On the fractional-order logistic equation

A.M.A. El-Sayed\textsuperscript{a}, A.E.M. El-Mesiry\textsuperscript{b}, H.A.A. El-Saka\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a} Faculty of Science, Alexandria University, Alexandria, Egypt
\textsuperscript{b} Department of Mathematics, Damietta Faculty of Science, New Damietta, Egypt

Received 14 November 2005; received in revised form 5 June 2006; accepted 4 August 2006

Abstract


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Keywords: Logistic equation; Fractional-order differential equations; Stability; Existence; Uniqueness; Numerical solution; Predictor–corrector method

1. Introduction

Now we give the definitions of fractional-order integration and fractional-order differentiation.

\textsuperscript{*} Corresponding address: Department of Mathematics, Damietta Faculty of Science, Mansoura University, 34517 New Damietta, Egypt. Fax: +20 57 403868.

E-mail addresses: amasayed@maktoob.com (A.M.A. El-Sayed), elmisiery@hotmail.com (A.E.M. El-Mesiry), halaelsaka@yahoo.com (H.A.A. El-Saka).

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doi:10.1016/j.aml.2006.08.013
**Definition 1.** The fractional integral of order $\beta \in R^+$ of the function $f(t)$, $t > 0$ is defined by

$$I^\beta f(t) = \int_0^t (t - s)^{\beta - 1} \Gamma(\beta) f(s) \, ds$$

(1)

and the fractional derivative of order $\alpha \in (n - 1, n)$ of $f(t)$, $t > 0$ is defined by

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t), \quad D = \frac{d}{dt}.$$  

(2)

The following properties are some of the main ones of the fractional derivatives and integrals (see [10–19]).

Let $\beta, \gamma \in R^+$ and $\alpha \in (0, 1)$. Then we have

(i) $I^\beta_0 : L^1 \to L^1$, and if $f(x) \in L^1$, then $I^\gamma_0 I^\beta_0 f(x) = I^\gamma + \beta_0 f(x)$.

(ii) $\lim_{\beta \to n} I^\beta_0 f(x) \text{ uniformly on } [a, b], n = 1, 2, 3, \ldots$, where $I^1_0 f(x) = \int_a^x f(s) \, ds$.

(iii) $\lim_{\beta \to 0} I^\beta_0 f(x) \text{ weakly}$.

(iv) If $f(x)$ is absolutely continuous on $[a, b]$, then $\lim_{\alpha \to 1} D^\alpha_0 f(x) = \frac{df(x)}{dx}$.

(v) If $f(x) = k \neq 0$, $k$ is a constant, then $D^\alpha_0 k = 0$.

The following lemma can be easily proved (see [15]).

**Lemma 1.1.** Let $\beta \in (0, 1)$; if $f \in C[0, T]$, then $I^\beta f(t)|_{t=0} = 0$.

In Section 2 we will study the stability of the fractional-order differential equation and obtain results which agree with [2,3,17].

In Section 3 we will study the stability of the fractional-order logistic equation.

In Section 4 we will study the existence and uniqueness of the fractional-order logistic equation.

In Section 5 we will apply the PECE method for solving the fractional-order logistic equation.

2. Equilibrium and stability

Let $\alpha \in (0, 1]$ and consider the initial value problem

$$D^\alpha x(t) = f(x(t)), \quad t > 0 \text{ and } x(0) = x_0.$$  

(3)

To evaluate the equilibrium points of (3) let

$$D^\alpha x(t) = 0,$$

then

$$f(x_{eq}) = 0.$$  

To evaluate the asymptotic stability, let

$$x(t) = x_{eq} + \varepsilon(t),$$

then

$$D^\alpha (x_{eq} + \varepsilon) = f(x_{eq} + \varepsilon)$$

which implies that

$$D^\alpha \varepsilon(t) = f(x_{eq} + \varepsilon)$$

but

$$f(x_{eq} + \varepsilon) \simeq f(x_{eq}) + f'(x_{eq}) \varepsilon + \cdots \Rightarrow$$

$$f(x_{eq} + \varepsilon) \simeq f'(x_{eq}) \varepsilon$$
where \( f(x_{eq}) = 0 \), and then
\[
D^\alpha \varepsilon(t) \simeq f'(x_{eq}) \varepsilon(t), \quad t > 0 \text{ and } \varepsilon(0) = x_o - x_{eq}.
\] (4)

Now let the solution \( \varepsilon(t) \) of (4) exist. So if \( \varepsilon(t) \) is increasing, then the equilibrium point \( x_{eq} \) is unstable and if \( \varepsilon(t) \) is decreasing, then the equilibrium point \( x_{eq} \) is locally asymptotically stable.

It must be noted that these results are the same results as for studying the stability of the initial value problem of the ordinary differential equation
\[
\frac{dx}{dt} = f(x(t)), \quad t > 0 \text{ and } x(0) = x_o.
\]
Also these results agree with [2,3,17].

