Cospectrality and similarity for a pair of matrices under multiplicative and additive composition with diagonal matrices

J.A. Dias da Silva a,*,1, Charles R. Johnson b,2

aDepartamento de Matemática, da Universidade de Lisboa-CELC, Av Gama Pinto 2, 1699 Lisboa Codex, Portugal
bDepartment of Mathematics, College of William and Mary, Williamsburg, VA 23187, USA

Received 20 July 1999; accepted 11 September 2000
Submitted by T.J. Laffey

Abstract

Given \( n \times n \) complex matrices \( A \) and \( B \) of equal nonzero determinant (of equal trace), we discuss when there exists an invertible diagonal matrix \( D \) (a diagonal matrix \( D \)) such that \( AD \) and \( BD \) (\( A + D \) and \( B + D \)) have equal spectrum and, also, when \( D \) may be chosen so that they are similar. In case \( D \) is relaxed to be a general nonsingular (general) matrix, then similarity may always be achieved. © 2001 Published by Elsevier Science Inc. All rights reserved.

AMS classification: 15A18; 15A29

1. Introduction

Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be \( n \times n \) complex matrices throughout. The pair \( (A, B) \) is called cospectral if the matrices \( A \) and \( B \) have the same eigenvalues, counting multiplicities, i.e., if they have the same characteristic polynomial. It is obviously necessary for cospectrality that \( A \) and \( B \) have the same determinant and the same trace.

* Corresponding author.
E-mail address: perdigao@hermite.cii.fc.ul.pt (J.A. Dias da Silva).
1 This research was done within the activities of “Centro de Álgebra da Universidade de Lisboa” and partially supported by PRAXIS XXI project “Álgebra e Matemáticas Discretas”.
2 The work was supported by PRAXIS XXI grant BCC/11955/97.

0024-3795/01/$ - see front matter © 2001 Published by Elsevier Science Inc. All rights reserved.
PII: S 0 0 2 4 - 3 7 9 5 ( 0 0 ) 0 0 2 9 8 - 6
It is necessary and sufficient that for each \( k = 1, \ldots, n \), the sum of the \( k \times k \) principal minors of \( A \) is the same as that for \( B \) as \( \pm \) this sum is the coefficient of \( \lambda^{n-k} \) in the characteristic polynomial. If \( A \) and \( B \) are randomly chosen, the probability is 0 that they be cospectral. It is natural, then, to ask how simple a common modification (multiplicative or additive) will produce cospectrality. We have two natural and parallel questions then:

1. for which pairs of invertible matrices \( A \) and \( B \) is there an invertible \( n \times n \) complex matrix \( C \) such that \( AC \) and \( BC \) are cospectral, and

2. for which pairs \( A \) and \( B \) is there a \( C \) such that \( A + C \) and \( B + C \) are cospectral?

For question (1) it is obviously necessary that \( \det A = \det B \) and for (2) that \( \text{tr} A = \text{tr} B \). Perhaps surprisingly, under the obvious necessary condition, the answer to each question is not only “yes”, but it may be shown that an arbitrary common spectrum may be achieved and, thus, similarity may be achieved. These problems are obviously related with the description of the set of spectrums of the product (the sum) of pairs of matrices with prescribed similarity classes, see [4].

We may then ask, under what further circumstances may \( C \) be chosen with special structure? Here, we ask when \( C \) may be chosen to be diagonal (and so we use \( D \) in place of \( C \)). Our questions are reminiscent of the multiplicative and additive inverse eigenvalue problems studied in [1,3]. However, there are notable differences, as well as parallels.

We are going to present some results proved in [3] that are going to be needed in the sequel. For the sake of completeness we present for Theorem 1.3 the proof of the existence of solutions. This proof and the proof of the remaining statements of this Theorem are similar to the proof of Theorem 4 of [3].

