M_2 is equal to its own radical. Theorem 5.1 applied to N yields the conclusion. \square

We now apply the previous theorem to the case of semiperfect modules.

Theorem 5.3. *Let M be a semiperfect module with a completely ADS projective cover P . Then M can be presented as $M = S \oplus T$ where S is semisimple and T is a sum of local modules. Moreover any partial sum in this decomposition contains a supplement of the remaining terms.*

Proof. Clearly P is semiperfect and projective (cf. Theorem 11.1.5 in [K]). Combining the statements in 42.5 in [W] and Corollary 4.54 in [MM], we get that P is discrete and is a direct sum of local modules. The remark preceding the present theorem then implies that we can write $P = S' \oplus T'$ where S' is semisimple and T' is a direct sum of indecomposable local modules. Let σ be an onto homomorphism from P to M with small kernel K . We thus have $M = \sigma(S') + \sigma(T')$. Since homomorphic images of M have projective covers, Lemma 4.40 [MM] shows that $\sigma(T')$ contains a supplement X of $\sigma(S')$. In particular, we have $\sigma(S') \cap X \ll X$. Since $\sigma(S')$ is semisimple we conclude that $\sigma(S') \cap X = 0$ and hence $M = \sigma(S') \oplus \sigma(T')$. Since homomorphic images of a local module are still local, we conclude that the terms appearing in $\sigma(T')$ are local modules. The last statement is a direct consequence of Lemma 4.40 [MM]. \square

Let us mention that local rings which are not uniform give examples of semiperfect completely ADS modules which are not CS and hence not quasi-continuous.

The following corollary characterizes semiperfect πc -rings providing a new proof of Theorem 2.4 in [GJ].

Theorem 5.4. *Let R be a semiperfect ring such that every cyclic module is quasi-continuous. Then $R = \bigoplus_{i \in I} A_i$ where each $A_i, i \in I$ is simple artinian or a valuation ring.*

Proof. Since R is semiperfect $R = B_1 \oplus B_2 \oplus \dots \oplus B_n$ a direct sum of indecomposable right ideals. In view of the fact that quasi-continuous modules are ADS, Theorem 5.1 gives a decomposition $R = e_1R \oplus e_2R \oplus \dots \oplus e_kR \oplus \dots \oplus e_nR$ where e_iR are simple right ideals for $1 \leq i \leq k$ and e_jR are local right ideals for $k < j \leq n$. Let σ be a homomorphism from e_sR to e_tR for some $1 \leq s, t \leq n$. Then $e_sR/\ker\sigma$ embeds in e_tR . Since $R/\ker(\sigma)$ is quasi-continuous, $e_sR/\ker\sigma$ is e_tR -injective and hence $e_sR/\ker(\sigma)$ splits in e_tR . This shows that either $e_sR/\ker(\sigma) \cong e_tR$ or $\ker(\sigma) = e_sR$, that is $\sigma = 0$. Since e_tR is projective, if $e_sR/\ker(\sigma) \cong e_tR$, then $\ker(\sigma)$ splits in e_sR , thus $\ker(\sigma) = 0$. In short we get that if $\sigma \neq 0$ then $e_sR \cong e_tR$, the latter isomorphism implies e_sR and e_tR are minimal right ideals (cf. Lemma 2.3 in [GJ]). By grouping the right ideals e_iR according to their isomorphism classes, we get $R = A_1 \oplus A_2 \oplus \dots \oplus A_l, l \leq n$, where each A_i is either a simple artinian ring or a local ring. We claim that if A_i is a local ring then it is a valuation ring. We thus have to show that any pair of two nonzero submodules C, D of the ring A_i are comparable. Let us consider the right submodules $\frac{C}{C \cap D}$ and $\frac{D}{C \cap D}$ of $\frac{R}{C \cap D}$. Since $A_i/(C \cap D)$ is a local quasi-continuous it is uniform, but $C/(C \cap D) \cap D/(C \cap D) = 0$. Therefore $C/(C \cap D)$ or $D/(C \cap D) = 0$ hence C and D are indeed comparable. \square

Let us conclude this paper with some questions:

1. It is known that if R_R and ${}_R R$ are both CS then R is Dedekind finite. What could be the analogue of this for ADS modules?

2. Does a directly finite ADS module have the internal cancellation property? (Cf. Theorem 2.33 in [MM], for the quasi-continuous case.)
3. What can be said of a module which is ADS and has the C_2 property?

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