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# Instability of Homogeneous Periodic Solutions of Parabolic-Delay Equations

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## 1. INTRODUCTION

Let $r \geqslant 0$ be a given real number, $f: C([-r, 0], \mathbf{R}^N) \to \mathbf{R}^N$ be a $\mathbf{C}^2$ map, $D$ be an $N \times N$ real matrix with eigenvalues in the right half plane $\lambda > 0$, and consider the system

$$\frac{\partial u}{\partial t}(x, t) = D\,\Delta u(x, t) + f(u_t(x, \cdot)), \qquad x \in \mathbf{R}^n, \quad t > 0, \tag{1.1}$$

where $u_t(x, \theta) = u(x, t + \theta)$, $x \in \mathbf{R}^n$, $-r \leqslant \theta \leqslant 0$ and $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the Laplacian operator.

Assume that the functional equation

$$\dot{v}(t) = f(v_t) \tag{1.2}$$

has a non-constant $\omega$-periodic solution $v = p(t)$. Then $u_0(x, t) = p(t)$, $x \in \mathbf{R}^n$, $t \in \mathbf{R}$, is a solution of (1.1) and we examine its stability or instability in the space $\chi$ of bounded uniformly continuous functions $u: \mathbf{R}^n \times [-r, 0] \to \mathbf{R}^N$ with the sup–norm.

If $p$ is unstable for (1.2), it is also unstable for (1.1) since the subspace of the constant functions in the $x$-variable in $\chi$ may be identified with $\mathbf{C}([-r, 0], \mathbf{R}^N)$ and (1.1) becomes (1.2) on this subspace. Suppose $p$ is orbitally asymptotically stable for (1.2) and, in fact, that the characteristic multipliers of the linearization of (1.2) around $p$—aside from the obvious multiplier 1, assumed to be simple—are strictly inside the unit circle. The solution $u_0(x, t) = p(t)$ of (1.1) may still not be stable for certain

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matrices $D$. In fact, as we show (cf. Section 3), (1.1) has a family of periodic travelling waves near $u_0$, so we cannot expect to prove orbital stability (in the usual sense) of $u_0$. However, this result suggests a weaker definition of orbital stability (cf. Remark 2 in Section 3), but Theorem 4.4 shows that, for certain matrices $D$, the solution $u_0$ is unstable even in this context.

The method we use to study Eq. (1.1) is semigroup theory, which consists of treating (1.1) as an evolution equation in a Banach space as delineated in [5]. Specifically, let $X = \mathbf{C}_{\text{unif}}(\mathbf{R}^n, \mathbf{R}^N)$ be the space of bounded uniformly continuous functions on $\mathbf{R}^n$. Then $A = D\Delta$ is the generator of an analytic semigroup $\{e^{At}: t \geq 0\}$ on $X$ and, if $F: \mathbf{R} \times \mathbf{C}([-r, 0], X) \to X$ is defined by $F(t, \varphi)(x) = f(\varphi(\cdot)(x))$, $x \in \mathbf{R}^n$, then Eq. (1.1) can be, at least formally, written in the form

$$\dot{u}(t) = Au(t) + F(t, u_t), \qquad t > 0, \tag{1.3}$$

where, of course, $u_t: [-r, 0] \to X$ denotes the function $u_t(\theta) = u(t+\theta)$, $-r \leq \theta \leq 0$. Equations like (1.3) are studied in Section 2 from an abstract point of view, at first looking for mild solutions and then for $\mathbf{C}^1$ solutions.

## 2. Existence and Uniqueness of Solutions

In this paragraph, $X$ is a (real or complex) Banach space, $r \geq 0$ a real number, and $\mathbf{C} = \mathbf{C}([-r, 0], X)$ the Banach space of the continuous functions $\varphi: [-r, 0] \to X$ with the sup-norm. $\{e^{At}: t \geq 0\}$ always means the analytic semigroup generated by the closed linear operator $A: D(A) \to X$ satisfying $\|e^{At}\|_{L(X)} \leq M$, for all $t \geq 0$ and some constant $M \geq 1$.

Let $F: \mathbf{R} \times \mathbf{C} \to X$ be continuous and $\varphi \in \mathbf{C}$. By a *solution* of (1.3) with initial condition $u_0 = \varphi$ we mean a continuous function $u: [-r, T) \to X$, with $T > r$, such that

(i) $u(t) = \varphi(t)$, for $-r \leq t \leq 0$;

(ii) for $0 < t \leq r$, $u$ is a solution of the integral equation

$$u(t) = e^{At}\varphi(0) + \int_0^t e^{A(t-s)} F(s, u_s)\, ds;$$

(iii) for $r < t < T$, the function $u$ is $\mathbf{C}^1$, has $u(t) \in D(A)$ and $\dot{u}(t) = Au(t) + F(t, u_t)$, for all $t \in (r, T)$.

*Remarks.* 1. As we show below (cf. Theorem 2.7) if, besides continuity in $[-r, 0]$, we suppose $\varphi$ is locally Hölder continuous on $(-r, 0]$ and $\varphi(0) \in D((-A)^\alpha)$ for some $\alpha > 0$, then a continuous function $u: [-r, T) \to X$, with $T > 0$, satisfying (i) on $[-r, 0]$ and (ii) on $[0, T)$ is a $\mathbf{C}^1$ function on

$(0, T)$. In this case, our definition of solution coincides with the usual one in the evolution equations theory.

2. The assumption $T > r$ is not too restrictive for the problem we study because, as we see later, solutions of (1.3) sufficiently close to a periodic solution are defined on arbitrarily large interval of times.

THEOREM 2.1. *Suppose $F\colon \mathbf{R} \times \mathbf{C} \to X$ is continuous and locally Lipschitzian in the second argument. Given $(s, \varphi) \in \mathbf{R} \times \mathbf{C}$, there exists a real number $\alpha = \alpha(s, \varphi) > 0$ and a unique continuous function $u\colon [s-r, s+\alpha] \to X$ such that $u_s = \varphi$ and*

$$u(t) = e^{A(t-s)}\varphi(0) + \int_s^t e^{A(t-\sigma)} F(\sigma, u_\sigma)\, d\sigma \tag{2.1}$$

*for all $s \leqslant t \leqslant s + \alpha$.*

The proof is a rather simple application of the Contraction Mapping Theorem, which the reader can supply. It is easy to see that, if $u, v; [s-r, s+\beta] \to X$ (any $\beta > 0$) are continuous solutions of (2.1) such that $u_s = v_s = \varphi$, then $u = v$ on $[s-r, s+\beta]$. This result allows us to consider the *maximal* solution $u(s, \varphi)$ of (2.1) through $(s, \varphi)$: for each $(s, \varphi)$, we define $\alpha^*(s, \varphi) = \sup\{\alpha > s : (2.1)$ has a continuous solution on $[s-r, \alpha]\}$ and $u(s, \varphi) : [s-r, \alpha^*(s, \varphi)) \to X$ by $u(s, \varphi)(t) = \varphi(t-s)$, if $s - r \leqslant t \leqslant s$ and, if $s < t < \alpha^*(s, \varphi)$, then $u(s, \varphi)(t) =$ the value at $t$ of a solution of (2.1) satisfying $u_s = \varphi$, defined on $[s-r, \alpha]$, with $t < \alpha$. By the previous result, $u$ is a well-defined continuous function on $[s-r, \alpha^*(s, \varphi))$ and is a solution of (2.1) satisfying $u_s = \varphi$. Any other solution $v$ of (2.1) satisfying the same initial condition is a restriction of $u(s, \varphi)$. Of course, the interval of existence of a maximal solution of (2.1) must be open to the right and the case $\alpha^*(s, \varphi) = \infty$ is not excluded.

LEMMA 2.2. *Suppose the solution $u = u(s, \varphi)$ of* (2.1) *with $u_s = \varphi$ is defined on $[s-r, \alpha)$, for some $\alpha > s$, and let $T$ be a real number such that $s < T < \alpha$. Then, there is a number $\delta > 0$ such that any solution $v = v(s, \psi)$ of* (2.1), *with $v_s = \psi$ and $\|\varphi - \psi\| < \delta$, is defined at least on $[s-r, T]$. Moreover, for a fixed $t$, $s \leqslant t \leqslant T$, the map $\varphi \mapsto u_t(s, \varphi)$ is continuous.*

*Proof.* There exists a neighborhood $V$ of $G := \{(t, u_t) : s \leqslant t \leqslant T\}$ and a positive real number $L$ such that $|F(t, \varphi_1) - F(t, \varphi_2)| \leqslant L\, \|\varphi_1 - \varphi_2\|$, for all $(t, \varphi_1)$, $(t, \varphi_2)$ in $V$. The neighborhood $V$ can be chosen in such a way that it satisfies the following condition: there is a $\rho > 0$ such that, if $\psi \in \mathbf{C}$ and $\|\psi - u_t\| \leqslant \rho$ for some $t \in [s, T]$, then $(t, \psi) \in V$.

Let $k = Me^{L(T-s)} \geqslant 1$ and $\delta = \rho/k$. If $\psi \in \mathbf{C}$ satisfies $\|\varphi - \psi\| < \delta$ and $v = v(s, \psi)$ is the solution of (2.1) with $v_s = \psi$, then, for all values of $t \in [s-r, T]$ for which the solution exists, we have

$$u(t) - v(t) = \begin{cases} e^{A(t-s)}[\varphi(0) - \psi(0)] + \int_s^t e^{A(t-s)}[F(\sigma, u_\sigma) - F(\sigma, v_\sigma)]\, d\sigma, \\ \qquad \text{if} \quad t - s > 0, \\ \varphi(t-s) - \psi(t-s), \qquad \text{if} \quad t - s \leqslant 0. \end{cases}$$

Hence, for $-r \leqslant \theta \leqslant 0$ and $t \in [s, T]$ for which $v$ exists, we have

$$|u(t+\theta) - v(t+\theta)| \leqslant M\, \|\varphi - \psi\| + \int_s^{t+\theta} M\, |F(\sigma, u_\sigma) - F(\sigma, v_\sigma)|\, d\sigma$$

and so

$$\|u_t - v_t\| \leqslant M\, \|\varphi - \psi\| + \int_s^t ML\, \|u_\sigma - v_\sigma\|\, d\sigma,$$

for all values of $t$ for which $(\sigma, v_\sigma) \in V$ for $s \leqslant \sigma \leqslant t$. By Gronwall's inequality, we have

$$\|u_t - v_t\| \leqslant M\, \|\varphi - \psi\|\, e^{ML(t-s)} \leqslant k\, \|\varphi - \psi\|,$$

for all $t \in [s, T]$ for which $(t, u_t) \in V$. Since $\|\varphi - \psi\| < \rho/k$, we have $\|u_t - v_t\| \leqslant \rho$ for $s \leqslant t \leqslant T$ and, hence, $(t, v_t) \in V$ and $v$ is defined on $[s, T]$. The continuity of the map $t \mapsto v_t(s, \varphi)$ follows from the above estimate.

LEMMA 2.3. *Let* $(s, \varphi) \in \mathbf{R} \times \mathbf{C}$ *and* $u\colon [s-r, \alpha^*) \to X$ *be the maximal solution of* (2.1) $u_s = \varphi$. *If* $\alpha^* < \infty$, *then* $\limsup_{t \to \alpha^{*-}} |F(t, u_t)|/(1 + \|u_t\|) = \infty$.

*Proof.* Suppose $\alpha^* < \infty$ and $\limsup_{t \to \alpha^{*-}} |F(t, u_t)|/(1 + \|u_t\|) < \infty$. Then there exists $B > 0$ such that $|F(t, u_t)| \leqslant B(1 + \|u_t\|)$, for all $s \leqslant t < \alpha^*$. If $t \geqslant s$ and $-r \leqslant \theta \leqslant 0$, we have

$$u(t+\theta) = \begin{cases} e^{A(t+\theta-s)}\varphi(0) + \int_s^{t+\theta} e^{A(t+\theta-\sigma)} F(\sigma, u_\sigma)\, d\sigma, \\ \qquad \text{if} \quad s < t + \theta < \alpha^*, \\ \varphi(t+\theta-s), \qquad \text{if} \quad s - r \leqslant t + \theta \leqslant s, \end{cases}$$

so

$$|u(t+\theta)| \leqslant M\, \|\varphi\| + \int_s^{t+\theta} M\, |F(\sigma, u_\sigma)|\, d\sigma \leqslant M\, \|\varphi\| + \int_s^t MB(1 + \|u_\sigma\|)\, d\sigma,$$

for all $t \in [s, \alpha^*)$ and $\theta \in [-r, 0]$. Therefore,

$$\|u_t\| \leqslant M \|\varphi\| + MB(\alpha^* - s) + \int_s^t MB \|u_\sigma\| \, d\sigma,$$

for all $t \geqslant s$. This implies $\|u_t\|$ is bounded for $t \in [s, \alpha^*)$, and so $|F(t, u_t)|$ is bounded on $[s, \alpha^*)$ and, therefore, there exists $B_1 > 0$ such that $|F(t, u_t)| \leqslant B_1$, for $s \leqslant t < \alpha^*$.

Let us prove that $|u(t) - u(\tau)| \to 0$ as $t, \tau \to \alpha^*-$. Indeed, given $\varepsilon > 0$, let $0 < \varepsilon_1 < \alpha^* - s$ be such that $\varepsilon_1 < \varepsilon/4B_1 M$ and let $\tau^* = \alpha^* - \varepsilon_1$. Choose $0 < \delta \leqslant \varepsilon_1$ such that $|(e^{A\tau} - e^{A\sigma})\, u(t^*)| \leqslant \varepsilon/2$ whenever $|\tau - \sigma| \leqslant \delta$ and $0 \leqslant \sigma, \tau \leqslant \varepsilon_1$. Looking at $u$ as the solution of the integral equation (2.1) with initial condition $(t^*, u_{t^*})$, we can write, for $t, \tau \in (t^*, \alpha^*)$,

$$u(t) = e^{A(t - t^*)} u(t^*) + \int_{t^*}^{t} e^{A(t - \sigma)} F(\sigma, u_\sigma) \, d\sigma,$$

$$u(\tau) = e^{A(\tau - t^*)} + \int_{t^*}^{\tau} e^{A(\tau - \sigma)} F(\sigma, u_\sigma) \, d\sigma,$$

from which we conclude

$$|u(t) - u(\tau)| \leqslant \varepsilon/2 + 2 \int_{t^*}^{\alpha^*} MB_1 \, d\sigma \leqslant \varepsilon/2 + 2MB_1 \varepsilon_1 \leqslant \varepsilon$$

whenever $t, \tau \in (t^*, \alpha^*)$ with $\alpha^* - \delta < t, \tau < \alpha^*$.