3. Fractional-order logistic equation

Now we study the equilibrium and stability of the fractional-order logistic equation.

Let \( \alpha \in (0, 1] \), \( \rho > 0 \) and \( x_o > 0 \); the initial value problem of the fractional-order logistic equation is given by
\[
D^\alpha x(t) = \rho x(t)(1 - x(t)), \quad t > 0 \text{ and } x(0) = x_o,
\] (5)
and to evaluate the equilibrium points, let
\[
D^\alpha x(t) = 0;
\]
then \( x = 0, 1 \) are the equilibrium points.

Now, to study the stability of the equilibrium points, we have (see Section 2)
\[
f'(x(t)) = \rho (1 - 2x(t)) \Rightarrow f'(0) = \rho \text{ and } f'(1) = -\rho.
\]

Now the solution of the initial value problem
\[
D^\alpha \varepsilon(t) = f'(x_{eq} = 0) \varepsilon(t) = \rho \varepsilon(t), \quad t > 0 \text{ and } \varepsilon(0) = x_o
\]
is given by (see [11])
\[
\varepsilon(t) = \sum_{n=0}^{\infty} \frac{\rho^n \Gamma(n\alpha)}{\Gamma(n\alpha + 1)} x_o
\] (6)
and then the equilibrium point \( x = 0 \) is unstable.

Also for the equilibrium point \( x = 1 \) we have the initial value problem
\[
D^\alpha \varepsilon(t) = f'(x_{eq} = 1) \varepsilon(t) = -\rho \varepsilon(t), \quad t > 0 \text{ and } \varepsilon(0) = x_o - 1
\]
which is (if \( x_o > 1 \)) the fractional-order relaxation equation and has the solution [13]
\[
\varepsilon(t) = \sum_{n=0}^{\infty} \frac{(-\rho)^n \Gamma(n\alpha)}{\Gamma(n\alpha + 1)} (x_o - 1)
\] (7)
and then the equilibrium point \( x = 1 \) is asymptotically stable.

4. Existence and uniqueness

Let \( I = [0, T], \ T < \infty \) and \( C(I) \) be the class of all continuous functions defined on \( I \), with norm
\[
\|x\| = \sup_{t} |e^{-Nt} x(t)|, \quad N > 0
\] (8)
which is equivalent to the sup-norm \( \|x\| = \sup_{t} |x(t)| \). When \( t > \sigma \geq 0 \) we write \( C(I_{\sigma}) \).

Consider the initial value problem of the fractional-order logistic equation (5).
Definition 2. We will define \( x(t) \) to be a solution of the initial value problem (5) if

1. \((t, x(t)) \in D, \ t \in I \) where \( D = I \times B, \ B = \{ x \in R : |x| \leq b \} \).
2. \( x(t) \) satisfies (5).

Theorem 4.1. The initial value problem (5) has a unique solution \( x \in C(I), x' \in X = \{ x \in L_1[0, T], \|x\| = \|e^{-Nt}x(t)\|_{L_1} \} \).

Proof. From the properties of fractional calculus the fractional-order differential equation in (5) can be written as

\[
I^{1-a} \frac{d}{dt} x(t) = \rho x(t)(1 - x(t)).
\]

Operating with \( I^\alpha \) we obtain

\[
x(t) = x_o + I^\alpha \rho (x(t) - x^2(t)).
\]

(9)

Now let the operator \( F : C(I) \rightarrow C(I) \) be defined by

\[
Fx(t) = x_o + I^\alpha \rho (x(t) - x^2(t)).
\]

(10)

Then

\[
e^{-Nt}(Fx - Fy) = \rho e^{-Nt} I^\alpha [(x(t) - y(t)) - (x^2(t) - y^2(t))]
\]

\[
\leq \rho \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} e^{-N(t-s)}(x(s) - y(s))(1 + (x(s) + y(s)))e^{-Ns} ds
\]

\[
\leq \frac{(1 + 2b) \rho}{N^\alpha} \|x - y\| \int_0^t s^{\alpha - 1} e^{-Ns} ds.
\]

This implies that

\[
\|Fx - Fy\| \leq \|x - y\| \frac{(1 + 2b) \rho}{N^\alpha}
\]

and it can be proved that if we choose \( N \) such that \( N^\alpha > (1 + 2b) \rho \), we obtain

\[
\|Fx - Fy\| < \|x - y\|
\]

and the operator \( F \) given by (10) has a unique fixed point.

Consequently the integral equation (9) has a unique solution \( x \in C(I) \). Also we can deduce that [15] \( I^\alpha (x - x^2) \big|_{t=0} = 0 \).

