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On the fractional-order logistic equation

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Abstract

The topic of fractional calculus (derivatives and integrals of arbitrary orders) is enjoying growing interest not only among mathematicians, but also among physicists and engineers (see [E.M. El-Mesiry, A.M.A. El-Sayed, H.A.A. El-Saka, Numerical methods for multi-term fractional (arbitrary) orders differential equations, Appl. Math. Comput. 160 (3) (2005) 683–699; A.M.A. El-Sayed, Fractional differential–difference equations, J. Fract. Calc. 10 (1996) 101–106; A.M.A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, Nonlinear Anal. 33 (2) (1998) 181–186; A.M.A. El-Sayed, F.M. Gaafar, Fractional order differential equations with memory and fractional-order relaxation–oscillation model, (PU.M.A) Pure Math. Appl. 12 (2001); A.M.A. El-Sayed, E.M. El-Mesiry, H.A.A. El-Saka, Numerical solution for multi-term fractional (arbitrary) orders differential equations, Comput. Appl. Math. 23 (1) (2004) 33–54; A.M.A. El-Sayed, F.M. Gaafar, H.H. Hashem, On the maximal and minimal solutions of arbitrary orders nonlinear functional integral and differential equations, Math. Sci. Res. J. 8 (11) (2004) 336–348; R. Gorenflo, F. Mainardi, Fractional calculus: Integral and differential equations of fractional order, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, Wien, 1997, pp. 223–276; D. Matignon, Stability results for fractional differential equations with applications to control processing, in: Computational Engineering in System Application, vol. 2, Lille, France, 1996, p. 963; I. Podlubny, A.M.A. El-Sayed, On Two Definitions of Fractional Calculus, Solvak Academy of science-institute of experimental phys, ISBN: 80-7099-252-2, 1996. UEF-03-96; I. Podlubny, Fractional Differential Equations, Academic Press, 1999] for example). In this work we are concerned with the fractional-order logistic equation. We study here the stability, existence, uniqueness and numerical solution of the fractional-order logistic equation.

Keywords: Logistic equation; Fractional-order differential equations; Stability; Existence; Uniqueness; Numerical solution; Predictor–corrector method

1. Introduction

Now we give the definitions of fractional-order integration and fractional-order differentiation.

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Definition 1. The fractional integral of order $\beta \in R^+$ of the function $f(t),\ t > 0$ is defined by

$$I^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\,\mathrm{d}s \tag{1}$$

and the fractional derivative of order $\alpha \in (n-1, n)$ of $f(t), t > 0$ is defined by

$$\mathrm{D}^\alpha f(t) = I^{n-\alpha}\mathrm{D}^n f(t), \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}. \tag{2}$$

The following properties are some of the main ones of the fractional derivatives and integrals (see [10–19]).

Let $\beta, \gamma \in R^+$ and $\alpha \in (0, 1)$. Then we have

(i) $I_a^\beta : L^1 \to L^1$, and if $f(x) \in L^1$, then $I_a^\gamma I_a^\beta f(x) = I_a^{\gamma+\beta} f(x)$.
(ii) $\lim_{\beta\to n} I_a^\beta f(x) = I_a^n f(x)$ uniformly on $[a, b]$, $n = 1, 2, 3, \ldots$, where $I_a^1 f(x) = \int_a^x f(s)\,\mathrm{d}s$.
(iii) $\lim_{\beta\to 0} I_a^\beta f(x) = f(x)$ weakly.
(iv) If $f(x)$ is absolutely continuous on $[a, b]$, then $\lim_{\alpha\to 1} \mathrm{D}_a^\alpha f(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}$.
(v) If $f(x) = k \neq 0$, k is a constant, then $\mathrm{D}_a^\alpha k = 0$.

The following lemma can be easily proved (see [15]).

Lemma 1.1. *Let $\beta \in (0, 1)$; if $f \in C[0, T]$, then $I^\beta f(t)|_{t=0} = 0$.*

In Section 2 we will study the stability of the fractional-order differential equation and obtain results which agree with [2,3,17].

In Section 3 we will study the stability of the fractional-order logistic equation.

In Section 4 we will study the existence and uniqueness of the fractional-order logistic equation.

In Section 5 we will apply the PECE method for solving the fractional-order logistic equation.