By Cauchy's criterion, there exists $\lim_{t \to \alpha^*-} u(t) := u_1 \in X$ and the function $\tilde{u}(t)$: $[\alpha^* - r, \alpha^*] \to X$ defined by $\tilde{u}(t) = u(t)$, if $\alpha^* \leqslant t < \alpha^*$ and $\tilde{u}(\alpha^*) = u_1$ is continuous.

Taking $(\alpha^*, u_{\alpha^*})$ as initial condition, we obtain a continuous solution of (2.1) $v$: $[\alpha^* - r, \alpha^* + \alpha] \to X$, for some $\alpha > 0$, such that $v_{\alpha^*} = \tilde{u}_{\alpha^*}$. Hence the function

$$u^*(t) = \begin{cases} u(t), & \text{if } s - r \leqslant t < \alpha^*, \\ v(t), & \text{if } \alpha^* \leqslant t \leqslant \alpha^* + \alpha, \end{cases}$$

is a solution of (2.1) satisfying $u_s^* = \varphi$, which contradicts the definition of $u(s, \varphi)$. This contradiction shows $\limsup_{t \to \alpha^*} |F(t, u_t)|/(1 + \|u_t\|) = \infty$ and the proof is complete.

COROLLARY 2.4. *In addition to the assumptions of Theorem* 2.1, *suppose $F$ satisfies the following hypothesis: $F(B)$ is a bounded set in $X$, for all bounded set $B$ contained in $\mathbf{R} \times \mathbf{C}$. Let $u(s, \varphi)$ and $\alpha^*(s, \varphi)$ be as above and assume $\alpha^*(s, \varphi) < \infty$. Then* $\limsup_{t \to \alpha^*(s, \varphi)-} \|u_t(s, \varphi)\| = \infty$.

THEOREM 2.5. *Suppose* $F: \mathbf{R} \times \mathbf{C} \to X$ *is* $\mathbf{C}^1$. *Let* $(s, \varphi) \in \mathbf{R} \times \mathbf{C}$, $u(s, \varphi)$: $[s-r, \alpha^*(s, \varphi)) \to X$ *be the solution of* (2.1) *through* $(s, \varphi)$ *and* $s < T < \alpha^*(s, \varphi)$. *Then there exists a neighborhood* $U$ *of* $\varphi$ *such that, for all* $\psi \in U$, *the solution* $u(s, \psi)$ *of* (2.1) *with* $u_s(s, \psi) = \psi$ *is defined at least on* $[s-r, T]$ *and, for each* $s \leqslant t \leqslant T$, *the map* $\psi \in U \mapsto u(s, \psi)(t) \in X$ *is* $\mathbf{C}^1$ *and its derivative* $(\partial u/\partial\psi)(s, \varphi) \cdot \xi)(t) \equiv v(t)$ *at* $(s, \varphi)$ *applied to* $\xi$ *is the solution of*

$$v(t) = e^{A(t-s)}\xi(0) + \int_s^t e^{A(t-\sigma)} \frac{\partial F}{\partial\varphi}(\sigma, u_\sigma(s, \varphi)) \cdot v_\sigma \, d\sigma \tag{2.2}$$

*on* $(s, T]$ *and* $v(t) = \xi(t-s)$ *on* $[s-r, s]$.

*Proof.* Define $\Phi$: $\mathbf{C}([s-r, T], X) \times \mathbf{C}([-r, 0], X) \to \mathbf{C}([s-r, T], X)$ by

$$\Phi(u, \psi) = \begin{cases} u(t) - e^{A(t-s)}\psi(0) - \int_s^t e^{A(t-\sigma)}F(\sigma, u_\sigma)\, d\sigma, \\ \qquad \text{if } \; s < t \leqslant T, \\ u(t) - \psi(t-s), \qquad \text{if } \; s-r \leqslant t \leqslant s. \end{cases}$$

We have $\Phi(u(s, \varphi), \varphi) = 0$ and a simple calculation shows that $\Phi$ is $\mathbf{C}^1$. In order to prove that $(\partial\Phi/\partial u)(u(s, \varphi), \varphi)$ is an isomorphism, we need to prove that for any $h \in \mathbf{C}([s-r, T], X)$ there exists a unique continuous solution $v$: $[s-r, T] \to X$ of the equation

$$v(t) - \int_s^t e^{A(t-s)} \frac{\partial F}{\partial\varphi}(\sigma, u_\sigma(s, \varphi)) \cdot v_\sigma \, d\sigma = h(t), s < t \leqslant T \tag{2.3}$$

with initial condition $v(t) = h(t)$, for $s-r \leqslant t \leqslant s$, and that $v$ depends continuously on $h$. Uniqueness follows immediately from Theorem 2.1. To prove existence, define $v(t) = h(t) + w(t)$, where $w$: $[s-r, T] \to X$ is the solution of

$$w(t) = \int_s^t e^{A(t-\sigma)} \frac{\partial F}{\partial\varphi}(\sigma, u_\sigma(s, \varphi)) \cdot w_\sigma \, d\sigma + \int_s^t e^{A(t-\sigma)} \frac{\partial F}{\partial\varphi}(\sigma, u_\sigma(s, \varphi)) \cdot h_\sigma \, d\sigma \tag{2.4}$$

on $(s, T]$ and $w(t) = 0$ for $s-r \leqslant t \leqslant s$ (the solution $w$ exists on $[s-r, T]$ because the right-hand side is globally Lipschitzian with respect to $w_t$ in $\mathbf{R} \times \mathbf{C}([-r, 0], X)$). Then we have

$$\begin{aligned} &v(t) - \int_s^t e^{A(t-\sigma)} \frac{\partial F}{\partial\varphi}(\sigma, u_\sigma(s, \varphi)) \cdot v_\sigma \, d\sigma \\ &\quad = w(t) + h(t) - \int_s^t e^{A(t-\sigma)} \frac{\partial F}{\partial\varphi}(\sigma, u_\sigma(s, \varphi)) \cdot (w_\sigma + h_\sigma)\, d\sigma = h(t), \end{aligned}$$

for $s < t \leqslant T$ and $v(t) = h(t)$, for $s - t \leqslant t \leqslant s$, so $v$ is the solution of (2.3) on $[s-r, T]$. From Gronwall's inequality, there exists $K > 0$ such that $\sup_{s-r \leqslant t \leqslant T} |v(t)| \leqslant K \sup_{s-r \leqslant t \leqslant T} |h(t)|$, showing the continuous dependence of $v$ on $h$.

By the Implicit Function Theorem, there exist neighborhoods $U$ of $\varphi$, $V$ of $u(s, \varphi)$ and a $\mathbf{C}^1$ map $\eta$: $U \to V$ such that $\eta(\varphi) = u(s, \varphi)$ and $\Phi(u, \psi) = 0$ on $U \times V$ if and only if $u = \eta(\psi)$. In particular, $\Phi(\eta(\psi), \psi) = 0$ for all $\psi \in U$, that is, $u = \eta(\psi)$ is the solution of

$$u(t) = \begin{cases} e^{A(t-s)}\psi(0) + \int_s^t e^{A(t-\sigma)} F(\sigma, u_\sigma)\, d\sigma, & \text{if } s < t \leqslant T, \\ \psi(t-s), & \text{if } s - r \leqslant t \leqslant s. \end{cases}$$

Furthermore, if $\xi \in \mathbf{C}([-r, 0], X)$ and $v(t) = (\eta'(\varphi) \cdot \xi)(t)$, then the identity

$$\frac{\partial \Phi}{\partial u}(u(s, \varphi), \varphi)\, \eta'(\varphi) + \frac{\partial \Phi}{\partial \varphi}(u(s, \varphi), \varphi) = 0$$

shows that $v$ satisfies $v_s = \xi$ and

$$v(t) = e^{A(t-s)}\xi(0) + \int_s^t e^{A(t-\sigma)} \frac{\partial F}{\partial \varphi}(\sigma, u_\sigma(s, \varphi)) \cdot v_\sigma\, d\sigma$$

on $(s, T]$, and the proof is complete.

*The Autonomous Case*

Suppose Eq. (2.1) is autonomous, that is, $F(t, \varphi) = g(\varphi)$ does not depend on t. If $u$: $[-r, \alpha^*(\varphi)) \to X$ is the maximal solution of (2.1) such that $u_0(\varphi) = \varphi$ and $s \in [0, \alpha^*(\varphi))$, then the function $v(t) = u(\varphi)(t+s)$, defined on $[-s-r, \alpha^*(\varphi) - s)$ is the solution of (2.1) satisfying $v_0 = u_s(\varphi)$, so $v(t) = u(u_s(\varphi))(t)$, for all $t \in [-r, \alpha^*(u_s(\varphi)))$. This implies $\alpha^*(\varphi) - s \leqslant \alpha^*(u_s(\varphi))$ and $u(\varphi)(t+s) = u(u_s(\varphi))(t)$ for all $t, s \geqslant 0$ such that $s - r \leqslant t + s < \alpha^*(\varphi)$. Therefore, if $-r \leqslant \theta \leqslant 0$, then $u_{t+s}(\varphi)(\theta) = u(\varphi)(t+s+\theta) = u(u_s(\varphi))(t+\theta) = u_t(u_s(\varphi))(\theta)$, and so, $u_{t+s}(\varphi) = u_t(u_s(\varphi))$ for all $t, s \geqslant 0$ such that $t + s < \alpha^*(\varphi)$. From these considerations and the previous results we conclude that, if (2.1) is autonomous and the solutions $u(\varphi)$ are defined on $[-r, \infty)$ for all $\varphi \in \mathbf{C}$, then the map $U(t)$: $\mathbf{C} \to \mathbf{C}$ given by $U(t)\varphi = u_t(\varphi)$ defines a (non-linear) strongly continuous semigroup $\{U(t): t \geqslant 0\}$ on $\mathbf{C}$.

Now, we will describe the relationship between $\{U(t): t \geqslant 0\}$ and $\{e^{At}: t \geqslant 0\}$. Let $\{T(t): t \geqslant 0\}$ be the strongly continuous semigroup defined on $\mathbf{C}$ by the operator $A$, that is,

$$(T(t)\varphi)(\theta) = \begin{cases} e^{A(t+\theta)}\varphi(0), & \text{if } t + \theta > 0, \\ \varphi(t+\theta), & \text{if } -r \leqslant t + \theta \leqslant 0. \end{cases}$$

If $u(\varphi)$ is the mild solution of $\dot{u}(t) = Au(t) + g(u_t)$ such that $u_0 = \varphi$, then

$$u(t) = \begin{cases} e^{At}\varphi(0) + \int_0^t e^{A(t-s)} g(u_s)\, ds, & \text{if } t > 0, \\ \varphi(t), & \text{if } -r \leqslant t \leqslant 0. \end{cases}$$

If $t \geqslant 0$ and $-r \leqslant \theta \leqslant 0$, we have

$$u_t(\varphi)(\theta) = \begin{cases} (T(t)\varphi)(\theta) + \int_0^{t+\theta} e^{A(t+\theta-s)} g(u_s(\varphi))\, ds, & \text{if } t+\theta > 0, \\ (T(t)\varphi)(\theta), & \text{if } -r \leqslant t+\theta \leqslant 0. \end{cases}$$

Letting $X_0 \colon [-r, 0] \to L(X)$ be defined by $X_0(\theta) = 0$, if $-r \leqslant \theta < 0$ and $X_0(0) = I$, the above integral can be written as

$$\int_0^{t+\theta} e^{A(t+\theta-s)} g(u_s(\varphi))\, ds = \int_0^t [T(t-s)X_0](\theta)\, g(u_s(\varphi))\, ds,$$

which justifies the equality

$$u_t(\varphi) = T(t)(\varphi) + \int_0^t [T(t-s)X_0]\, g(u_s(\varphi))\, ds$$

for $t \geqslant 0$. Here, we define

$$[T(t)X_0](\theta) = \begin{cases} e^{A(t+\theta)}, & \text{if } t+\theta > 0 \\ 0, & \text{if } t+\theta \leqslant 0' \end{cases}$$

which is (formally) the former definition.

*Differentiability with Respect to t.* In this section, we obtain sufficient conditions for a solution of (2.1) to be a solution of (1.3). We assume that $F\colon \mathbf{R} \times \mathbf{C}([-r, 0], X) \to X$ is locally Hölder continuous in $t$ and locally Lipschitzian in $\varphi$. The next result is basic and the reader can find the corresponding proof in [7].

LEMMA 2.6. *Suppose $\{e^{At} \colon t \geqslant 0\}$ is an analytic semigroup in a Banach space $X$ and let $f\colon (0, T) \to X$ be locally Hölder continuous with $\int_0^\rho \|f(s)\|\, ds < \infty$ for some $\rho > 0$. For $0 \leqslant t < T$, define $F(t) = \int_0^t e^{A(t-s)} f(s)\, ds$. Then, $F$ is continuous on $[0, T)$, continuously differentiable on $(0, T)$, with $F(t) \in D(A)$ for $0 < t < T$, $\dot{F}(t) = AF(t) + f(t)$ on $(0, T)$ and $F(t) \to 0$ in $X$ as $t \to 0+$.*

In the next results the function $u: [-r, T] \to X$ a solution of (2.1) on $[0, T]$ with initial condition $u_0 = \varphi$. As usual, $D((-A)^\delta)$ is the domain of the fractional power of operator $-A$(cf. [7]).

THEOREM 2.7. *Suppose $\varphi: [-r, 0] \to X$ is continuous and locally Hölder continuous on $(-r, 0]$ and $\varphi(0) \in D((-A)^\delta)$ for some $0 < \delta < 1$. Then $t \mapsto u(t) : (-r, T] \to X$ and $t \mapsto F(t, u_t) : (0, T] \to X$ are locally Hölder continuous and, therefore, $t \mapsto u(t)$ is $\mathbf{C}^1$ on $0 < t < T$.*

*Proof.* Let us show that $u$ is Hölder continuous on $[s, T]$, for any $-r < s \leqslant 0$. Let $0 < \alpha < 1$, $L, B_s > 0$ be such that $|\varphi(\theta_1) - \varphi(\theta_2)| \leqslant B_s |\theta_1 - \theta_2|^\alpha$, for $\theta_1, \theta_2 \in [s, 0]$ and $|F(t, \psi) - F(\tilde{t}, \tilde{\psi})| \leqslant L(|t - \tilde{t}|^\alpha + \|\psi - \tilde{\psi}\|)$, for $(t, \psi)$, $(\tilde{t}, \tilde{\psi})$ in a neighborhood of the curve $\{(t, u_t): 0 \leqslant t \leqslant T\} \subset \mathbf{R} \times \mathbf{C}$.