**Lemma 1.1.** Let \( f_t \in \mathbb{C}[X_1, \ldots, X_n] \), where

\[
    f_t = \sum_{1 \leq i_1 < \cdots < i_t \leq n} a_{i_1 \cdots i_t} x_{i_1} \cdots x_{i_t}, \quad t = 1, \ldots, n,
\]

with \( a_{i_1, \ldots, i_t} \not= 0 \), \( t = 1, \ldots, n \), \( 1 \leq i_1 < \cdots < i_t \leq n \). Then, there exist homogeneous polynomials \( h_{ij} \), \( i = 1, \ldots, n \), \( j = 1, \ldots, n \), and an integer \( q \) such that

\[
    X_i^q = h_{i1} f_1 + \cdots + h_{in} f_n, \quad i = 1, \ldots, n, \tag{1}
\]

\[
    \deg (h_{ij}) = q - j \text{ or } \deg (h_{ij}) = -\infty, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n, \tag{2}
\]

and

\[
    x_i^p | h_{i,n-p}, \quad i = 1, \ldots, n, \quad p = 1, \ldots, n-1. \tag{3}
\]

**Lemma 1.2.** If \( h_{ij}, i = 1, \ldots, n, j = 1, \ldots, n \) and \( q \) meet the conditions of Lemma 1 and

\[
    R_i(X_1, \ldots, X_n) = X_i^q + P_i(X_1, \ldots, X_n), \quad i = 1, \ldots, n
\]

with \( \deg P_i(X_1, \ldots, X_n) < q \), then

\[
    \det (H_{ij}) \not\in (R_1, \ldots, R_n).
\]
Theorem 1.3. Let \( f_t \in \mathbb{C}[X_1, \ldots, X_n] \), where
\[
f_t = \sum_{1 \leq i_1 < \cdots < i_t \leq n} a_{i_1 \cdots i_t} X_{i_1} \cdots X_{i_t}, \quad t = 1, \ldots, n
\] (4)
with \( a_{i_1 \cdots i_t} \neq 0 \), \( t = 1, \ldots, n \), \( 1 \leq i_1 < \cdots < i_t \leq n \). Let
\[
(\varphi_1(X_1, \ldots, X_n), \ldots, \varphi_n(X_1, \ldots, X_n))
\]
be an \( n \)-tuple of elements of \( \mathbb{C}[X_1, \ldots, X_n] \) such that \( \deg \varphi_i < i \), \( i = 1, \ldots, n \). Then the system of \( n \) polynomial equations
\[
f_t(X_1, \ldots, X_n) + \varphi_t(X_1, \ldots, X_n) = 0, \quad t = 1, \ldots, n
\] has a solution. Moreover, the number of solutions is finite and does not exceed \( n! \).

Proof. We present just the proof of the existence of solutions.

Lemma 1.1 guarantees the existence of homogeneous polynomials \( h_{ij} \), \( i = 1, \ldots, n \), \( j = 1, \ldots, n \) and an integer \( q \) satisfying (1)–(3). Define
\[
g_t(X_1, \ldots, X_n) = f_t(X_1, \ldots, X_n) + \varphi_t(X_1, \ldots, X_n), \quad t = 1, \ldots, n
\] (5)
and
\[
R_t = X_t^q + \varphi_1 h_{t1} + \cdots + \varphi_n h_{tn}, \quad t = 1, \ldots, n.
\] (6)
Then \( R_t \) is a polynomial of the form
\[
R_t(X_1, \ldots, X_n) = X_t^q + P_t(X_1, \ldots, X_n)
\] with \( \deg P_t < q \). From (1), (5) and (6) we get
\[
R_t = g_1 h_{t1} + g_2 h_{t2} + \cdots + g_n h_{tn}, \quad t = 1, \ldots, n.
\] (7)
Let \( H := (h_{ij}) \). The above equality can be written
\[
[R_1 \cdots R_n]^T = H[g_1 \cdots g_n]^T.
\] (8)
Let \( Z = (z_{ij}) \) be the adjoint of \( H \). Multiplying both sides of (8) by \( Z \) on the left, we obtain
\[
det H g_t = \sum_{k=1}^{n} z_{tk} R_k, \quad t = 1, \ldots, n.
\]
If \( g_1, \ldots, g_n \) does not have any common zero, by the Hilbert Nullstellensatz we know that there exist \( v_t \in K[X_1, \ldots, X_n] \), \( t = 1, \ldots, n \), satisfying
\[
1 = \sum_{t=1}^{n} v_t g_t.
\]
Multiplying both sides of the former equality by \( \det H \) we would get \( \det H \in (R_1, \ldots, R_n) \) if \( g_1, \ldots, g_n \) have no common zero. But, this would contradict Lemma 1.2, and we conclude that \( g_1, \ldots, g_n \) have a common zero, as was to be shown. \( \square \)
2. The multiplicative problem