2. Equilibrium and stability

Let $\alpha \in (0, 1]$ and consider the initial value problem

$$\mathrm{D}^\alpha x(t) = f(x(t)), \quad t > 0 \text{ and } x(0) = x_o. \tag{3}$$

To evaluate the equilibrium points of (3) let

$$\mathrm{D}^\alpha x(t) = 0,$$

then

$$f(x_{\mathrm{eq}}) = 0.$$

To evaluate the asymptotic stability, let

$$x(t) = x_{\mathrm{eq}} + \varepsilon(t),$$

then

$$\mathrm{D}^\alpha (x_{\mathrm{eq}} + \varepsilon) = f(x_{\mathrm{eq}} + \varepsilon)$$

which implies that

$$\mathrm{D}^\alpha \varepsilon(t) = f(x_{\mathrm{eq}} + \varepsilon)$$

but

$$f(x_{\mathrm{eq}} + \varepsilon) \simeq f(x_{\mathrm{eq}}) + f'(x_{\mathrm{eq}})\varepsilon + \cdots \Rightarrow$$
$$f(x_{\mathrm{eq}} + \varepsilon) \simeq f'(x_{\mathrm{eq}})\varepsilon$$

where $f(x_{\text{eq}}) = 0$, and then

$$\text{D}^{\alpha}\varepsilon(t) \simeq f'(x_{\text{eq}})\varepsilon(t), \quad t > 0 \text{ and } \varepsilon(0) = x_o - x_{\text{eq}}. \tag{4}$$

Now let the solution $\varepsilon(t)$ of (4) exist. So if $\varepsilon(t)$ is increasing, then the equilibrium point x_{eq} is unstable and if $\varepsilon(t)$ is decreasing, then the equilibrium point x_{eq} is locally asymptotically stable.

It must be noted that these results are the same results as for studying the stability of the initial value problem of the ordinary differential equation

$$\frac{\text{d}}{\text{d}t}x(t) = f(x(t)), \quad t > 0 \text{ and } x(0) = x_o.$$

Also these results agree with [2,3,17].

3. Fractional-order logistic equation

Now we study the equilibrium and stability of the fractional-order logistic equation.

Let $\alpha \in (0, 1]$, $\rho > 0$ and $x_o > 0$; the initial value problem of the fractional-order logistic equation is given by

$$\text{D}^{\alpha}x(t) = \rho\, x(t)(1 - x(t)), \quad t > 0 \text{ and } x(0) = x_0, \tag{5}$$

and to evaluate the equilibrium points, let

$$\text{D}^{\alpha}x(t) = 0;$$

then $x = 0, 1$ are the equilibrium points.

Now, to study the stability of the equilibrium points, we have (see Section 2)

$$f'(x(t)) = \rho(1 - 2x(t)) \Rightarrow f'(0) = \rho \quad \text{and} \quad f'(1) = -\rho.$$

Now the solution of the initial value problem

$$\text{D}^{\alpha}\varepsilon(t) = f'(x_{\text{eq}} = 0)\varepsilon(t) = \rho\, \varepsilon(t), \quad t > 0 \text{ and } \varepsilon(0) = x_o$$

is given by (see [11])

$$\varepsilon(t) = \sum_{n=0}^{\infty} \frac{\rho^n t^{n\alpha}}{\Gamma(n\alpha + 1)} x_o \tag{6}$$

and then the equilibrium point $x = 0$ is unstable.

Also for the equilibrium point $x = 1$ we have the initial value problem

$$\text{D}^{\alpha}\varepsilon(t) = f'(x_{\text{eq}} = 1)\varepsilon(t) = -\rho\, \varepsilon(t), \quad t > 0 \text{ and } \varepsilon(0) = x_o - 1$$

which is (if $x_o > 1$) the fractional-order relaxation equation and has the solution [13]

$$\varepsilon(t) = \sum_{n=0}^{\infty} \frac{(-\rho)^n\, t^{n\alpha}}{\Gamma(n\alpha + 1)} (x_o - 1) \tag{7}$$

and then the equilibrium point $x = 1$ is asymptotically stable.

4. Existence and uniqueness

Let $I = [0, T]$, $T < \infty$ and $C(I)$ be the class of all continuous functions defined on I, with norm

$$\|x\| = \sup_t |\text{e}^{-Nt}x(t)|, \quad N > 0 \tag{8}$$

which is equivalent to the sup-norm $\|x\| = \sup_t |x(t)|$. When $t > \sigma \geq 0$ we write $C(I_\sigma)$.