If $s \leqslant t < t + h \leqslant T$ and $0 < h \leqslant r$, then

$$u(t+h) - u(t) = \begin{cases} \varphi(t+h) - \varphi(t), & \text{if } s \leqslant t \leqslant -h, \\ e^{(t+h)}\varphi(0) - \varphi(t) + \int_0^{t+h} e^{A(t+h-s)} F(u_s, u_s)\, ds, & \\ \qquad \text{if } -h < t \leqslant 0, & \\ (e^{Ah} - I)\, e^{At}\varphi(0) + \int_0^t (e^{Ah} - I)\, F(s, u_s)\, ds & \\ \qquad + \int_t^{t+h} e^{A(t+h-s)} F(s, u_s)\, ds, & \text{if } t > 0. \end{cases}$$

Therefore, we have

(a) If $s \leqslant t \leqslant -h$, then $|u(t+h) - u(t)| \leqslant B_s h^\alpha$.

(b) If $-h < t \leqslant 0$, then $t \leqslant 0 < t + h$ and, therefore,

$$|u(t+h) - u(t)| \leqslant |\varphi(0) - \varphi(t)| + |(e^{A(t+h)} - I)\, \varphi(0)| + \left| \int_0^t e^{A(t+h-s)} F(s, u_s)\, ds \right|.$$

Since $\varphi(0) \in D((-A)^\delta)$, we have

$$|(e^{A(t+h)} - I)\, \varphi(0)| \leqslant C(t+h)^\delta\, |(-A)^\delta\, \varphi(0)| \leqslant K h^\delta,$$

for $t \leqslant 0 \leqslant t + h$, where $C$ and $K$ are constants. The integral can be estimated as

$$\left| \int_0^{t+h} e^{A(t+h-s)} F(s, u_s)\, ds \right| \leqslant (t+h) M \sup_{0 \leqslant s \leqslant t+h} |F(s, u_s)| \leqslant Ch,$$

for some constant $C$. Therefore, for $t \leqslant 0 \leqslant t+h$ we have

$$|u(t+h)-u(t)| \leqslant B_s h^\alpha + K h^\delta + Ch.$$

(c) If $t>0$, then

$$\begin{aligned}|u(t+h)-u(t)| \leqslant & |(e^{Ah}-I)\, e^{At}\varphi(0)| \\ & + \left|\int_0^t (e^{Ah}-I)\, e^{A(t-s)}F(s, u_s)\, ds\right| \\ & + \left|\int_t^{t+h} e^{A(t+h-s)}F(s, u_s)\, ds\right|.\end{aligned}$$

The first and the last terms above can be estimated as in (b). For the second, remember that

$$|(e^{Ah}-I)\, e^{A(t-s)}F(s, u_s)| \leqslant Ch^\alpha (t-s)^{-\alpha} \sup_{0 \leqslant s \leqslant t} |F(s, u_s)|,$$

for some constant $C$, and therefore, if $t>0$, we have

$$\begin{aligned}|u(t+h)-u(t)| &\leqslant MKh^\delta + C\int_0^t h^\alpha (t-s)^{-\alpha}\, ds + Ch \\ &\leqslant (\text{Const.})h^\delta + (\text{Const.})h^\alpha + (\text{Const.})h.\end{aligned}$$

In any case, we have

$$\sup_{s \leqslant t \leqslant T-h} |u(t+h)-u(t)| \leqslant \max\{B_s h^\alpha, C_1 h^\alpha + C_2 h^\delta + C_3 h\} \leqslant (\text{Const.})h^\beta,$$

where $\beta = \min\{\delta, \alpha\}$ and $C_j$ are constants. This shows that $u$ is locally Hölder continuous on $(-r, T]$ and $t \mapsto u_t$: $(0, T] \to \mathbf{C}$ is locally Hölder continuous. Since $F$ is locally Hölder continuous in $t$ and locally Lipschitzian in $\varphi$, it follows that $t \mapsto F(t, u_t)$ is locally Hölder continuous on $(0, T]$ and, by Lemma 2.6, $u$: $(0, T) \to X$ is $\mathbf{C}^1$.

THEOREM 2.8. *Suppose* $\varphi$: $[-r, 0] \to X$ *is continuous and* $u$ *is defined on* $[-r, T]$, *for some* $T>r$. *Then,* $u$ *is locally Hölder continuous on* $(0, r]$, $u(r) \in D((-A)^\alpha)$ *for every* $0<\alpha<1$ *and therefore,* $u$ *is* $\mathbf{C}^1$ *on* $r<t \leqslant T$.

*Proof.* Suppose $0<t<t+h \leqslant T$. Then

$$u(t+h)-u(t) = (e^{Ah}-I)\, e^{At}\varphi(0) + \int_0^t (e^{Ah}-I)\, e^{A(t-s)}F(s, u_s)\, ds + \int_t^{t+h} e^{A(t+h-s)}F(s, u_s)\, ds.$$

As in the previous result, we have

$$|u(t+h)-u(t)| \leqslant Ch^{\alpha}t^{-\alpha} + C\int_0^t h^{\alpha}(t-s)^{-\alpha}\,ds + Ch \leqslant C'h^{\alpha}t^{-\alpha},$$

where $C$ and $C'$ are constants. Therefore, $u$ is locally Hölder continuous on $(0, T]$.

On the other hand,

$$u(r) = e^{Ar}\varphi(0) + \int_0^r e^{A(r-s)}F(s, u_s)\,ds,$$

and therefore,

$$|u(r)|_{\alpha} := |(-A)^{\alpha}\,u(r)| \leqslant (\text{Const.})|\,\varphi(0)) + \int_0^r (\text{Const.})(r-s)^{-\alpha}\,ds < \infty,$$

for every $0<\alpha<1$.

Finally, if $r < s \leqslant t+h \leqslant T$, we have

$$|F(t+h, u_{t+h}) - F(t, u_t)| \leqslant L(h^{\alpha} + \|u_{t+h} - u_t\|) \leqslant C_s h^{\alpha}$$

for some constant $C_s$ (depending only on $s$), since

$$\|u_{t+h} - u_t\| = \sup_{t-r\leqslant\tau\leqslant t} \|u(\tau+h) - u(\tau)\| \leqslant C_s' h^{\alpha}(s-r)^{-\alpha}$$

for $s \leqslant t < T$. Therefore, $t \mapsto F(t, u_t)$ is locally Hölder continuous on $r < t \leqslant T$ and the proof follows from Lemma 2.6.

*The Equation* $\partial u/\partial t = D\,\Delta u + f(u_t)$

Let us now apply the previous results to the equation

$$\frac{\partial u}{\partial t} = D\,\Delta u + f(u_t), \qquad x \in \mathbf{R}^n, \quad t > 0. \tag{2.5}$$

As mentioned in the Introduction, $D$ is an $N \times N$ real matrix with eigenvalues in the half-plane $\operatorname{Re}\lambda > 0$, $f\colon \mathbf{C}([-r, 0], R^N) \to R^N$ is $\mathbf{C}^2$ and $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$.

Let $X = \mathbf{C}_{\text{unif}}(\mathbf{R}^n, \mathbf{R}^N)$ be the Banach space of bounded and uniformly continuous functions $u_0\colon \mathbf{R}^n \to \mathbf{R}^N$ endowed with the sup-norm. We are going to show that the equation

$$\frac{\partial u}{\partial t} = D\,\Delta u, \qquad x \in \mathbf{R}^n, \quad t > 0, \tag{2.6}$$

with initial condition $u(\cdot, 0) = u_0 \in X$ defines an analytic semigroup on $X$. In fact, taking formally the Fourier transform of (2.6) with respect to the $x$-variable, we obtain

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) = -|\xi|^2 D\hat{u}(\xi, t), \qquad \hat{u}(\xi, 0) = \hat{u}_0(\xi),$$

and so, $\hat{u}(\xi, t) = e^{-|\xi|^2 Dt}\hat{u}_0(\xi)$ for all $\xi \in \mathbf{R}^n$ and $t > 0$. Since $\operatorname{Re} \sigma(D) > 0$, for each $t > 0$, the map $\xi \in \mathbf{R}^n \mapsto e^{-|\xi|^2 Dt} \in L(\mathbf{R}^n)$ is in the Schwartz class $\mathscr{S} = \mathscr{S}(\mathbf{R}^n, \mathbf{R}^{N \times N})$ of the rapidly decreasing functions at infinity and, since the Fourier transform is a bijection of $\mathscr{S}$, there exists a unique $K: \mathbf{R}^n \times (0, \infty) \to L(C^n)$ such that $K(\cdot, t) \in \mathscr{S}$, for $t > 0$ and $\int_{\mathbf{R}^n} K(z, t)\, e^{-i\xi \cdot z}\, dz = e^{-|\xi|^2 Dt}$. We claim that

$$K(z, t) = (4\pi t)^{-n/2} D^{-n/2} e^{-|z|^2 D^{-1}/4t},$$

for all $z \in \mathbf{R}^n$ and $t > 0$. Indeed, by the Spectral Mapping Theorem, for $z \in \mathbf{R}^n$ and $t > 0$, we have

$$e^{-|z|^2 D^{-1}/4t} = \frac{1}{2\pi i} \int_\Gamma e^{-|z|^2 \lambda^{-1}/4t} (\lambda - D)^{-1}\, d\lambda,$$

where $\Gamma$ is a simple closed curve containing $\sigma(D)$ in its interior. By changing the order of integration, we have

$$\begin{aligned}\int_{\mathbf{R}^n} e^{-|z|^2 D^{-1}/4t} e^{-i\xi \cdot z}\, dz &= \frac{1}{2\pi i} \int_\Gamma \left( \int_{\mathbf{R}^n} e^{-|z|^2/4\lambda t} e^{-i\xi \cdot z}\, dz \right) (\lambda - D)^{-1}\, d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma (4\pi t)^{n/2} \lambda^{n/2} e^{-|\xi|^2 \lambda t} (\lambda - D)^{-1}\, d\lambda \\ &= (4\pi t)^{n/2} D^{n/2} e^{-|\xi|^2 Dt},\end{aligned}$$

for all $z \in \mathbf{R}^n$ and $t > 0$, and the result follows.

Still from the Spectral Mapping Theorem, we have

$$D^{-n/2} = \frac{1}{2\pi i} \int_\gamma \lambda^{-n/2} (\lambda - D)^{-1}\, d\lambda,$$

where $\gamma$ is a simple closed curve containing $\sigma(D)$ in its interior, symmetric with respect to the real axis, as shown in Fig. 1.

This implies that $D^{-n/2}$ is real and therefore, $K(z, t)$ is a real matrix, for all $z \in \mathbf{R}^n$ and $t > 0$.

Note that $K$ is $\mathbf{C}^\infty$ in $(z, t) \in \mathbf{R}^n \times (0, \infty)$, $\int_{\mathbf{R}^n} K(z, t)\, dz = I$, the identity matrix, and there exist positive constants $\delta$, $M$, $M'$ such that $|K(z, t)| \leqslant M' t^{-n/2} e^{-\delta |z|^2/t}$ and $\int_{\mathbf{R}^n} |K(z, t)|\, dz \leqslant M$, for all $z \in \mathbf{R}^n$ and $t > 0$.

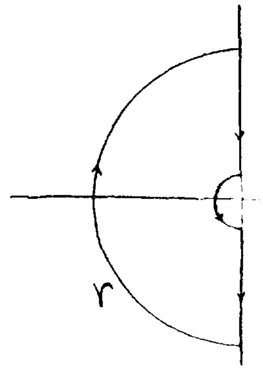


FIGURE 1

If $u_0 \in X$, it follows that

$$u(x, t) = \int_{\mathbf{R}^n} K(x-y, t)\, u_0(y)\, dy \tag{2.7}$$

is a solution of (2.6), has $u(\cdot, t) \in X$ for each $t > 0$ and $u(\cdot, t) \to u_0$ in $X$ as $t \to 0_+$. As in the scalar case, we can show that the equation (2.6) has a unique solution in the class of bounded functions on $\mathbf{R}^n \times [0 \times T)$; therefore, the solution is given by (2.7) and from this representation we can conclude that $(e^{tD\Delta}(u_0))(x) := u(x, t)$ is a strongly continuous semigroup on $X$ satisfying $\|e^{tD\Delta}\| \leqslant M$, for all $t \geqslant 0$.

Now, let us show that $\{e^{tD\Delta}: t \geqslant 0\}$ is actually an analytic semigroup. Since $\operatorname{Re} \sigma(D) > 0$, we have $\operatorname{Re} \sigma(D^{-1}) > 0$ and therefore, there exists $\varepsilon > 0$ such that $\arg \sigma(D) \subset (-\pi/2 + \varepsilon, \pi/2 - \varepsilon)$. Note that if $\theta \in C$ with $|\theta| = 1$, then the eigenvalues of $\theta D^{-1}$ have real part $= |\mu| \cos(\arg \mu + \arg \theta)$, where $\mu$ is an eigenvalue of $D^{-1}$. Therefore, if $|\theta| = 1$ and $|\arg \theta| < \varepsilon/2$, we have $\operatorname{Re} \sigma(\theta D^{-1}) > \min |\sigma(D^{-1})| \cos(\pi/2 - \varepsilon/2) = \delta > 0$. Therefore, there exists a constant $C \geqslant 1$ such that $|e^{-s\theta D^{-1}}| \leqslant Ce^{-\delta s}$, for all $s \geqslant 0$, uniformly in $\theta \in C$ with $|\theta| = 1$ and $|\arg \theta| < \varepsilon/2$.