We first consider the multiplicative problem. Since we wish to equate the kth principal minor sum in $AD$ with that in $BD$, the differences of corresponding individual principal minors will be important. Let $N = \{1, \ldots, n\}$, and let $A[\alpha]$ denote the principal submatrix of $A$ lying in rows and columns $\alpha$. We define

$$c_\alpha = \det A[\alpha] - \det B[\alpha].$$

Let $D = \text{diag} (X_1, \ldots, X_n)$, the diagonal matrix with diagonal entries $X_1, \ldots, X_n$. It is then an exercise to see that the sum of the $k \times k$ principal minors of $AD$ is that of $BD$ if and only if

$$\sum_{|\alpha|=k} c_\alpha \prod_{i \in \alpha} X_i = 0 \quad (9)$$

and that $AD$ and $BD$ are cospectral if and only if (9) holds for $k = 1, \ldots, n$. When we require that $D$ be invertible, the $n$th equation holds if and only if (9) holds for $k = n$ if and only if $\det A = \det B$, which we henceforth assume for the multiplicative problem. As we shall see and as occurred in the multiplicative inverse eigenvalue problem, some regularity conditions on our coefficients are needed in order that there be a nonzero solution. We say that the pair $(A, B)$ is generic (for the multiplicative problem) if

$$\det (A) = \det (B) \neq 0$$

and

$$\det A[\alpha] \neq \det B[\alpha]$$

for each proper subset $\alpha$ of $N$. We often use multiplicatively generic for short and the second requirement is just that $c_\alpha \neq 0$. Notice that if $(A, B)$ is a multiplicatively generic pair, then so is $(EA, EB)$ for any invertible diagonal matrix $E$.

Similarly, if our problem has a solution for the given pair $(A, B)$ (regardless of assumptions), then it has also for $(EA, EB)$ when $E$ is an invertible diagonal matrix. Our principal multiplicative result is then the following. Recall that two elements of a vector space are projectively distinct if neither is a multiple of the other. Note that if $D$ is a solution to our problem so is $tD$, $0 \neq t \in C$.

**Theorem 2.1.** If $(A, B)$ is a multiplicatively generic pair of $n \times n$ complex matrices, then there is an invertible $n \times n$ complex diagonal matrix $D$ such that $AD$ and $BD$ are cospectral. Furthermore, for $n \geq 2$, there are at most (and almost always) $(n - 1)!$ projectively distinct such $D$’s.

**Proof.** Consider the systems

$$\sum_{|\alpha|=k} c_\alpha \prod_{i \in \alpha} X_i = 0, \quad k = 1, \ldots, n - 1 \quad (10)$$
and

\[
\begin{align*}
\sum_{|\alpha|=k} c_\alpha \prod_{i \in \alpha} X_i &= 0, \quad k = 1, \ldots, n - 1, \\
\quad (\det A)X_1 \cdots X_n &= 1.
\end{align*}
\]

(11)

If an \(n\)-tuple of complex numbers is a solution of (11), then it is a system of coordinates of a projective solution of (10). Conversely if an \(n\)-tuple of complex numbers is a system of coordinates of a projective solution of (10) and the product of its coordinates is \((\det A)^{-1}\), then it is a solution of (11).