Consider the initial value problem of the fractional-order logistic equation (5).

Definition 2. We will define $x(t)$ to be a solution of the initial value problem (5) if
(1) $(t, x(t)) \in \mathrm{D},\ t \in I$ where $\mathrm{D} = I \times B,\ \ B = \{x \in R : |x| \leq b\}$.
(2) $x(t)$ satisfies (5).

Theorem 4.1. *The initial value problem* (5) *has a unique solution* $x \in C(I), x' \in X = \{x \in L_1[0, T], \|x\| = \|\mathrm{e}^{-Nt} x(t)\|_{L_1}\}$.

Proof. From the properties of fractional calculus the fractional-order differential equation in (5) can be written as

$$I^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} x(t) = \rho x(t)(1 - x(t)).$$

Operating with I^{α} we obtain

$$x(t) = x_o + I^{\alpha} \rho (x(t) - x^2(t)). \tag{9}$$

Now let the operator $F : C(I) \to C(I)$ be defined by

$$F x(t) = x_o + I^{\alpha}\ \rho (x(t) - x^2(t)). \tag{10}$$

Then

$$\begin{aligned} \mathrm{e}^{-Nt}(Fx - Fy) &= \rho\, \mathrm{e}^{-Nt}\ I^{\alpha}\ [(x(t) - y(t)) - (x^2(t) - y^2(t))] \\ &\leq \rho \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mathrm{e}^{-N(t-s)} (x(s) - y(s))(1 + (x(s) + y(s)))\mathrm{e}^{-Ns} \mathrm{d}s \\ &\leq \frac{(1+2b)\ \rho}{N^{\alpha}} \|x - y\| \int_0^t \frac{s^{\alpha-1}\mathrm{e}^{-Ns}}{\Gamma(\alpha)} \mathrm{d}s. \end{aligned}$$

This implies that

$$\|Fx - Fy\| \leq \|x - y\| \frac{(1+2b)\ \rho}{N^{\alpha}}$$

and it can be proved that if we choose N such that $N^{\alpha} > (1 + 2b)\ \rho$, we obtain

$$\|Fx - Fy\| < \ \|x - y\|$$

and the operator F given by (10) has a unique fixed point.

Consequently the integral equation (9) has a unique solution $x \in C(I)$. Also we can deduce that [15] $I^{\alpha}(x - x^2)|_{t=0} = 0$.

Now from Eq. (9) we formally have

$$x(t) = x_o + \rho \left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} (x_o - x_o^2) + I^{\alpha+1}\ (x'(t) - 2x(t)x'(t)) \right]$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \rho \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} (x_o - x_o^2) + I^{\alpha}(x'(t) - 2x(t)x'(t)) \right],$$

$$\mathrm{e}^{-Nt} x'(t) = \rho \mathrm{e}^{-Nt} \left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} (x_o - x_o^2) + I^{\alpha}(x'(t) - 2x(t)x'(t)) \right]$$

from which we can deduce that $x' \in C(I_{\sigma})$ and $x' \in X$.

Now from Eq. (9), we get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \rho \frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha}[x(t) - x^2(t)]$$

$$I^{1-\alpha} \frac{\mathrm{d}x}{\mathrm{d}t} = \rho\ I^{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} I^{\alpha}[x(t) - x^2(t)]$$

$$I^{1-\alpha} \frac{\mathrm{d}x}{\mathrm{d}t} = \rho\ \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} I^{\alpha}[x(t) - x^2(t)]$$

$$D^\alpha x(t) = \rho \frac{d}{dt} I[x(t) - x^2(t)]$$
$$D^\alpha x(t) = \rho\, [x(t) - x^2(t)]$$

and

$$x(0) = x_o + I^\alpha \rho\, [x(t) - x^2(t)]|_{t=0}$$
$$x(0) = x_o + 0$$
$$x(0) = x_o$$

Then the integral equation (9) is equivalent to the initial value problem (5) and the theorem is proved. □

5. Numerical methods and results

An Adams-type predictor–corrector method has been introduced in [4,5] and investigated further in [1,6–10,14]. In this work we use an Adams-type predictor–corrector method for the numerical solution of a fractional integral equation.