Let $S = \{t \in C: \operatorname{Re} t > 0 \text{ and } |\arg t| < \varepsilon/2\}$. For each $t \in S$ and $z \in \mathbf{R}^n \backslash \{0\}$, let $y = |z|^2/4t$, $s = |y|$ and $\theta = y/|y|$; then we have $|\arg \theta| < \varepsilon/2$ and therefore

$$|e^{-|z|^2 D^{-1}/4t}| = |e^{-s\theta D^{-1}}| \leqslant Ce^{-\delta |z|^2/4|t|},$$

for all $z \in \mathbf{R}^n$ and $t \in S$ (this inequality is obviously true for $z = 0$). From

$$t \frac{\partial K}{\partial t}(z, t) = \left[\frac{-n}{2} I + \frac{|z|^2}{4t} D^{-1}\right] K(z, t),$$

for all $z \in \mathbf{R}^n$ and $t > 0$, it follows that there exist constants $C' > 0$, $\beta > 0$ such that

$$\left|\frac{\partial K}{\partial t}(z, t)\right| \leqslant C' \, |t|^{-n/2-1} \, e^{-\beta |z|^2/4 |t|},$$

for all $z \in \mathbf{R}^n$ and $t \in S$. These considerations shows that $t \in S \mapsto u(\cdot, t) \in X$ is differentiable and $|(\partial u/\partial t)(\cdot, t)| \leqslant (C'/|t|) \, |u_0|$, for $t \in S$ and this implies that $\{e^{tD\Delta}: t \geqslant 0\}$ is analytic.

*Remark.* In opposition to the scalar case, it is not always true that $K(z, t) \geqslant 0$ for all $z \in \mathbf{R}^n$ and $t > 0$. The following example is due to D. Henry and illustrates this fact: let $n = 1$ and

$$D = \frac{1}{(\alpha^2 + \beta^2)^2} \begin{pmatrix} \alpha^2 - \beta^2 & -2\alpha\beta \\ 2\alpha\beta & \alpha^2 - \beta^2 \end{pmatrix},$$

where $\alpha > |\beta| > 0$. Then

$$\sigma(D) = \left\{ \frac{\alpha^2 - \beta^2}{(\alpha^2 + \beta^2)^2} \pm i \frac{2\alpha\beta}{(\alpha^2 + \alpha^2)^2} \right\}, \qquad D^{-1/2} = B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

and therefore

$$D^{-1/2} e^{-sD^{-1}} = B e^{-sB^2} = e^{-s(\alpha^2 - \beta^2)} \begin{pmatrix} \alpha \cos\theta + \beta \sin\theta & \beta \cos\theta - \alpha \sin\theta \\ -\beta \cos\theta + \alpha \sin\theta & \alpha \cos\theta + \beta \sin\theta \end{pmatrix},$$

where $\theta = 2\alpha\beta s$, which is not a matrix with non-negative entries for all $s \geqslant 0$.

On the other hand, if $D$ is real, has all eigenvalues in the half plane $\operatorname{Re} \lambda > 0$ and all entries $D_{ij} \leqslant 0$ for $i \neq j$, then all entries of $D^{-n/2}$ are non-negative. Indeed, for any $\alpha > 0$ sufficiently small, we have

$$D^{-1/2} = \alpha^{1/2} (\alpha D)^{-1/2} = \alpha^{1/2} [I - (I - \alpha D)]^{-1/2} = \alpha^{1/2} \sum_{k=0}^{\infty} c_k (I - \alpha D)^k,$$

where $c_0 = 1$ and $c_k = 1 \cdot 2 \cdot 3 \cdot 5 \cdots (2k-1)/2 \cdot 4 \cdot 6 \cdots 2k > 0$, for $k \geqslant 1$ (the above series is convergent because $\sigma(I - \alpha D) = \{1 - \alpha d : d \in \sigma(D)\}$, so the spectral radius of $I - \alpha D$ is equal to $\max\{1 - \alpha \operatorname{Re} d + O(\alpha^2) : d \in \sigma(D)\}$, which can be made strictly less than 1 by choosing $\alpha > 0$ sufficiently small). Since $D_{ij} \leqslant 0$ for $i \neq j$ implies $I - \alpha D \geqslant 0$, it follows that $D^{-1/2}$ (and hence, $D^{-n/2}$) has all entries $\geqslant 0$.

Another result which we quote from [1] is the following: for $s \geqslant 0$, all entries of $e^{-sD^{-1}}$ are non-negative if and only if $D_{ij}^{-1} \leqslant 0$, for $i \neq j$ (i.e., the outside of diagonal entries of $D^{-1}$ are non-negative). Hence, it follows that $D_{ij} \leqslant 0$ and $D_{ij}^{-1} \leqslant 0$ for $i \neq j$ together the hypothesis $\operatorname{Re} \sigma(D) > 0$ is a sufficient condition for $K(z, t) \geqslant 0$ for all $z \in \mathbf{R}^n$ and $t > 0$. In the case $N = 2$ these conditions imply that $D$ is a diagonal matrix.

Now, let $F: \mathbf{C}([-r, 0], X) \to X$ be defined by $F(\varphi)(x) = f(\varphi(\cdot)(x))$, $x \in \mathbf{R}^n$. It is easy to prove the following result:

LEMMA 2.9. *If $f: \mathbf{C}([-r, 0], \mathbf{R}^n) \to \mathbf{R}^n$ is a $\mathbf{C}^1$ map and $f'$ is uniformly continuous on bounded sets, then $F$ is a $C^1$ map.*

## 3. EXISTENCE OF TRAVELING WAVES

In this paragraph we consider the existence of travelling waves for (1.1), that is, solutions of (1.1) of the special form $u(x, t) = v(\xi \cdot x + t)$, where $v: \mathbf{R} \to \mathbf{R}^N$ is a $\mathbf{C}^2$ bounded function and $\xi \in \mathbf{R}^n$ is a fixed vector. Of course, $v$ must be a bounded solution of the equation

$$v'(s) = \varepsilon D v''(s) + f(v_s), \tag{3.1}$$

where "$'$" denotes the derivative with respect to the argument $s = \xi \cdot x + t$ of $v$ and $\varepsilon = |\xi|^2$.

The main result here is the extension to Eq. (1.1), the one previously obtained by [10], which asserts that, if the reaction equation

$$\dot{u}(t) = f(u_t) \tag{3.2}$$

has a *simple* non-constant periodic solution, then (1.1) has a family of travelling waves.

THEOREM 3.1. *Assume* (3.2) *has a simple non-constant $\omega$-periodic solution $u = p(t)$ and $D$ is a real matrix with eigenvalues in the halfplane $\operatorname{Re} \lambda > 0$. Then,* (1.1) *has a family of travelling waves $u(x, t; \xi)$, for $|\xi|$ sufficiently small, periodic with respect to $t$ of period $\omega(\xi)$, such that $u(x, t; 0) = p(t)$, $u(x, t; \xi) \to p(t)$ uniformly on compact sets of $\mathbf{R}^n \times \mathbf{R}$ and $\omega(\xi) \to \omega$ as $\xi \to 0$.*

The method we use to prove Theorem 3.1 is very similar to the one in [6] on the persistence of a periodic orbit with respect to perturbation of the vector field. First, we state some results related to the linear variational equation of (3.2) around $p_t$, that is,

$$\dot{y}(t) = f'(p_t)\, y_t. \tag{3.3}$$

Recall that the *periodic map* of (3.3) is the linear operator $U$ on $\mathbf{C}([-r, 0], \mathbf{R}^N)$ defined by $U\varphi = y_\omega(\cdot, \varphi)$, where $y(t, \varphi)$ is the solution of (3.3) with initial condition $y_0(\cdot, \varphi) = \varphi$. Since $\dot{p}$ satisfies (3.3), we have $U\dot{p} = \dot{p}$, so $\mu = 1$ is an eigenvalue of $U$. Recall that $p$ is a *simple* periodic solution of (3.2) if $\mu = 1$ is a simple eigenvalue of $U$ and, in this case, the generalized eigenspace $\mathcal{M}_{\mu=1}$ corresponding to $\mu = 1$ is unidimensional and is generated by $\dot{p}$. Furthermore, the *adjoint equation* of (3.3) has also a non-constant $\omega$-periodic solution $q$ (row vector), which we can choose such that $\int_0^\omega |q(\tau)|^2 \, d\tau = 1$.

Let $\mathcal{P}_\omega$ be the Banach space of continuous and $\omega$-periodic functions $h: \mathbf{R} \to \mathbf{R}^N$ with the sup-norm. Let $\gamma: \mathcal{P}_\omega \to \mathbf{R}$ and $\pi: \mathcal{P}_\omega \to \mathcal{P}_\omega$ be the continuous maps defined by $\gamma(h) = \int_0^\omega q(\tau) \, h(\tau) \, d\tau$ and $\pi(h)(t) = \dot{p}(t) \, \gamma(h)/\gamma(\dot{p})$, respectively. For the proof of the following result, the reader if referred to [6, Thm. 9.1.2].

LEMMA 3.2. *Let $h \in \mathcal{P}_\omega$. Then, the equation*

$$\dot{y}(t) = f'(p_t) + h(t) \tag{3.4}$$

*has a solution in $\mathcal{P}_\omega$ if and only if $\gamma(h) = 0$. Moreover, there exists a bounded linear operator $\mathcal{K}: \mathcal{N}(\gamma) \to \mathcal{P}_\omega$ such that $y = \mathcal{K}h$ is the unique $\omega$-periodic solution of* (3.4) *satisfying $\pi(\mathcal{K}h) = 0$, for all $h \in N(\gamma)$.*

LEMMA 3.3. *Let $J(t) = \dot{p}(t) - f'(p_t)(\cdot)\,\dot{p}_t$, $t \in \mathbf{R}$. Then, $\int_0^\omega q(t) \, J(t) \, dt \neq 0$. (here, $f'(p_t)(\cdot)\,\dot{p}_t$ denotes $f'(p_t)$ applied to the function $\theta \mapsto \theta\dot{p}(t+\theta)$ in $\mathbf{C}([-r, 0], \mathbf{R}^N)$).*

*Proof.* It is easy to verify that $J(t + \omega) = J(t)$, for all $t$ and that the function $z(t) = tp(t)$ is a solution of the equation

$$\dot{z}(t) = f'(p_t) z_t + J(t). \tag{3.5}$$

If $\int_0^\omega q(t) \, J(t) \, dt = 0$ then, by Lemma 3.2, Eq. (3.5) has a $\omega$-periodic solution $z^*$ and therefore, $y(t) := z^*(t) - tp(t)$ is a solution of the homogeneous equation (3.3). Let $S$ be the subspace of $\mathbf{C}([-r, 0], \mathbf{R}^N)$ generated by the restrictions of $\dot{p}$ and $y$ to $[-r, 0]$ and consider the restriction of the period map $U$ of (3.3) to $S$. Since $U\dot{p}_0 = \dot{p}_0$ and $Uy_0 = y_0 - \omega\dot{p}_0$, the matrix of $U|_S$ with respect to the basis $\{\dot{p}_0, y_0\}$ is $\left(\begin{smallmatrix} 1 & -\omega \\ 0 & 1 \end{smallmatrix}\right)$. But this implies that $\mu = 1$ is an eigenvalue of $U$ with multiplicity $\geqslant 2$, which contradicts the hypothesis that $p$ is simple. Therefore, $\int_0^\omega q(t) \, J(t) \, dt \neq 0$ and the proof is complete.

*Proof of Theorem* 3.1. Let $u(x, t) = v(s)$, where $s = \xi \cdot x + t$ and $\xi \in \mathbf{R}^n$ is a fixed vector. Then, $u$ is a solution of (1.1) if and only if $v$ is a solution of

$$v'(s) = \varepsilon D v''(s) + f(v_s), \tag{3.1}$$

where $\varepsilon = |\xi|^2$ and "$'$" $= d/ds$.

On the other hand, Eq. (3.1), for $\varepsilon > 0$, has a periodic solution $v$ if and only if $v$ is a periodic solution of

$$v'(s) = \frac{D^{-1}}{\varepsilon} \int_s^{\infty} e^{(D^{-1}/\varepsilon)(s-\eta)} f(v_\eta)\, d\eta. \tag{3.6}$$

It is sufficient to show that, for $\varepsilon > 0$ sufficiently small, (3.6) has a periodic solution satisfying the conditions of the Theorem 3.1.

For each real number $\beta > -1$, let $s = (1+\beta)\tau$ and $v(s) = y(\tau)$. Then

$$v(\eta + \theta) = y\left(\frac{\eta}{1+\beta} + \frac{\theta}{1+\beta}\right), \quad -r \leqslant \theta \leqslant 0.$$

Let us define $y_{\eta,\beta}(\theta) = y(\eta + \theta/(1+\beta))$, $-r \leqslant \theta \leqslant 0$. Equation (3.6) becomes

$$\frac{dy}{d\tau} = (1+\beta) \frac{D^{-1}}{\varepsilon} \int_{(1+\beta)\tau}^{\infty} e^{(D^{-1}/\varepsilon)((1+\beta)\tau - \eta)} f(y_{\eta/(1+\beta),\beta})\, d\eta \tag{3.7}$$

or, making the change of variables $\eta = (1+\beta)\tau - \varepsilon v$,

$$\frac{dy}{d\tau} = (1+\beta) D^{-1} \int_{-\infty}^{0} e^{D^{-1}v} f(y_\tau - \varepsilon v/(1+\beta), \beta)\, dv. \tag{3.7$'$}$$

Note that if $y(t)$ is a $T$-periodic solution of (3.7) (or (3.7)$'$) then $v(s) = y(s/(1+\beta))$ is a $T(1+\beta)$-periodic solution of (3.6) and vice versa. We show that if $\varepsilon > 0$ is sufficiently small, then, there exists $\beta = \beta(\varepsilon) > -1/2$ such that (3.7)$'$ has a periodic solution of fixed period $\omega$. For this, take $|\beta| < 1/2$ and consider (3.7)$'$ in the space $\mathbf{C}(2r) = \mathbf{C}([-2r, 0], \mathbf{R}^N)$ through the natural extension $\tilde{f}$ of $f$ to $\mathbf{C}(2r)$: $\tilde{f}(\varphi) = f(\varphi_\beta)$—in the following we still indicate by $f$ that extension.