Using Theorem 1.3 we conclude that there exists a solution of the system (11). We can also conclude, using the same type of argument used in [3], that the number of these solutions is finite, is at most \(n!\) and, almost always, is exactly \(n!\). The number of solutions of (11) in the same projective point of \(C^{n-1}\) is \(n\). Therefore, the number of distinct projective solutions of (10) is at most \((n - 1)!\) and almost always is \((n - 1)!\) \(\square\)

It may happen, however, even for a multiplicative generic pair, that there is no invertible diagonal \(D\) such that \(AD\) and \(BD\) are similar (even though they may be made cospectral).

**Example.** For \(n = 2\), let

\[
A = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
0 & 1 \\
-1 & 2
\end{bmatrix}.
\]

Then, the pair \((A, B)\) is multiplicatively generic and the projectively unique invertible solution guaranteed by Theorem 2.1 is \(D = I\). Then \(AD\) and \(BD\) have the common spectrum \((1, 1)\), but \(AD\) and \(BD\) are not similar. For larger \(n\), we may have fewer than \((n - 1)!\) solutions, none of which gives similarity. For \(n = 3\), let

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
0 & 1 & -1 \\
1 & 0 & 1 \\
2 & -2 & 3
\end{bmatrix}.
\]

Then direct calculation shows that the only projectively distinct solution (guaranteed by Theorem 2.1, as this pair is also multiplicatively generic) is again \(D = I\), but for the common spectrum \((1, 1, 1)\), \(BD\) has Jordan blocks of sizes 1 and 2, so that, in particular, \(AD\) and \(BD\) are not similar.

Under what broad circumstances then may we be sure of achieving similarity? If the spectrum common to \(AD\) and \(BD\) consists of distinct eigenvalues, then similarity is automatic. But, it is difficult to predict the possible common spectra, and, as the prior example shows, there may be none consisting of distinct eigenvalues; and it may be that one of \(AD\) and \(BD\) is diagonalizable, while the other is not. We consider the following property satisfied by “almost all” matrices. We call the \(n \times n\) complex
matrix $A$ strongly nonderogatory if for each diagonal matrix $D$, $A + D$ is nonderogatory (in the classical sense that the geometric multiplicity of all eigenvalues is 1). Recall that, in the example, the identity matrix $A$ in each pair is not (strongly) nonderogatory. However, many large classes of matrices naturally are. For example all irreducible Hessenberg matrices are.

**Corollary 2.2.** If $A$ and $B$ are a multiplicatively generic pair of strongly nonderogatory $n \times n$ complex matrices, then there is an invertible $n \times n$ diagonal complex matrix $D$ such that $AD$ is similar to $BD$. Again, there are at most (and almost always) $(n - 1)!$ projectively distinct such $D$'s.

**Proof.** By Theorem 2.1, there is an invertible diagonal $D$ such that $AD$ and $BD$ are cospectral. Suppose that one of $AD$ or $BD$ was not nonderogatory, say $AD$. Then there is an eigenvalue $\lambda$ of $AD$ such that $\text{rank}(AD - \lambda I) < n - 1$. But $AD - \lambda I = (A - \lambda D^{-1})D$, and because of the invertibility of $D$, $\text{rank}(A - \lambda D^{-1}) \leq n - 2$. Since $\lambda D^{-1}$ is diagonal, this contradicts the assumption that $A$ is strongly nonderogatory. We conclude that $AD$ and $BD$ are cospectral and both nonderogatory, from which it follows that they are similar. The remaining assertion also follows from Theorem 2.1. □

**Remark.** It is clear from the proof that any solution $D$ to the multiplicative cospectrality problem is also a solution to the multiplicative similarity problem when both $A$ and $B$ are strongly nonderogatory.