The key to the derivation of the method is replacing the original fractional differential equation in (3) by the fractional integral equation (11)

$$x(t) = x_o + I^\alpha f(x(t)). \tag{11}$$

The product trapezoidal quadrature formula is used with nodes t_j $(j = 0, 1, \ldots, k+1)$, taken with respect to the weight function $(t_{k+1} - .)^{\alpha-1}$. In other words, one applies the approximation

$$\int_{t_0}^{t_{k+1}} (t_{k+1} - u)^{\alpha-1} g(u) du \approx \int_{t_0}^{t_{k+1}} (t_{k+1} - u)^{\alpha-1} g_{k+1}(u) du$$
$$= \sum_{j=0}^{k+1} a_{j,k+1} g(t_j),$$

where

$$a_{j,k+1} = \begin{cases} \dfrac{h^\alpha}{\alpha(\alpha+1)}[k^{\alpha+1} - (k-\alpha)(k+1)^\alpha] & \text{if } j = 0, \\ \dfrac{h^\alpha}{\alpha(\alpha+1)} & \text{if } j = k+1 \end{cases}$$

and h is a step size, and for $1 \le j \le k$

$$a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha+1)}[(k-j+2)^{\alpha+1} - 2(k-j+1)^{\alpha+1} + (k-j)^{\alpha+1}].$$

This yields the corrector formula, i.e. the fractional variant of the one-step Adams–Moulton method

$$x_{k+1} = x_0 + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^{k} a_{j,k+1} f(x_j) + a_{k+1,k+1} f(x_{k+1}^p) \right). \tag{12}$$

The remaining problem is the determination of the predictor formula that is needed to calculate the value x_{k+1}^p. The idea used to generalize the one-step Adams–Bashforth method is the same as the one described above for the Adams–Moulton technique: the integral on the right-hand side of Eq. (11) is replaced by the product rectangle rule, i.e.

$$\int_{t_0}^{t_{k+1}} (t_{k+1} - u)^{\alpha-1} g(u) du \approx \sum_{j=0}^{k} b_{j,k+1} g(t_j),$$

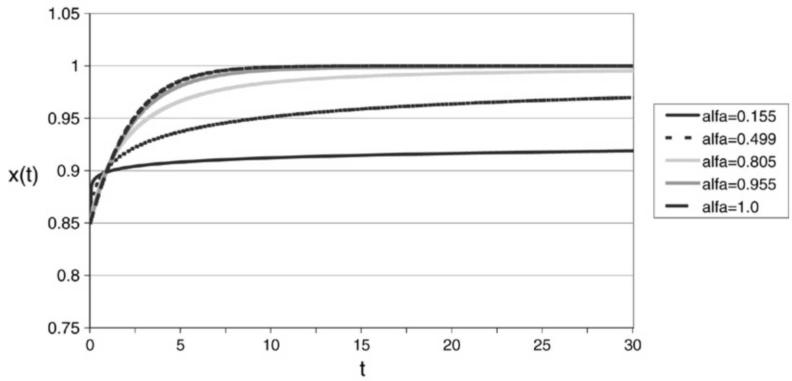

Fig. 1.

where

$$b_{j,k+1} = \frac{h^\alpha}{\alpha}[(k+1-j)^\alpha - (k-j)^\alpha].$$

Thus, the predictor x_{k+1}^p is determined by the fractional Adams–Bashforth method:

$$x_{k+1}^p = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(x_j). \tag{13}$$

This completes the description of the basic algorithm, namely, the fractional version of the one-step Adams–Bashforth Moulton method. Recapitulating, one first calculates the predictor x_{k+1}^p according to Eq. (13), then evaluates $f(x_{k+1}^p)$, uses this to determine the corrector x_{k+1} by means of Eq. (12), and finally evaluates $f(x_{k+1})$ which is then used in the next integration step. Methods of this type are usually called predictor–corrector or, more precisely, **PECE** (Predict, Evaluate, Correct, Evaluate) methods.

Now we apply the **PECE** method to the fractional logistic equation in (5).

The approximate solutions are displayed in Fig. 1 for the step size 0.05 and different values of α. In Fig. 1 we take $\rho = 0.5$ and $x_0 = 0.85$.

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