Writing $y(\tau) = p(\tau) + z(\tau)$, $z$ satisfies

$$\frac{dz}{d\tau} = f'(p_\tau)\, z_{\tau,0} + H(z, \beta, \varepsilon)(\tau), \tag{3.8}$$

where

$$H(z, \beta, \varepsilon)(\tau) = (1+\beta) D^{-1} \int_{-\infty}^{0} e^{vD^{-1}} f(z_{\tau - \varepsilon v/(1+\beta), \beta} + p_{\tau - \varepsilon v/(1+\beta), \beta})\, dv$$
$$- f(p_\tau) - f'(p_\tau)\, z_{\tau, 0}.$$

Let us consider the equation

$$\frac{dz}{d\tau} = f'(p_\tau)\, z_{\tau,0} + H(z, \beta, \varepsilon)(\tau) - \gamma(H(z, \beta, \varepsilon))\, q(\tau)^T \tag{3.9}$$

obtained from (3.8) by adding the term $-\gamma(H)\, q(\tau)^T$, where $\gamma$ and $q$ are as above mentioned. By Lemma 3.2., (3.9) has a $\omega$-periodic solution $z$ with $\pi z = 0$ if and only if

$$R(z, \beta, \varepsilon)(\tau) := z(\tau) - \mathscr{K}[H(z, \beta, \varepsilon) - \gamma(H(z, \beta, \varepsilon)) q^T](\tau) = 0.$$

We intend to apply the Implicit Function Theorem. In order to do that, let us define the spaces envolved. Let $\mathscr{P}^1_\omega$ be the Banach space of $\mathbf{C}^1$ $\omega$-periodic functions endowed with the $\mathbf{C}^1$-norm. A careful examination of the expressions of the maps $H: \mathscr{P}^1_\omega \times (-1/2, 1/2) \times (-\varepsilon_0, \varepsilon_0) \to \mathscr{P}_\omega$ and $R: \mathscr{P}^1_\omega \times (-1/2, 1/2) \times (-\varepsilon_0, \varepsilon_0) \to \mathscr{P}^1_\omega$ shows that they are $\mathbf{C}^1$ maps, and we have

$$\begin{aligned} &\frac{\partial H}{\partial z}(0,0,0) = 0 \qquad \frac{\partial H}{\partial \beta}(0,0,0)(t) = \dot{p}(t) - f'(p_t)(\cdot)\, \dot{p}_t \\ &R(0,0,0) = 0 \qquad\quad \frac{\partial R}{\partial z}(0,0,0) = I. \end{aligned} \tag{3.10}$$

Therefore, for $|\beta|$ and $|\varepsilon|$ sufficiently small, (3.9) has a unique $\omega$-periodic solution $z = z^*(\beta, \varepsilon)$ such that $z^*(0,0) = 0$ and is $\mathbf{C}^1$ in $\beta, \varepsilon$. In order for $z^*(\beta, \varepsilon)$ to be a solution of the original equation (3.8), it is necessary and sufficient that

$$B(\beta, \varepsilon) := \int_0^\omega q(\tau)\, H(z^*(\beta, \varepsilon), \beta, \varepsilon)(\tau)\, d\tau \equiv 0.$$

Now, we have a $\mathbf{C}^1$ map such that $B(0,0) = 0$. The proof would be complete if one could apply the Implicit Function Theorem once again to obtain $\beta = \beta^*(\varepsilon)$ ($\varepsilon > 0$ sufficiently small) as the roots of $B(\beta, \varepsilon) = 0$. To compute $(\partial B/\partial\beta)(0,0)$, observe that $\sigma=(\partial z^*/\partial\beta)(\beta,\varepsilon)$ is an $\omega$-periodic solution of the equation

$$\frac{d\sigma}{d\tau}=f'(p_\tau)\sigma_\tau+\frac{\partial H}{\partial z}(z^*(\beta,\varepsilon),\beta,\varepsilon)\sigma+\frac{\partial H}{\partial\beta}(z^*(\beta,\varepsilon),\beta,\varepsilon)-\frac{\partial B}{\partial\beta}(\beta,\varepsilon)\,q(\tau)^T.$$

Therefore, by Lemma 3.2., we have

$$\int_0^\omega q(\tau)\left[\frac{\partial H}{\partial z}(z^*(\beta,\varepsilon),\beta,\varepsilon)\frac{\partial z^*}{\partial\beta}(\beta,\varepsilon)\right.$$
$$\left.+\frac{\partial H}{\partial\beta}(z^*(\beta,\varepsilon),\beta,\varepsilon)-\frac{\partial B}{\partial\beta}(\beta,\varepsilon)\,q(\tau)^T\right]d\tau=0$$

and since $\int_0^\omega |q(\tau)|^2\,d\tau=1$, we get from (3.10)

$$\frac{\partial B}{\partial\beta}(0,0)=\int_0^\omega q(\tau)[\dot p(\tau)-f'(p_\tau)(\cdot)\dot p_\tau]\,d\tau,$$

and, by Lemma 3.3, we have $(\partial B/\partial\beta)(0,0)\neq 0$.

*Remark* 1. The conclusion of Theorem 3.1 remains true even when $D$ has no eigenvalue in the imaginary axis and the proof is similar. Indeed, we decompose $\mathbf{R}^N=E\oplus F$ and $D=\operatorname{diag}(D_1,D_2)$ where $\operatorname{Re}\sigma(D_1)>0$ and $\operatorname{Re}\sigma(D_2)<0$; if $f=\binom{f^1}{f^2}$ and $v=\binom{v^1}{v^2}$ with respect to this decomposition, then travelling waves of (1.1) are solutions of

$$\begin{cases}(v^1)'(s)=\varepsilon D_1(v^1)''(s)+f^1(v_s^1,v_s^2),\\ (v^2)'(s)=\varepsilon D_2(v^2)''(s)+f^2(v_s^1,v_s^2),\end{cases}$$

and bounded solutions of these equations are solutions of the system

$$\begin{cases}(v^1)'(s)=\dfrac{D_1^{-1}}{\varepsilon}\displaystyle\int_s^\infty e^{D_1^{-1}/\varepsilon(s-\eta)}f^1(v_\eta^1,v_\eta^2)\,d\eta,\\ (v^2)'(s)=\dfrac{D_2^{-1}}{\varepsilon}\displaystyle\int_{-\infty}^s e^{D_2^{-1}/\varepsilon(s-\eta)}f^2(v_\eta^1,v_\eta^2)\,d\eta,\end{cases}$$

which can be treated as before.

*Remark* 2. Existence of travelling waves for (1.1) suggests that, at least for unbounded regions, it could be useful to define a weaker condition of stability imposing only that $|u(x,t)-p(t+\psi(x,t))|$ is uniformly small on $\mathbf{R}^n\times\mathbf{R}^+$ for some slowly varying function $\psi\colon\mathbf{R}^n\times\mathbf{R}^+\to\mathbf{R}$.

## 4. An Instability Result

Since the existence of travelling waves near $u_0$ shows one cannot expect to prove orbital stability of $u_0$ and suggests a weaker definition of stability, it seems desirable to have a stronger result on instability. In order to study this problem, we consider first the variational equation of (1.1) around $p_t$, that is,

$$\frac{\partial w}{\partial t} = D\,\Delta w + f'(p_t) w_t. \tag{4.1}$$

From the results of Section 2, for each $s \in \mathbf{R}$ and $\varphi \in \chi$, there exists a unique solution $w(s, \varphi)$: $\mathbf{R}^n \times [s-r, \infty) \to \mathbf{R}^N$ of (4.1) such that $w_s(s, \varphi) = \varphi$ and, for each $t \geqslant s$, the map $L(t, s)\varphi = w_t(s, \varphi)$ defines a bounded linear operator on $\chi$. The family $\{T(t, s)\colon t \geqslant s\}$ is called *evolution operators* of (4.1) and the following properties are satisfied:

(a) $L(s, s) = I$ and $L(t, s)\, L(s, \tau) = L(t, \tau)$

(b) $L(t+\omega, s+\omega) = L(t, s)$

(c) $L(t+\omega, s) = L(t, s)\, L(s+\omega, s)$

for all $\tau \leqslant s \leqslant t$.

It follows from (b) that $L(s+n\omega, s) = L(s+\omega, s)^n$, for each positive integer $n$. Also, if $t \geqslant s$, then there exist unique integer $n$ and $0 \leqslant \tau < \omega$ such that $t = s + nw + \tau$ and therefore $L(t, s) = L(s+n\omega+\tau, s+n\omega)\, L(s+n\omega, s) = L(s+\tau, s)\, L(s+\omega, s)^n$. Hence, since $\|L(s+\tau, s)\|$ is uniformly bounded for $0 \leqslant \tau \leqslant \omega$ and $s$ fixed, the asymptotic behaviour of $L(t, s)$ depends only on $\|L(s+\omega, s)^n\|$ or, at ultimately of the spectral radius of $L(s+\omega, s)$.

The proof of the next result can be read in [7].

Lemma 4.1. *Let $U(s) := L(s+\omega, s)$ for $s \in \mathbf{R}$. Then, $U(s+\omega) = U(s)$, for all $s \in \mathbf{R}$ and $\sigma(U(s)) \setminus \{0\}$ is independent of $s$.*

The map $L(\omega) := L(\omega, 0) = U(0)$ is called *period map* of (4.1) and the non-zero eigenvalues of $L(\omega)$ are called *characteristic multipliers* of (4.1).

Consider now the $\omega$-periodic linear functional equations

$$\frac{d\hat{w}}{dt} = f'(p_t)\hat{w}_t - |\xi|^2 D\hat{w}(t) \tag{4.2}$$

depending on the parameter $\xi \in \mathbf{R}^n$. One defines the period map $T_\xi(\omega)$ and the characteristic multipliers of (4.2) in the same way as for (4.1). If $\mu(|\xi|^2) = e^{\lambda(|\xi|^2)\omega}$ is a non-zero characteristic multiplier of (4.2) corresponding to the value $\xi$, then there exists a $\omega$-periodic function $q: \mathbf{R} \to \mathbf{R}^N$ such that $\hat{w}(t) = e^{\lambda(|\xi|^2)t} q(t)$ is a solution of (4.2) and so $w(x, t) = e^{\lambda(|\xi|^2)t + i\xi \cdot x} q(t)$ is a solution of (4.1) with initial value $w_0(x, \theta) = e^{\lambda(|\xi|^2)\theta + i\xi \cdot x} p(\theta)$, $x \in \mathbf{R}^n$, $-r \leqslant \theta \leqslant 0$. Since $L(\omega) w_0 = e^{\lambda(|\xi|^2)\omega} w_0$, it follows that $e^{\lambda(|\xi|^2)\omega}$ is an eigenvalue of $L(\omega)$ and therefore

$$\sigma(L(\omega)) \setminus \{0\} \supseteq \bigcup_{\xi \in \mathbf{R}^n} \{\mu(|\xi|^2) : \mu(|\xi|^2) \in \sigma(T_\xi(\omega))\} \setminus \{0\}.$$

The next result is important in the calculus of the spectral radius of the period map of (4.2) and the reader is referred to [4] for the corresponding proof.

LEMMA 4.2. *Suppose* $x: [-r, \infty) \to \mathbf{R}$ *is a non-negative function satisfying*

$$\dot{x}(t) = -\alpha x(t) + \beta \sup_{-r \leqslant \theta \leqslant 0} x(t+\theta),$$

*for* $t > 0$ *and assume* $\alpha > \beta > 0$. *Then,* $0 \leqslant x(t) \leqslant e^{-kt} \|x_0\|$, *for all* $t \geqslant 0$, *where* $k > 0$ *is the unique real solution of the equation* $k - \alpha + \beta e^{kr} = 0$.

LEMMA 4.3. *Suppose* $u = p(t)$ *is an orbitally stable non-constant* $\omega$*-periodic solution of* (1.2) *and* $D$ *is an* $N \times N$ *real matrix with eigenvalues in the halfplane* $\operatorname{Re} \lambda > 0$. *Let* $r(|\xi|^2)$ *be the spectral radius of the period map of* (4.2). *Then,* $r(0) = 1$ *and* $r(|\xi|^2) \to 0$ *as* $|\xi| \to \infty$.

*Proof.* When $\xi = 0$, (4.2) is the linear variational equation of (1.1) around $p$ and the hypotheses imply $r(0) = 1$. Suppose $\xi \neq 0$ and let $\alpha$, $M$, $K$ be positive constants such that $\|f'(p_t)\| \leqslant M$ and $|e^{-|\xi|^2 Dt}| \leqslant K e^{-\alpha|\xi|^2 t}$, for $t \geqslant 0$. If $\hat{w}(\cdot, \xi)$ is the solution of (4.2) such that $\hat{w}_0(\cdot, \varphi) = \varphi$, then

$$\hat{w}(t, \xi) = \begin{cases} e^{-|\xi|^2 Dt} \varphi(0) + \int_0^t e^{-|\xi|^2 D(t-s)} f'(p_s)\, \hat{w}_s(\cdot, \xi)\, ds, & \text{if } t > 0, \\ \varphi(t), & \text{if } -r \leqslant t \leqslant 0. \end{cases}$$

Therefore, $|\hat{w}(t, \xi)| \leqslant z(t)$, for all $t \geqslant -r$, where $z$ is the non-negative function satisfying

$$z(t) = \begin{cases} K e^{-\alpha|\xi|^2 t} |\varphi(0)| + \int_0^t K M e^{-\alpha|\xi|^2 (t-s)} |\hat{w}_s(\cdot, \xi)|\, ds, & \text{if } t > 0, \\ |\varphi(t)|, & \text{if } -r \leqslant t \leqslant 0. \end{cases}$$

For $t > 0$ we have

$$\dot{z}(t) = -\alpha\,|\xi|^2\, z(t) + KM\,|\hat{w}_t(\cdot, \xi)| \leqslant -\alpha\,|\xi|^2\, z(t) + KM \sup_{-r \leqslant \theta \leqslant 0} z(t+\theta).$$

Taking $|\xi|^2 > KM/\alpha$ and using Lemma 4.2, we obtain $|\hat{w}(t, \xi)| \leqslant e^{-kt}\, \|\hat{w}_0(\cdot, \xi)\|$, for all $t > 0$, where $k = k(|\xi|^2) > 0$ is the unique real solution of $k - \alpha\,|\xi|^2 + KMe^{kr} = 0$. Therefore, if $t \geqslant r$ and $|\xi|^2 > KM/\alpha$, then

$$\|\hat{w}_t(\cdot, \xi)\| \leqslant e^{-k(|\xi|^2)(t-r)}\, \|\hat{w}_0(\cdot, \xi)\|.$$

This implies that

$$\|T_\xi(\omega)^j\|^{1/j} \leqslant e^{-k(|\xi|^2)(\omega - r/j)},$$

for all integer $j$ sufficiently large and so, $r(|\xi|^2) \leqslant e^{-k(|\xi|^2)\omega}$, if $|\xi| > KM/\alpha$. To complete the proof we observe that $k(|\xi|^2) \to \infty$ as $|\xi| \to \infty$. In fact, $k(|\xi|^2) \approx (1/r)\log((\alpha/M)\,|\xi|^2)$ as $|\xi| \to \infty$.