What may happen then, when $A$ and $B$ are not a multiplicatively generic pair? Of course $\det A = \det B$ remains necessary and we assume that this common value is nonzero and require that the common diagonal multiplier is invertible. The $2 \times 2$ case is straightforward to analyze. If $A[i] = B[i]$ for one value of $i$, then it must also be for the other, and, $DA$ and $DB$ are cospectral for all invertible diagonal $D$ and may be made similar at least by almost all such $D$. A complete analysis of the $3 \times 3$ case is more enlightening. In this case, we simply want to know when there exist $x_1, x_2, x_3 \neq 0$ such that

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$$

and

$$c_{12} x_1 x_2 + c_{13} x_1 x_3 + c_{23} x_2 x_3 = 0.$$  

Here, we use $c_i$ in place of $c_{[i]}$, etc. The case in which all $c$'s are nonzero is analyzed in Theorem 2.1, but we may also distinguish whether there are one or two (the generic case) projectively distinct solutions. Algebraically,

$$c_1 c_{13} x_1^2 + \alpha x_1 x_2 + c_2 c_{23} x_2^2 = 0$$

(14)
(in which \( \alpha = c_1c_{23} + c_2c_{13} - c_3c_{12} \)) follows from (12) and (13), and (12) and (14) are equivalent to (12) and (13), at least when all \( \{c_1, c_2, c_3\} \) are nonzero. Projectively there are then two solutions, unless
\[
\alpha^2 = 4c_1c_{23}c_2c_{13}
\]
or equivalently
\[
c_1^2c_{23}^2 + c_2^2c_{13}^2 + c_3^2c_{12}^2 = 2(c_1c_{23}c_2c_{13} + c_1c_{23}c_3c_{12} + c_2c_{13}c_3c_{12})
\]
(in which case there is one), and, in either event, they may be written down by setting \( x_1 = 1 \), solving the quadratic (14) for \( x_2 \) and solving (12) for \( x_3 \). If all of \( \{c_{12}, c_{13}, c_{23}\} \) are nonzero, the “solutions” will be totally nonzero (i.e. solutions in our sense). If exactly one of them is 0, then exactly one of the (one or two) “solutions” will have a component equal to 0. Thus, in case (15) holds, there will be no solutions in our sense.

There are several possibilities for subsets of the \( c \)'s to be 0. However, because of permutation similarity and Jacobi’s identity [2], which interchanges the roles of \( 1 \times 1 \) and \( 2 \times 2 \) principal minors in the \( 3 \times 3 \) case, many cases are qualitatively duplicated. For example, only one case in which exactly one of the \( c \)'s is 0 need be considered, and this has already been discussed above. If exactly two \( c \)'s of the same type (either \( \{c_1, c_2, c_3\} \) or \( \{c_{12}, c_{13}, c_{23}\} \)) are 0, then there is obviously no solution in our sense. If exactly two \( c \)'s are 0, then there are two interesting cases. Without loss of generality, either \( c_1 = c_{12} = 0 \) or \( c_1 = c_{23} = 0 \). In the former case, there is the unique (totally nonzero) projective solution: \( x_1 = 1 \), \( x_2 = -(c_{13}/c_{23}) \), \( x_3 = c_{2c13}/c_{3c23} \). In the latter, there are no (totally nonzero) solutions, unless the vectors \( (c_2, c_3) \) and \( (c_{12}, c_{13}) \) are linearly dependent, in which case there are (projectively) infinitely many. All remaining possibilities are easily analyzed. If exactly three of the \( c \)'s are 0, then either two are of the same type (and there are no solutions, as above) or they are all of the same type. If three of the same type are 0, there are (projectively) infinitely many solutions, unless exactly two of the other type are 0 (no solutions). Thus, exactly four \( c \)'s or five or six equal to 0 are also covered.