Now from Eq. (9) we formally have

\[
x(t) = x_o + \rho \left[ \frac{t^\alpha}{\Gamma(\alpha + 1)} (x_o - x_o^2) + I^{\alpha + 1} (x'(t) - 2x(t)x'(t)) \right]
\]

and

\[
\frac{dx}{dt} = \rho \left[ \frac{t^{\alpha - 1}}{\Gamma(\alpha)} (x_o - x_o^2) + I^\alpha (x'(t) - 2x(t)x'(t)) \right],
\]

\[
e^{-Nt} x'(t) = \rho e^{-Nt} \left[ \frac{t^{\alpha - 1}}{\Gamma(\alpha)} (x_o - x_o^2) + I^\alpha (x'(t) - 2x(t)x'(t)) \right]
\]

from which we can deduce that \( x' \in C(I_o) \) and \( x' \in X \).

Now from Eq. (9), we get

\[
\frac{dx}{dt} = \rho \frac{d}{dt} I^\alpha [x(t) - x^2(t)]
\]

\[
I^{1-\alpha} \frac{dx}{dt} = \rho \ I^{1-\alpha} \frac{d}{dt} I^\alpha [x(t) - x^2(t)]
\]

\[
I^{1-\alpha} \frac{dx}{dt} = \rho \ \frac{d}{dt} I^{1-\alpha} I^\alpha [x(t) - x^2(t)]
\]
\[ D^\alpha x(t) = \rho \frac{d}{dt} I[x(t) - x^2(t)] \]
\[ D^\alpha x(t) = \rho [x(t) - x^2(t)] \]

and

\[ x(0) = x_0 + I^\alpha [x(t) - x^2(t)] \]
\[ x(0) = x_0 + 0 \]
\[ x(0) = x_0 \]

Then the integral equation (9) is equivalent to the initial value problem (5) and the theorem is proved. □

5. Numerical methods and results

An Adams-type predictor–corrector method has been introduced in [4,5] and investigated further in [1,6–10,14]. In this work we use an Adams-type predictor–corrector method for the numerical solution of a fractional integral equation.

The key to the derivation of the method is replacing the original fractional differential equation in (3) by the fractional integral equation (11)

\[ x(t) = x_0 + I^\alpha f(x(t)). \] (11)

The product trapezoidal quadrature formula is used with nodes \(t_j\) (\(j = 0, 1, \ldots, k + 1\)), taken with respect to the weight function \((tk+1 - .)^{\alpha-1}\). In other words, one applies the approximation

\[ \int_{t_0}^{tk+1} (tk+1 - u)^{\alpha-1} g(u)du \approx \sum_{j=0}^{k+1} a_{j,k+1} g(t_j), \]

where

\[ a_{j,k+1} = \begin{cases} \frac{h^\alpha}{\alpha(\alpha + 1)} \left[ k^{\alpha+1} - (k - \alpha)(k + 1)^\alpha \right] & \text{if } j = 0, \\ \frac{h^\alpha}{\alpha(\alpha + 1)} & \text{if } j = k + 1 \end{cases} \]

and \(h\) is a step size, and for \(1 \leq j \leq k\)

\[ a_{j,k+1} = \frac{h^\alpha}{\alpha(\alpha + 1)} [(k - j + 2)^{\alpha+1} - 2(k - j + 1)^{\alpha+1} + (k - j)^{\alpha+1}]. \]

This yields the corrector formula, i.e. the fractional variant of the one-step Adams–Moulton method

\[ x_{k+1} = x_0 + \frac{1}{I(\alpha)} \left( \sum_{j=0}^{k} a_{j,k+1} f(x_j) + a_{k+1,k+1} f(x_{k+1}^p) \right). \] (12)

The remaining problem is the determination of the predictor formula that is needed to calculate the value \(x_{k+1}^p\). The idea used to generalize the one-step Adams–Bashforth method is the same as the one described above for the Adams–Moulton technique: the integral on the right-hand side of Eq. (11) is replaced by the product rectangle rule, i.e.

\[ \int_{t_0}^{tk+1} (tk+1 - u)^{\alpha-1} g(u)du \approx \sum_{j=0}^{k} p_{j,k+1} g(t_j), \]
where

\[ b_{j,k+1} = \frac{h^\alpha}{\alpha} \left[ (k+1-j)^\alpha - (k-j)^\alpha \right]. \]

Thus, the predictor \( x_{k+1}^p \) is determined by the fractional Adams–Bashforth method:

\[ x_{k+1}^p = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(x_j). \] (13)

This completes the description of the basic algorithm, namely, the fractional version of the one-step Adams–Bashforth Moulton method. Recapitulating, one first calculates the predictor \( x_{k+1}^p \) according to Eq. (13), then evaluates \( f(x_{k+1}^p) \), uses this to determine the corrector \( x_{k+1} \) by means of Eq. (12), and finally evaluates \( f(x_{k+1}) \) which is then used in the next integration step. Methods of this type are usually called predictor–corrector or, more precisely, PECE (Predict, Evaluate, Correct, Evaluate) methods.

Now we apply the PECE method to the fractional logistic equation in (5).

The approximate solutions are displayed in Fig. 1 for the step size 0.05 and different values of \( \alpha \). In Fig. 1 we take \( \rho = 0.5 \) and \( x_0 = 0.85 \).

References