THEOREM 4.4. *In addition to the hypotheses of the previous lemma, suppose that $r(|\xi|^2) > 1$ for some $\xi \in \mathbf{R}^n$. Then, the solution $u(x, t) = p(t)$ of* (1.1) *is unstable. Moreover, there exist $R > 0$ and $a > 0$ such that, for any $\varepsilon > 0$, there are $\varphi \in \mathbf{C}(\mathbf{R}^n \times [-r, 0], \mathbf{R}^N)$ with $\|\varphi - p_0\| \leqslant \varepsilon$ and $t_\varepsilon > 0$ such that the solution $u$ of* (1.1) *with initial condition $u_0 = \varphi$ satisfies $\|u_{t_\varepsilon} - p_{t_\varepsilon}\| \geqslant a$, and, in fact,*

$$\sup_{-r \leqslant \theta \leqslant 0}\ \sup_{|x_1| \leqslant R,\, |x_2| \leqslant R} |u(x_1, t_\varepsilon + \theta) - u(x_2, t_\varepsilon + \theta)| \geqslant 2a.$$

Observe that the solution $u$ cannot be interpreted as being close to $p(t + \psi(x, t))$ for some slowly varying phase function $\psi$. Spatial inhomogeneities develop of definite amplitude $\geqslant a$ and wavelength $\leqslant R$

Before proving Theorem 4.4, we prove the following result, which is related to a more general result in [7, 8]. In our situation, we have more detailed information about the direction in which a given point leaves a neighborhood of the origin, as we see in the proof.

LEMMA 4.5. *Let $X$ be a real Banach space, $L\colon X \to X$ be a bounded linear operator with spectral radius $r = r(L) > 1$ and assume that there is an eigenvalue $\lambda$ of $L$ such that $|\lambda| = r$. If $T\colon U \subset X \to X$ is a map defined in a neighborhood $U$ of the origin such that $T(0) = 0$ and $\|T(x) - Lx\| = O(\|x\|^p)$ for some $p > 1$, then 0 is an unstable fixed point of $T$. In fact, there exists $u \in X$, $\|u\| = 1$ with the following property: there exist constants $C$, $\sigma_0 > 0$ such that if $0 < \sigma < \sigma_0$, then there exist arbitrarily small $\varepsilon > 0$ and arbitrarily large integers $N$ such that* (i) *$x_0 = \varepsilon u$, $x_{k+1} = T(x_k)$, $0 \leqslant k \leqslant N$ are defined and* (ii) *$\|x_N\| \geqslant \sigma/4$ and $\|x_N - \sigma u\| \leqslant C\sigma^p$.*

*Proof.* Let $0<\eta<r^p-r$; since $r=\lim_{n\to\infty}\|L^n\|^{1/n}$, there is a constant $K>0$ such that $\|L^n\|\leqslant K(r+\eta)^n$, for all $n\in\mathbf{N}$. Choose positive constants $a$ and $b$ such that $T(x)$ is defined and $\|T(x)-Lx\|\leqslant b\,\|x\|^p$ whenever $\|x\|\leqslant a$. Let $C=2^p bK/(r^p-r-\eta)$ and choose $\sigma>0$ such that $\sigma\leqslant a/2$ and $C\sigma^{p-1}\leqslant 1/2$.

Let $\lambda=re^{i\theta}$ and $\xi=u+iv\in X+iX$ (the complexification of $X$) such that $\|u\|=1$, $\|v\|\leqslant 1$ and $L\xi=\lambda\xi$. Given a positive integer $N_0$ and a real number $\delta$ such that $0<\delta<1/2$, let $N\geqslant N_0$ be a positive integer such that $\cos N\theta-|\sin N\theta|\geqslant 1-\delta$ and put $\varepsilon=\sigma/r^N$.

Since $L^n u=r^n(u\cos n\theta-v\sin n\theta)$, for all $n\in\mathbf{N}$, we have $\|L^n\|\leqslant r^n(|\cos n\theta|+|\sin n\theta|)\leqslant\sqrt{2}\,r^n$ for $0\leqslant n\leqslant N$ and $\|L^N u\|\geqslant r^N(\cos N\theta-|\sin N\theta|)\geqslant(1-\delta)r^N$. Note that $\|L^N u-r^N u\|\leqslant\delta r^N$.

Taking $x_0=\varepsilon u$, let us show that $x_n=T^n(x_0)$ is defined and satisfies $\|x_n\|\leqslant 2\varepsilon r^n$ whenever $0\leqslant n\leqslant N$ and $\|x_N\|\geqslant\sigma(1/2-\delta)$. This shows the instability of the origin as a fixed point of $T$. First, let us show that $\|x_n\|\leqslant 2\varepsilon r^n$, for $0\leqslant n\leqslant N$. Obviously, this inequality holds for $n=0$. Assume that $\|x_k\|\leqslant 2\varepsilon r^k$, for $0\leqslant k\leqslant n-1<N$; since $x_{k+1}=T(x_k)$ and $\|x_k\|\leqslant 2\varepsilon r^k<2\varepsilon r^N=2\sigma\leqslant a$, we have

$$x_n=L^n x_0+\sum_{k=0}^{n-1}L^{n-1-k}(T(x_k)-Lx_k).$$

The summation can be estimated as follows:

$$\begin{aligned}\left\|\sum_{k=0}^{n-1}L^{n-1-k}(T(x_k)-Lx_k)\right\| &\leqslant\sum_{k=0}^{n-1}K(r+\eta)^{n-k-1}\,b\,\|x_k\|^p\\ &=Kbr^{(n-1)p}\sum_{k=0}^{n-1}\left(\frac{r+\eta}{r^p}\right)^{n-k-1}\|x_k\|^p\,r^{-kp}\\ &\leqslant\frac{Kb2^p\varepsilon^p r^{(n-1)p}}{1-(r+\eta)/r^p}\\ &=\frac{Kb2^p\varepsilon^p r^{np}}{r^p-r-\eta}=C(\varepsilon r^n)^{p-1}\,\varepsilon r^n\\ &\leqslant C\sigma^{p-1}\varepsilon r^n\leqslant\frac{1}{2}\,\varepsilon r^n.\end{aligned}$$

Thus, $\|x_n\|\leqslant\sqrt{2}\,\varepsilon r^n+\frac{1}{2}\varepsilon r^n<2\varepsilon r^n$ and, by induction, the inequality holds for $0\leqslant n\leqslant N$. Finally,

$$\begin{aligned}\|x_N\|&\geqslant\|L^N x_0\|-\left\|\sum_{k=0}^{N-1}L^{N-k-1}(T(x_k)-Lx_k)\right\|\\ &\geqslant\varepsilon(1-\delta)r^N-\frac{1}{2}\varepsilon r^N=(1-\delta)\sigma-\frac{\sigma}{2}=\sigma\left(\frac{1}{2}-\delta\right),\end{aligned}$$

and the proof is complete.

*Proof of Theorem* 4.4. Let $u(x, t) = p(t) + v(x, t)$, so

$$\frac{\partial v}{\partial t} = D\,\Delta v + f'(p_t)v_t + R(t, v_t), \tag{4.3}$$

where $R(t, \varphi) = f(p_t + \varphi) - f(p_t) - f'(p_t)\varphi$. It is sufficient to prove the instability of the null solution of (4.3), and we verify that the hypotheses of Lemma 4.5 are satisfied by the period map $T(\omega)$ of (4.3).

Let $m \geqslant 1$ be such that $\|L(t, s)\| \leqslant m$ for $0 \leqslant s \leqslant t \leqslant \omega$. Since $f$ is $\mathbf{C}^2$, we have $R(t, \varphi) = O(|\varphi|^2)$ as $\varphi \to 0$ and therefore, there exist constants $a > 0$ and $b > 0$ such that $|R(t, \varphi)| \leqslant b\, \|\varphi\|^2$ whenever $\|\varphi\| \leqslant a$.

If $\varphi \in \chi$ satisfies $\|\varphi\| \leqslant \min\{1/2m^2b\omega, a/2m\} := \rho$ and $v$ is the solution of (4.3) with initial condition $v_0 = \varphi$, then for all $t \geqslant 0$, as long as $\|v_s\| \leqslant a$, $0 \leqslant s \leqslant t$, we have

$$v_t = L(t, 0)\varphi + \int_0^t [L(t, s)X_0]\, R(s, v_s)\, ds$$

and so, $\|v_t\| \leqslant m\, \|\varphi\| + \int_0^t bm\|v_s\|^2\, ds$, for all $t \geqslant 0$ as long as $\|v_s\| \leqslant a$, $0 \leqslant s \leqslant t$. This implies that $\|v_t\| \leqslant m\, \|\varphi\|/1 - m^2b\, \|\varphi\|\, t$, as long as $0 \leqslant t < 1/m^2b\, \|\varphi\|$ and $\|\varphi\| \leqslant a$. Since $\|\varphi\| \leqslant 1/2bm^2\omega$, we have $2\omega \leqslant 1/bm^2\, \|\varphi\|$ and $1 - m^2b\, \|\varphi\|\, t \geqslant 1 - bm^2t/2bm^2\omega \geqslant \frac{1}{2}$, for all $0 \leqslant t \leqslant \omega$. Therefore, $\|v_t\| \leqslant 2m\, \|\varphi\| \leqslant a$ for all $0 \leqslant t \leqslant \omega$.

On the other hand,

$$\begin{aligned}\|T(\omega)\varphi - L(\omega)\varphi\| &= \left\| \int_0^\omega [L(\omega, s)X_0]\, R(s, v_s)\, ds \right\| \\ &\leqslant \int_0^\omega bm\, \|v_s\|^2\, ds \leqslant 4bm^3\omega\, \|\varphi\|^2,\end{aligned}$$

for $\|\varphi\| \leqslant \rho$, which shows that $\|T(\omega)\varphi - L(\omega)\varphi\| = O(\|\varphi\|^2)$ as $\varphi \to 0$.

From Lemma 4.3, there exist $\xi_0 \in \mathbf{R}^n$ and $\mu > 0$ such that $r(|\xi_0|^2) = \max_{\xi \in \mathbf{R}^n} r(|\xi|^2) = e^{\mu\omega}$. The hypotheses imply that $\xi_0 \neq 0$ and the compactness of the period map of (4.1) (or some of its powers) implies that there are $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda = \mu$ and a non-constant $\omega$-periodic function $t \mapsto \psi(t) \in \mathbf{C}^N$ such that $\hat{w}(t) = e^{\lambda t}\psi(t)$ satisfies (4.2) and therefore, $w(x, t) = \operatorname{Re}(e^{\lambda t + i\xi_0 \cdot x}\psi(t))$ is a solution of (4.1). We can assume that $\sup_{-r \leqslant \theta \leqslant 0} |e^{\lambda\theta}\psi(\theta)| = 1$.

Let $R = 2\pi/|\xi_0|$. Note that $\sup_{-r \leqslant \theta \leqslant 0} \sup_{|x_i| \leqslant R} |w(x_1, t + \theta) - w(x_2, t + \theta)| = 2e^{\mu t}$, for all $t > 0$. As in the Lemma 4.5, let $C = 8bm^4\omega/(e^{\mu\omega} - 1)$. Choose $\sigma > 0$ such that $\sigma \leqslant \rho/2$ and $C\sigma \leqslant 1/2$ and put $a = \sigma/2$. Given arbitrarily small $\varepsilon > 0$, choose a positive integer $N_\varepsilon$ such that

$\sigma e^{-\mu\omega N_\varepsilon} \leq \varepsilon$ and let $v^\varepsilon(x, t)$ be the solution of (4.1) with initial condition $v_0^\varepsilon(x, 0) = \mathrm{Re}(\varepsilon e^{\lambda\theta + i\xi_0 \cdot x}\psi(\theta)) = \varepsilon w(x, \theta)$, $x \in \mathbf{R}^n$, $-r \leq \theta \leq 0$.

Then we have

$$\sup_{-r \leq \theta \leq 0} \sup_{x \in \mathbf{R}^n} |v^\varepsilon(x, t+\theta) - \mathrm{Re}[\varepsilon e^{\lambda(t+\theta) + i\xi_0 \cdot x}\psi(t+\theta)]| \leq C\sigma^2,$$

for all $0 \leq t \leq t_\varepsilon$ and $\|v^\varepsilon(\cdot, t_\varepsilon)\| \geq \sigma/2 = a$, where $t_\varepsilon = \omega N_\varepsilon$. On the other hand, if $|x_i| \leq R$, $i = 1, 2$, we have

$$\sup_{-r \leq \theta \leq 0} \sup_{|x_i| \leq R} |v^\varepsilon(x_1, t_\varepsilon + \theta) - v^\varepsilon(x_2, t_\varepsilon + \theta)| \geq 2\varepsilon e^{\mu t_\varepsilon} - 2C\sigma^2 \geq \sigma = 2a,$$

which completes the proof.

In the next result, we compute $\beta = (dr(|\xi|^2)/d|\xi|^2)|_{\xi=0}$ when the solution $u = p(t)$ of (1.2) has 1 as a *simple* characteristic multiplier. The condition $\beta > 0$ implies that $r(|\xi|^2) > 1$ for every $\xi \neq 0$ sufficiently small, giving thus a sufficient condition for instability of $u(x, t) = p(t)$ as a solution of (1.1), according to Theorem 4.4. This in an instability condition equivalent to the one in [3], where the case $N = 2$ and $r = 0$ is considered.