We note that if one of the \( c \)'s is 0 it is always possible that the remaining data be such that there exist no solutions (in contrast to the case in which all the \( c \)'s are nonzero). We also note that we have implicitly used the easily proven fact that if there are cospectrality (or similarity) solutions for the pair \((A, B)\), there are also for the pair \((A^{-1}, B^{-1})\) (and the solutions are the inverses of the \((A, B)\) solutions).

3. The additive problem

We next turn to the problem of cospectrality of \( A + D \) and \( B + D \). There are strong analogies to the multiplicative case, and despite some similarities to the additive inverse eigenvalue problem, there are more differences from it. Recall that in the classical complex additive inverse eigenvalue problem, given \( A \) and a spectrum \( \sigma \) a diagonal matrix \( D \) is sought so that \( \sigma(A + D) = \sigma \). For this problem, there are
no restrictions on $A$ and there is always a solution. There are restrictions on $A$ for the classical multiplicative analog (choose $D$ so that $\sigma(AD) = \sigma$ for given $A$ and $\sigma$), e.g., nonzero proper principal minors in $A$, rather like those in our case.

For cospectrality of $A + D$ and $B + D$ it is necessary that $\text{tr} \, A = \text{tr} \, B$ (analogous to the determinant in the multiplicative case). We do not require nonzero traces or that $D$ meet any additional requirement (unlike our multiplicative problem) besides being diagonal. Like the multiplicative problem (and unlike the classical additive problem), some requirements on the pair $(A, B)$ are needed. We replace our multiplicative generic pair assumption with an analogous additive generic pair assumption. We say that the pair $(A, B)$ is generic for the additive problem if

$$\text{tr} \, A = \text{tr} \, B$$

and

$$\text{tr} \, A[\alpha] \neq \text{tr} \, B[\alpha]$$

for all proper subsets $\alpha$ of $N$. Analogously, we use “additively generic” for short.

Our principal result for the additive problem is then analogous to the multiplicative case. Now the coefficients $\text{tr} \, A[\alpha] - \text{tr} \, B[\alpha]$ arise and we define this difference to be $\gamma_\alpha$.

For the additive problem, analogous to the multiplicative problem, if $A + D$ and $B + D$ are cospectral, then so are $A + (D + tI)$ and $B + (D + tI)$ for any complex $t$. We say that two $n \times n$ diagonal matrices are translationally distinct if they do not differ by a scalar matrix. Notice that if there is any solution to the additive problem for a given pair $(A, B)$, then there will be an invertible solution though we do not require this. Also, the notion “additively generic” is unchanged by common translations of $A$ and $B.$

**Theorem 3.1.** *If $(A, B)$ is an additively generic pair of $n \times n$ complex matrices, then there is an $n \times n$ complex diagonal matrix $D$ such that $A + D$ and $B + D$ are cospectral. Furthermore for $n > 1$ there are at most (and almost always) $(n - 1)!$ translationally distinct such $D$'s.***

**Proof.** Using elementary properties of the determinant we have

$$\det (\lambda I - (A + \text{diag}(X_1, \ldots, X_n)))$$

$$= \lambda^n + \sum_{t=0}^{n-1} (-1)^{n-t} \left( \sum_{\alpha \subseteq N, |\alpha| = n-t} \det ((A + \text{diag}(X_1, \ldots, X_n))[\alpha]) \right) \lambda^t$$

$$= \lambda^n + \sum_{t=0}^{n-1} (-1)^{n-t} \left( \sum_{\alpha \subseteq N, |\alpha| = n-t} \sum_{\ell=0}^{n-t} \sum_{\beta \subseteq \alpha, |\beta| = \ell} \det A[\alpha \setminus \beta] \prod_{j \in \beta} X_j \right) \lambda^t$$
\[
\begin{align*}
&= \lambda^n \sum_{t=0}^{n-1} (-1)^{n-t} \left( \sum_{\ell=0}^{n-t} \sum_{\beta \subseteq \alpha \subseteq N \atop |\beta|=\ell} \sum_{j \in \beta} \det A[\alpha \backslash \beta] \prod_{j \in \beta} X_j \right) \lambda^t.
\end{align*}
\]