THEOREM 4.6. *Assume p is a simple non-constant ω-periodic solution of* (1.2) *and D is a real matrix. Then, for each sufficiently small real number λ, there exists an unique characteristic multiplier μ(λ) near* 1 *of the equation*

$$\dot{v}(t) = f'(p_t)v_t - \lambda D v(t). \tag{4.4$_\lambda$}$$

*The map $\lambda \mapsto \mu(\lambda)$ is differentiable (in fact, analytic) in a neighborhood of $\lambda = 0$, satisfies $\mu(0) = 1$, and*

$$\frac{d\mu}{d\lambda}(0) = -\int_0^\omega w(t)\, D\dot{p}(t)\, dt,$$

*where w (row vector) is the ω-periodic solution of the adjoint equation of* (3.3) *satisfying the condition*

$$\frac{1}{\omega}\int_0^\omega w(t)[\dot{p}(t) - f'(p_t)(\cdot)\dot{p}_t]\, dt = 1. \tag{4.5}$$

*If all the other characteristic multipliers of* (3.3) *have modulus less than* 1, *then $r(|\xi|^2) = \mu(|\xi|^2)$, when $|\xi|$ is close to zero. In any case, $r(|\xi|^2) \geq \mu(|\xi|^2)$.*

*Proof.* The period map $T_\lambda(\omega)$ of $(4.4)_\lambda$ depends analytically on $\lambda$ and, since $T_0(\omega)$ has $\mu = 1$ as a simple real eigenvalue, it follows that, for $\lambda$

sufficiently small, $T_\lambda(\omega)$ has a simple eigenvalue $\mu(\lambda)$ with $\mu(0)=1$ and the map $\lambda \mapsto \mu(\lambda)$ is analytic (see, e.g., [9]).

Let $\rho(\lambda)=(1/\omega)\log\mu(\lambda)$. Since $\mu(\lambda)$ is a characteristic multiplier of $(4.4)_\lambda$, there exists a function $q(t,\lambda)$, $\omega$-periodic in $t$ and analytic in $\lambda$, with $q(t,0)=\dot{p}(t)$ such that $v(t,\lambda)=e^{\rho(\lambda)t}q(t,\lambda)$ is a solution of $(4.4)_\lambda$. This implies that $q(t,\lambda)$ is a $\omega$-periodic solution of

$$\dot{q}(t)=f'(p_t)\,q_t+f'(p_t)[e^{\rho(\lambda)\cdot}-1]\,q_t-[\rho(\lambda)+\lambda D]\,q(t)$$

and, from Lemma 3.2, we have

$$\int_0^\omega w(t)[f'(p_t)(e^{\rho(\lambda)\cdot}-1)q_t(\cdot,\lambda)-(\rho(\lambda)+\lambda D)q(t,\lambda)]\,dt=0,$$

for all $\omega$-periodic solution $w(t)$ of the adjoint equation of (3.3). By taking derivative with respect to $\lambda$ and setting $\lambda=0$, we have

$$\int_0^\omega w(t)[\rho'(0)f'(p_t)(\cdot)\dot{p}_t-(\rho'(0)+D)\,\dot{p}(t)]\,dt=0,$$

and choosing $w$ so that (4.5) is satisfied, we obtain $\omega\rho'(0)=-\int_0^\omega w(t)\,D\dot{p}(t)\,dt$. Since $\omega\rho'(0)=\mu'(0)$, the proof is complete.

In the next theorem we give a result connecting $\mu'(0)$ with the periods of periodic solutions of a family of equations related to (1.2). This is an extension of the results contained in [11] to parabolic-delay equations.

THEOREM 4.7. *Assume $p$ is a simple non-constant $\omega$-periodic solution of Eq.* (1.2) *and $D$ is a real matrix. If $\varepsilon$ is sufficiently small, then the equation*

$$\dot{v}(t)=(I+\varepsilon D)^{-1}f(v_t) \tag{4.6}$$

*has a unique periodic solution $v(\cdot,\varepsilon)$ of periodic $T(\varepsilon)$ such that $v(\cdot,0)=p$ and $T(0)=\omega$. Moreover, the maps $\varepsilon\mapsto v(\cdot,\varepsilon)$ and $\varepsilon\mapsto T(\varepsilon)$ are differentiable in a neighborhood of zero and $T'(0)=-\mu'(0)$.*

*Proof.* The proof is a slight modification of the proof of Theorem 3.1. First, we choose the $\omega$-periodic solution $w$ of the adjoint equation of (3.3) such that (4.5) is satisfied.

Let $\gamma$ and $\pi$ be the continuous functional and the continuous projection defined on $\mathscr{P}_\omega$ by $\gamma(h)=\int_0^\omega w(t)\,h(t)\,dt$ and $(\pi h)(t)=\dot{p}(t)\,\gamma(h)/\gamma(\dot{p})$, respectively. By changing variables $t=(1+\beta)\,\tau$ and $v(t)=y(\tau)$, where $|\beta|<1/2$

and using the same notations as in the proof of Theorem 3.1, (4.6) can be written as

$$\frac{dy}{d\tau} = (1+\beta)(I+\varepsilon D)^{-1} f(y_{\tau,\beta}). \tag{4.7}$$

Letting $y(\tau) = p(\tau) + z(\tau)$, $z$ then satisfies

$$\frac{dz}{d\tau} = f'(p_\tau)\, z_{\tau,0} + H(z, \beta, \varepsilon)(\tau), \tag{4.8}$$

where $H(z, \beta, \varepsilon)(\tau) = (1+\beta)(I+\varepsilon D)^{-1} f(p_{\tau,\beta} + z_{\tau,\beta}) - f(p_\tau) - f'(p_\tau) z_{\tau,0}$.

Let $a = \int_0^\omega |w(\tau)|^2\, d\tau = \gamma(w^T)$. As in the proof of Theorem 3.1, we can show that the equation

$$\frac{dz}{d\tau} = f'(p_\tau)\, z_{\tau,0} + H(z, \beta, \varepsilon)(\tau) - \gamma(H(z, \beta, \varepsilon))\, \omega(\tau)^T/a$$

has a unique $\omega$-periodic solution $z = z^*(\beta, \varepsilon)$ such that $z^*(0,0) = 0$ and $z^*$ is $\mathbf{C}^1$ in a neighborhood of $(0, 0)$. In order that $z = z^*(\beta, \varepsilon)$ to be a solution of (4.8) it is necessary and sufficient that $\gamma(H(z^*(\beta, \varepsilon), \beta, \varepsilon)) = 0$, that is,

$$B(\beta, \varepsilon) := \int_0^\omega w(\tau)\, H(z^*(\beta, \varepsilon), \beta, \varepsilon))\, d\tau = 0.$$

Since $B$ is $\mathbf{C}^1$, $B(0,0) = 0$ and $(\partial B/\partial\beta)(0,0) = \int_0^\omega w(\tau)[\dot{p}(\tau) - f'(p_\tau)(\cdot)\, \dot{p}(\tau)]\, d\tau = \omega$, we apply the Implicit Function Theorem to solve $B(\beta, \varepsilon) = 0$ to get $\beta = \beta(\varepsilon)$ as a $\mathbf{C}^1$ function of $\varepsilon$ in a small neighborhood of $(0, 0)$.

Therefore, $v(t, \varepsilon) = p(\tau) + z^*(\beta(\varepsilon), \varepsilon)(\tau)$ is the unique periodic solution of (4.6) of period $T(\varepsilon) = \omega(1 + \beta(\varepsilon))$ such that $v(\cdot, 0) = p$ and $T(0) = \omega$. Since $T'(0) = \omega\beta'(0) = \int_0^\omega w(\tau)\, D\dot{p}(\tau)\, d\tau$, from Theorem 4.6, we have $T'(0) = -\mu'(0)$ and the proof is complete.

## 5. Examples

1. Consider $N = 2$ and $f(\varphi) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \varphi(0) + (1 - \varphi_1(0)^2 - \varphi_2(0)^2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varphi(-2\pi)$, where $a, b, c, d$ are real numbers. Then $p(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ is a $2\pi$-periodic solution of (1.2) and the variational equation around $p_t$ is

$$\dot{v}(t) = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - 2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos^2 t & \sin t \cos t \\ \sin t \cos t & \sin^2 t \end{pmatrix} \right\} v(t), \tag{5.1}$$

so the characteristic multipliers are 1 and $\mu = e^{-2\pi(a+d)}$ and, therefore, $p$ is orbitally stable with respect to the flow of (1.2) when $a+d>0$.

Let us choose $b=c=0$, $a=2$ and $d=-1$. In this case, the adjoint equation of (5.1) is

$$\dot{w}(t) = w(t)\begin{pmatrix} 4\cos^2 t & -1-\sin 2t \\ 1+2\sin 2t & -2\sin^2 t \end{pmatrix} := -w(t)\, A(t). \tag{5.2}$$

We are looking for the $2\pi$-periodic solution of (5.2) satisfying $\int_0^\omega w(t)\,\dot{p}(t)\,dt = 2\pi$. Since $w(t)\,\dot{p}(t)$ is constant, we must look for the $2\pi$-periodic solution of (5.2) such that $-w_1(t)\sin t + w_2(t)\cos t = 1$, for all $t$.

By changing variables

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

the above condition means $y(t) \equiv 1$ and $x$ satisfies

$$\dot{x}(t) = (1 + 3\cos 2t)\, x(t) - 3\sin 2t. \tag{5.3}$$

Equation (5.3) has a unique $\pi$-periodic solution, which we write as a Fourier series $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{2int}$. The coefficients $c_n$ satisfy $c_{-n} = \bar{c}_n$ and

$$c_1 + \tfrac{3}{2}(c_0 + c_2 + i) = 2ic_1$$

$$c_n + \tfrac{3}{2}(c_{n-1} + c_{n+1}) = 2nic_n, \qquad \text{if} \quad n \neq \pm 1.$$

Let $c_1 = (1/2\pi)\int_0^{2\pi} x(t)\, e^{-2it}\, dt := p + iq$ $(p, q \in \mathbf{R})$. Since $c_{n-1} + c_{n+1} = \frac{2}{3}(2in-1)\, c_n$, for $n \neq \pm 1$, letting $D_n = c_{n-1}/c_n$, we obtain $D_n = z_n - 1/D_{n+1}$, where $z_n = \frac{2}{3}(2in-1)$ and therefore

$$D_n = z_n - 1\Big/\left(z_{n+1} - 1\Big/\left(z_{n+2} - \frac{1}{z_{n+3} - \cdots}\right)\right)$$

In particular,

$$\frac{1}{D_2} = 1\Big/\left(z_2 - 1\Big/\left(z_3 - \frac{1}{z_4 - \cdots}\right)\right) := A + Bi.$$

On the other hand,

$$\frac{1}{D_2} = \frac{c_2}{c_1} = \frac{2}{3}(2i-1) + \frac{3p-i}{p+qi}$$

and thus

$$\frac{3p-i}{p+qi}=(A+Bi)-\frac{2}{3}(2i-1),$$

so, $p$ and $q$ are solutions of the equations

$$\begin{cases}(A-\frac{7}{3})p-(B-\frac{4}{3})q=0\\(B-\frac{4}{3})p+(A+\frac{2}{3})q=-1.\end{cases}$$

Cutting off the continuous fraction defining $1/D_2$ in the $z_5$-term, we obtain $A+B_i\approx -0.0716-0.3292i$ and, therefore, $p\approx 1.2470$ and $q\approx 1.8039$. Thus, $\int_0^\pi x(t)\cos 2t\,dt\approx 1.2470\pi$ and $\int_0^\pi x(t)\sin 2t\,dt\approx 1.8039\pi$.

Finally, we obtain

$$\mu'(0)=-\int_0^{2\pi}(x(t)\cos t-\sin t,\ x(t)\sin t+\cos t)\begin{pmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{pmatrix}\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}dt$$

$$\approx -\pi(2.8039d_{11}-2.4941d_{12}+4.9883d_{21}-0.8039d_{22}).$$

If we take $D=\left(\begin{smallmatrix}1/\varepsilon & 0\\ 0 & \varepsilon\end{smallmatrix}\right)$, we have $\mu'(0)>0$ (therefore, instability) when $\varepsilon>1.88$.

When $\varepsilon=1$, we have $D=I$ and the fundmental solution $X(t,\xi)$ of (4.2) satisfying $X(0,\xi)=I$ is given by $X(t,\xi)=X_0(t)\,e^{-t|\xi|^2}$ for all $\xi\in\mathbf{R}^n$, where $X_0(t)$ is the fundamental solution of (4.2) with $\xi=0$. When $|\varepsilon-1|$ is sufficiently small, there exist constants $C>0$ and $\beta>0$ such that $|X(t,\xi)|\leqslant Ce^{-\beta t|\xi|^2}$, for all $\xi\in\mathbf{R}^n$ and $t\geqslant 0$. Letting $K(x,t):=(2\pi)^{-n}\int_{\mathbf{R}^n}X(t,\xi)e^{i\xi\cdot x}\,d\xi$, we can show (see the Appendix) that $\int_{\mathbf{R}^n}|K(x,t)|\,dx<\infty$, so the solution of (4.1), for this example, is given by

$$w(x,t)=\int_{\mathbf{R}^n}K(x-y,t)\,w_0(y)\,dy,$$

and this implies that the linear equation (4.1) is stable when $\varepsilon$ is sufficiently close to 1.

Still in this example, we note that $\mu'(0)<0$ for $\varepsilon>0$ sufficiently small and, therefore, we cannot use Theorem 4.6 to conclude instability of $p$. However, we show that Theorem 4.4 can be used to prove instability of $p$ as a solution of (1.1) when $\varepsilon$ is sufficiently small. Actually, we show that Eq. (4.2), with $|\xi|=1$, has a characteristic multiplier whose modulus is greater than 1 when $\varepsilon\to 0+$. In this case, with $|\xi|=1$, Eq. (4.2) is written as

$$\frac{d\hat{v}}{dt}(t)=\begin{pmatrix}-1/\varepsilon-4\cos^2 t & -1-2\sin 2t\\ 1+\sin 2t & -\varepsilon+2\sin^2 t\end{pmatrix}\hat{v}(t)\tag{5.4}$$

Let $\hat{v} = \binom{\hat{v}_1}{\hat{v}_2}$ be a solution of (5.4) and $x = \binom{x_1}{x_2}$ be the function given by

$$x_1(t) = Ae^{-t/\varepsilon} - 2Ae^{-t/\varepsilon}(t + \tfrac{1}{2}\sin 2t) - \varepsilon C(1 + 2\sin 2t)\, e^{\int_0^t \sin^2 s\, ds}$$

and

$$x_2 = Ce^{2\int_0^t \sin^2 s\, ds},$$

where $A = \hat{v}_1(0) + \varepsilon\hat{v}_2(0)$ and $C = \hat{v}_2(0)$.

It can be shown that $\hat{v}_1(t) = x_1(t) + O(\varepsilon) + e^{-t/\varepsilon}O(1)$ and $\hat{v}_2 = x_2(t) + O(\varepsilon)$, uniformly on $0 \leqslant t \leqslant 2\pi$ as $\varepsilon \to 0+$. Thus, it follows that the period map $T_\varepsilon$ of (5.4) has matrix given by

$$[T_\varepsilon] = \begin{pmatrix} (1-4\pi)\, e^{-2\pi/\varepsilon} & \varepsilon[(1-4\pi)\, e^{-2\pi/\varepsilon} - 2\pi] \\ 0 & e^{2\pi} \end{pmatrix} + O(\varepsilon) + e^{-2\pi/\varepsilon}O(1),$$

and so $[T_\varepsilon] \to \begin{pmatrix} 0 & 0 \\ 0 & e^{2\pi} \end{pmatrix}$ as $\varepsilon \to 0+$, and the result follows.

**2.** Consider the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_{11}\, \Delta u + d_{12}\, \Delta v + g(u(x, t-1)), \\[2ex] \dfrac{\partial v}{\partial t} = d_{21}\, \Delta u + d_{22}\, \Delta v + \alpha(u(x, t-1) - \gamma v(x, t)), \end{cases} \tag{5.5}$$

where $\alpha, \gamma$ are positive constants and $g\colon \mathbf{R} \to \mathbf{R}$ is an odd $\mathbf{C}^2$ function satisfying the conditions $g'(0) < -\pi/2 < \lim_{u \to \infty} g(u)/u$, and $g'(u) < 0$ and $g''(u) > 0$ for all $u > 0$.

Under these conditions, the reaction equation

$$\begin{cases} \dot{u} = g(u(x, t-1)) \\ \dot{v} = \alpha(u(x, t-1) - \gamma v(x, t)) \end{cases} \tag{5.6}$$

has a 4-periodic solution $\phi(t) = (p(t), q(t))$ satisfying the symmetry condition $\phi(t-2) = \phi(t)$ for all $t \in \mathbf{R}$. Indeed, the existence of $p$ follows from the results in [2] and $q$ is given by

$$q(t) = \alpha \int_{-\infty}^{t} e^{-\alpha\gamma(t-s)} p(s-1)\, ds = \frac{1}{\gamma} \int_{-\infty}^{0} e^s p\left(t - 1 + \frac{s}{\alpha\gamma}\right) ds. \tag{5.7}$$

We assume that $p(-1) = 0$ and $p(t) \geqslant 0$ for $t \in [-1, 1]$.

By using the results contained in [2], it is not difficult to prove that $\phi$ is exponentially asymptotically stable for (5.6) and therefore the hypotheses of Theorem 4.6 are satisfied for this example. We are going to show that $\phi$ is unstable for (5.5) if $d_{12} > 0$ and $\alpha$ is sufficiently large.

Since the linearized equation around $\phi$ is

$$\begin{cases} \dot{u} = g'(p(t-1))\, u(t-1), \\ \dot{v} = \alpha u(t-1) - \alpha\gamma v(t); \end{cases} \tag{5.8}$$

the corresponding adjoint equation is given by

$$\begin{cases} \dot{x} = -g'(p(t))\, x(t+1) - \alpha y(t+1), \\ \dot{y} = \alpha\gamma y(t). \end{cases} \tag{5.9}$$

The 4-periodic solutions of the adjoint equation are given by $w(t) = k(\dot{p}(t-1), 0)$, where $k$ is a constant. In order to satisfy condition (4.5) we choose $k = 2(\int_{-1}^{1} g'(p(t))\, \dot{p}(t)^2\, dt)^{-1}$. It follows that

$$\begin{aligned} -\mu'(0) &= k \int_0^4 \dot{p}(t-1)[d_{11}\dot{p}(t) + d_{12}\dot{q}(t)]\, dt = -2kd_{12} \int_0^2 \ddot{p}(t-1)\, q(t)\, dt \\ &= -2kd_{12} \int_0^2 \ddot{p}(t-1) \frac{1}{\gamma} \int_{-\infty}^0 e^s p\left(t - 1 + \frac{s}{\alpha\gamma}\right) ds\, dt. \end{aligned}$$

Since $\int_{-\infty}^0 e^s p(t - 1 + s/\alpha\gamma)\, ds \to p(t-1)$ as $\alpha \to \infty$, uniformly on compact sets, the right-hand side tends to

$$\frac{-2k}{\gamma} d_{12} \int_0^2 \ddot{p}(t-1)\, p(t-1)\, dt = \frac{2k}{\gamma} d_{12} \int_0^2 \dot{p}(t-1)^2\, dt = \frac{2k}{\gamma} d_{12} \int_{-1}^1 \dot{p}(t)^2\, dt.$$

Therefore, if $d_{12} > 0$ and $\alpha$ is sufficiently large, we have $\mu'(0) > 0$, and the result follows from Theorem 4.6.

## Appendix

The main purpose of this Appendix is to state and prove some results we had used in Example 1. Since the linear equations envolved in the example do not contain delays, we will restrict ourselves to the linear O.D.E. case. The general case will be considered elsewhere. I am indebted to Professor D. Henry for many of the computations below.

Let $A\colon \mathbf{R} \to L(\mathbf{R}^N)$ be an $\omega$-periodic matrix and $D$ an $N \times N$ real matrix with eigenvalues in the half plane $\operatorname{Re} \lambda > 0$. Let $X_0(t)$ be the fundamental matrix solution of

$$\dot{X}_0(t) = A(t)\, X_0(t), \qquad X_0(0) = I, \tag{A.1}$$

and assume that 1 is a *simple* eigenvalue of $X_0(\omega)$ and the others have modulus $< 1$, so that $|X_0(t)\, X_0(s)^{-1}| \leqslant M$, for some positive constant $M$ and all $t \geqslant s$.

For $\xi \in \mathbf{R}^n$, let $X(t, s; |\xi|^2)$ be the fundamental solution of

$$\frac{\partial X}{\partial t}(t, s; |\xi|^2) = (A(t) - |\xi|^2 D)\, X(t, s; |\xi|^2), \qquad X(s, s; |\xi|^2) = I. \tag{A.2}$$

LEMMA A.1. *If $D$ is sufficiently close to the identity matrix, then there exist positive constants $C$ and $\beta$ such that*

$$|X(t, s; |\xi|^2)| \leqslant C e^{-\beta |\xi|^2 (t-s)},$$

*for all $t \geqslant s \geqslant 0$ and $\xi \in \mathbf{R}^n$.*

*Proof.* Since $X(t, s; |\xi|^2)$ is the fundamental matrix solution of a periodic system, it is sufficient to find estimates for $X(t, 0; |\xi|^2) := X(t; |\xi|^2)$. Letting $R = D - I$, we have

$$X(t; |\xi|^2) = X_0(t)\, e^{-|\xi|^2 t} - \int_0^t X_0(t)\, X_0(s)^{-1}\, e^{-|\xi|^2 (t-s)}\, |\xi|^2\, RX(s; |\xi|^2)\, ds,$$

and so,

$$|X(t; |\xi|^2)|\, e^{|\xi|^2 t} \leqslant M + \int_0^t M\, |R|\, |\xi|^2\, e^{|\xi|^2 s}\, |X(s; |\xi|^2)|\, ds,$$

for $t \geqslant 0$. Gronwall's lemma implies $|X(t, |\xi|^2)| \leqslant Me^{-|\xi|^2 t(1 - M|R|)}$ for $t \geqslant 0$, and the proof is complete by taking $|R| < 1/M$ and $\beta = 1 - M|R|$.

In the following, the notation $\Delta_\xi^m X(t, s; |\xi|^2)$ means the $m$th iteration of the Laplacian of $X(t, s; |\xi|^2)$ with respect to the $\xi$-variable, and $L_2$-norms are taken with respect to $\xi \in \mathbf{R}^n$.

LEMMA A.2. *Suppose $D$ as in Lemma A.1. Then, there is a positive constant $A_0$ such that*

$$\|X(t, s; |\xi|^2)\|_{L_2} \leqslant A_0 (t-s)^{-n/4},$$

*for $t > s$. Furthermore, for each $m > n/4$, there is a positive constant $A_m$ such that*

$$\|\Delta_\xi^m X(t, s; |\xi|^2)\|_{L_2} \leqslant A_m (t-s)^{m - n/4},$$

*for $t > s$.*

*Proof.* As before, it is sufficient to find estimates for $X(t, 0; |\xi|^2) := X(t; |\xi|^2)$. Define $X^{(0)}(t; \lambda) = X(t; \lambda)$ and $X^{(k)}(t; \lambda) = (d^k X/d\lambda^k)(t; \lambda)$, for $\lambda \in \mathbf{R}_+$ and $k \in N$. By induction, it is easy to show that

$$\frac{\partial X^{(k)}}{\partial t}(t; \lambda) = (A(t) - \lambda D)\, X^{(k)}(t; \lambda) - kDX^{(k-1)}(t; \lambda),$$

and $X^{(k)}(0; \lambda) = 0$, for $k > 0$ and $X^{(0)}(0; \lambda) = I$. It follows that, for each $k \in \mathbf{N}$, there is a positive constant $C_k$ such that $|X^{(k)}(t; \lambda)| \leqslant C_k t^k e^{-\beta\lambda t}$, for all $t \geqslant 0$ and $\lambda \in \mathbf{R}$.

Now, we have

$$\begin{aligned} \Delta_\xi \{ |\xi|^{2j} X^{(k)}(t; |\xi|^2)\} &= 2j(2j+n-2)\, |\xi|^{2(j-1)}\, X^{(k)}(t; |\xi|^2) \\ &\quad + 2(4j+n)\, |\xi|^2\, X^{(k+1)}(t; |\xi|^2) \\ &\quad + 4\, |\xi|^{2(j+1)}\, X^{(k+2)}(t; |\xi|^2). \end{aligned}$$

Note that we have a sum of terms $|\xi|^{2p} X^{(q)}(t; |\xi|^2)$ with $q - p = k - j + 1$, $k \leqslant q \leqslant k+2$, $p \geqslant 0$, and $j - 1 \leqslant p \leqslant j+1$ (if $j = 0$, the term $|\xi|^{2(j-1)} X^{(k)}(t; |\xi|^2)$ is absent).

It follows that $\Delta_\xi^m(|\xi|^{2j} X(t; |\xi|^2))$ is a sum of terms of the form $|\xi|^{2(q+j-m)} X^{(q)}(t; |\xi|^2)$, with $m - j \leqslant q \leqslant 2m$. In particular,

$$\Delta_\xi^m X(t; |\xi|^2) = \sum_{q=m}^{2m} a_q\, |\xi|^{2(q-m)}\, X^{(q)}(t; |\xi|^2),$$

for some real constants $a_q$. Thus, for $t > 0$, we have

$$\begin{aligned} \|\Delta_\xi^m X(t; |\xi|^2)\|_{L_2} &\leqslant \sum_{q=m}^{2m} |a_q|\, C_q t^q \left( \int_{\mathbf{R}^n} |\xi|^{4(q-m)}\, e^{-2\beta |\xi|^2 t}\, d\xi \right)^{1/2} \\ &= \sum_{q=m}^{2m} |a_q|\, C_q t^q C_{m,q} t^{-n/4+m-q} = A_m t^{m-n/4}, \end{aligned}$$

for some positive constant $A_m$, and the proof is complete.

LEMMA A.3. *Suppose D as in Lemma* A.1 *and let*

$$K(t, s; x) := (2\pi)^{-n} \int_{\mathbf{R}^n} X(t, s; |\xi|^2)\, e^{i\xi \cdot x}\, d\xi.$$

*Then there is a constant C such that* $\int_{\mathbf{R}^n} |K(t, s; x)|\, dx \leqslant C$, *for all* $t > s$.

*Proof.* Let $m > n/4$ be a positive integer and $R$ any positive real number. Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_{\mathbf{R}^n} |K(t,x)|\,dx &= \int_{|x|\leqslant R} 1\cdot|K(t,x)|\,dx + \int_{|x|\geqslant R} |x|^{-2m}\,|x|^{2m}\,|K(t,x)|\,dx\\ &\leqslant \left(\int_{|x|\leqslant R} 1\,dx\right)^{1/2}\left(\int_{|x|\leqslant R} |K(t,x)|^2\,dx\right)^{1/2}\\ &\quad + \left(\int_{|x|\geqslant R} |x|^{-4m}\,dx\right)^{1/2}\left(\int_{|x|\geqslant R} |x|^{4m}\,|K(t,x)|^2\,dx\right)^{1/2}\\ &\leqslant \omega_n^{1/2} R^{n/2}\,\|K(t,\cdot)\|_{L_2} + \frac{\omega_n^{1/2} R^{n/2-2m}}{(4m-n)^{1/2}}\,\|\,|x|^{2m}\,K(t,x)\|_{L_2},\end{aligned}$$

where $\omega_n$ is the area of the unit sphere on $\mathbf{R}^n$.

Since $X(t,|\xi|^2)$ and $\Delta_\xi^m X(t;|\xi|^2)$ are the Fourier transforms of the maps $x \mapsto K(t,x)$ and $x \mapsto |x|^{2m} K(t,x)$, respectively, it follows from Plancherel's Theorem that

$$\int_{\mathbf{R}^n} |K(t,x)|\,dx \leqslant C_n R^{n/2}\,\|X(t,\cdot)\|_{L_2} + C'_{n,m} R^{n/2-2m}\,\|\Delta_\xi^m X(t,\cdot)\|_{L_2},$$

for some constants $C_n$ and $C'_{n,m}$, and by Lemma A.2, we have

$$\begin{aligned}\int_{\mathbf{R}^n} |K(t,x)| &\leqslant C_n R^{n/2} A_0 t^{-n/4} + C'_{n,m} R^{n/2-2m} A_m t^{m-n/4}\\ &= C_n A_0 (R/\sqrt{t})^{n/2} + C'_{n,m} A_m (t/R^2)^{m-n/4},\end{aligned}$$

for all $t>0$, $m>n/4$ and $R>0$.

Taking $R=\sqrt{t}$, we have $\int_{\mathbf{R}^n} |K(t,x)|\,dx \leqslant C$, for some constant $C$ and all $t>0$ and the proof is complete.

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