Define
\[
\chi^A_{t,\ell}(X_1, \ldots, X_n)
= \sum_{\beta \subseteq N \atop |\beta|=\ell} \sum_{\beta \subseteq \alpha \subseteq N \atop |\alpha|=n-t} \det A[\alpha \backslash \beta] \prod_{j \in \beta} X_j, \quad t = 0, \ldots, n - 1, \quad \ell = 0, \ldots, n - t.
\]

Then
\[
\det(\lambda I - (A + \text{diag}(X_1, \ldots, X_n)))
= \lambda^n + \sum_{t=0}^{n-1} (-1)^{n-t} \left( \sum_{\ell=0}^{n-t} \chi^A_{t,\ell}(X_1, \ldots, X_n) \right) \lambda^t.
\]

It is also easy to see that
\begin{enumerate}
\item[(i)] \(\chi^A_{t,n-t}(X_1, \ldots, X_n) = s_{n-t}(X_1, \ldots, X_n)\),
\item[(ii)] \(\chi^A_{t,n-t-1}(X_1, \ldots, X_n) = \sum_{\beta \subseteq N \atop |\beta|=n-t-1} \Tr A[\beta] \prod_{j \in \beta} X_j\),
\item[(iii)] \(\deg \chi^A_{t,\ell} = \ell\),
\end{enumerate}

where \(\overline{\beta}\) denotes the complement of \(\beta\) with respect to \(N\).

Bearing in mind these properties we can prove the equivalence of the following equalities:
\[
\sigma(A + \text{diag}(X_1, \ldots, X_n)) = \sigma(B + \text{diag}(X_1, \ldots, X_n))
\]
\[
\Downarrow
\]
\[
\det(\lambda I - (A + \text{diag}(X_1, \ldots, X_n))) = \det(\lambda I - (B + \text{diag}(X_1, \ldots, X_n))
\]
\[
\Downarrow
\]
\[
\sum_{\ell=0}^{n-t} \chi^A_{t,\ell}(X_1, \ldots, X_n) = \chi^B_{t,\ell}(X_1, \ldots, X_n)
= 0,
\]
\[
\Downarrow
\]
\[
t = 0, \ldots, n - 1
\]
\[
\sum_{\beta \subseteq N \mid |\beta| = n - t - 1} (\text{tr } A^\beta \text{ tr } B[\beta]) \prod_{j \in \beta} X_j \\
+ \varphi_t(X_1, \ldots, X_n) = 0, \quad t = 0, \ldots, n - 2,
\]
where \(\deg \varphi_t < n - t - 1\).

The diagonal matrix \(D = \text{diag}(d_1, \ldots, d_n)\) satisfies
\[
\sigma(A + D) = \sigma(B + D)
\]
if and only if
\[
\sum_{\beta \subseteq N \mid |\beta| = n - t - 1} \gamma^\beta \prod_{j \in \beta} d_j + \varphi_t(d_1, \ldots, d_n) = 0, \quad t = 0, \ldots, n - 2.
\]

We also get from the above remark that if \(D\) is a solution of
\[
\sum_{\beta \subseteq N \mid |\beta| = n - t - 1} \gamma^\beta \prod_{j \in \beta} d_j + \varphi_t(d_1, \ldots, d_n) = 0, \quad t = 0, \ldots, n - 2,
\]
then \(D + \xi I\) is also a solution of this system of polynomial equations for every \(\xi \in \mathbb{C}\).

Therefore, using Theorem 1.3, we conclude that if \((A, B)\) is an additively generic pair, then the system
\[
(S_z) = \begin{cases} 
  s_n(X_1, \ldots, X_n) = z, \\
  \sum_{\beta \subseteq N \mid |\beta| = n - t - 1} \gamma^\beta \prod_{j \in \beta} X_j + \varphi_t(X_1, \ldots, X_n) = 0, \quad t = 0, \ldots, n - 2
\end{cases}
\]
is solvable and the number of solutions is less than or equal to \(n!\). Let \(z_1 \in \mathbb{C}\) and let \(\xi\) be a solution of the polynomial of \(\mathbb{C}[\lambda]\
\lambda^n + \lambda^{n-1}s_1(d_1, \ldots, d_n) + \cdots + s_{n-1}(d_1, \ldots, d_n)\lambda + z - z_1 = 0. \quad (17)
\]

It is now easy to see that \(D + \xi I\) is a solution of \((S_{z_1})\).

In the sequel we are going to denote by \(\mathcal{E}_n\) the subgroup of the additive group of \(M_n(\mathbb{C})\) consisting of the scalar matrices and by \(\mathcal{S}_z\) the set of solutions of \((S_z)\).

Let \(z_1, z_2 \in \mathbb{C}\). If \(D_1, D_2 \in \mathcal{S}_{z_1}\), then there exist \(\xi_1, \xi_2 \in \mathbb{C}\) such that \(E_1 = (D_1 + \xi_1 I)\) and \(E_2 = (D_2 + \xi_2 I)\) are elements of \(\mathcal{S}_{z_2}\). It is easy to see that \(D_1 \neq D_2\), mod \(\mathcal{E}_n\) if and only if \(E_1 \neq E_2\), mod \(\mathcal{E}_n\). Therefore
\[
\{D + \xi \mathcal{E}_n \mid (D + \xi \mathcal{E}_n) \cap \mathcal{S}_{z_1} \neq \emptyset\} = \{D + \xi \mathcal{E}_n \mid (D + \xi \mathcal{E}_n) \cap \mathcal{S}_{z_2} \neq \emptyset\} = \{D + \xi \mathcal{E}_n \mid \sigma(A + D) = \sigma(B + D)\}.
\]

Moreover, since there exists \(z \in \mathbb{C}\) such that Eq. (17) has \(n\) distinct roots we conclude from Theorem 1.3, that there are at most (and almost always) \((n - 1)!\) translationally
distinct such $D$’s such that $A + D$ and $B + D$ are cospectral. Recall that we assume that $(A, B)$ is an additively generic pair and use arguments similar to the ones used in [3]. □

Again it can happen that $A + D$ and $B + D$ may be made cospectral, but may not be similar, even when $A$ and $B$ are additively generic.

**Example.** Consider the additively generic pairs

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}
\]

and

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.
\]

In each case, direct calculation shows that the only solution to the additive problem is $D = 0$ (up to translation). But, in neither case is $A$ similar to $B$, though they are, of course, cospectral. These are the same pairs as in the multiplicative example. In view of the example it is not surprising that under the same broad circumstances, similarity of $A + D$ and $B + D$ may be achieved.

**Corollary 3.2.** If $A$ and $B$ are an additively generic pair of strongly nonderegotary $n \times n$ complex matrices, then there is an invertible $n \times n$ complex diagonal matrix $D$ such that $A + D$ is similar to $B + D$. Again there are at most (and almost always) $(n - 1)!$ translationally distinct such $D$’s.

**Proof.** Use Theorem 3.1 and recall that if $A + D$ and $B + D$ are nonderogatory then they are similar if and only if they have the same characteristic polynomial. □

**Remark.** Again any solution $D$ to the additive cospectrality problem is a solution to the additive similarity problem when $A$ and $B$ are strongly nonderogatory.

What may happen then, when $A$ and $B$ are not an additively generic pair? Of course, tr $A = \text{tr} B$ remains necessary, and under this assumption in the 2-by-2 case, if $a_{11} = b_{11}$, then $a_{22} = b_{22}$ also (and vice versa). In this event, we must have det $A = \text{det} B$ and, then, any $D$ is a solution to the additive cospectrality problem and in most cases all $D$’s will be a solution to the additive similarity problem.

**References**


