JOURNAL OF COMBINATORIAL THEORY (B) 16, 139-144 (1974)

Bounding the Diameter of a Distance-Transitive Graph

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Communicated by W. T. Tutte

Received April 5, 1973

A graph Γ is distance-transitive if for all vertices u, v, x, y such that $d(u, v) = d(x, y)$ there is an automorphism h of Γ such that $uh = x$, $vh = y$. We show how to find a bound for the diameter of a bipartite distance-transitive graph given a bound for the order $|G_\alpha|$ of the stabilizer of a vertex.

In [10] Tutte initiated a study of s-transitive graphs by proving that for an s-transitive trivalent graph $s \leqslant 5$. Sims [4] generalized Tutte's result to find a bound for the order $|G_\alpha|$ of the stabilizer of a primitive permutation group with a suborbit of length 3. Sims, Thompson, and Quirin [5, 3] dealt with the case of a suborbit of length 4. Gardiner [2] gives bounds for $|G_\alpha|$ for an s-transitive graph of valency $p + 1$ (p prime).

A natural question is to ask when it is possible to convert a bound for $|G_\alpha|$ into a bound for the diameter of a graph. In fact for s-transitive graphs it is possible to show that for a particular valency and value of s there can be infinitely many s-transitive graphs and so no bound for the diameter in terms of $|G_\alpha|$ is possible. We work in the more restricted class of distance-transitive graphs. In the case of valency 2 the distance-transitive graphs are just circuits and no bound for the diameter is possible. In the case of valencies 3 and 4, bounds for the diameter were found in [1], [6], [7], and [8] together with a complete list of distance-transitive graphs of valencies 3 and 4.

We consider distance-transitive graphs of valency greater than 4 and assume that a bound for $|G_\alpha|$ exists. The purpose of this paper is to show how to find a bound for the diameter of a bipartite distance-transitive graph of valency >4 given a bound for $|G_\alpha|$. Notice that if we attempt to extend the result to any distance-transitive graph we can assume that the automorphism group acts primitively on the vertices, since if it acts imprimitively the graph is either bipartite or antipodal [9, Theorem 2]. If it is antipodal but not bipartite the derived graph is primitive and a bound for the diameter in the primitive case would imply a bound for the diameter in the antipodal case [9, Theorem 3].

DEFINITION. Γ is a distance-transitive graph if for all vertices u, v, x, y such that $d(u, v) = d(x, y)$ there is an automorphism h of Γ such that $uh = x$, $vh = y$.

DEFINITION. Let Γ be a distance-transitive graph and for any vertex u let $\Gamma_i(u) = \{v \mid d(u,v) = i\}$. We define the intersection array of Γ by

$$P(\Gamma) = \begin{Bmatrix} * & c_1 & c_2 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ k & b_1 & b_2 & \cdots & b_i & \cdots & b_{d-1} & * \end{Bmatrix}$$

where d is the diameter of the graph, k is the valency and if u and v are vertices such that $d(u, v) = i$ then

$$\begin{aligned} c_i &= | \Gamma_{i-1}(u) \cap \Gamma_1(v)|, \\ a_i &= | \Gamma_i(u) \cap \Gamma_1(v)|, \\ b_i &= | \Gamma_{i+1}(u) \cap \Gamma_1(v)|. \end{aligned}$$

These numbers are independent of the choices of u and v. Clearly $c_1 = 1$, $c_i + a_i + b_i = k$. It was shown in [9] that $1 \leqslant c_2 \leqslant c_3 \leqslant \cdots \leqslant c_d$ and $k \geqslant b_1 \geqslant b_2 \geqslant \cdots \geqslant b_{d-1}$. We write $k_i = | \Gamma_i(u)|$. A simple counting argument shows $k_i b_i = k_{i+1} c_{i+1}$.

Now suppose Γ to be bipartite and the intersection array to be

$$\begin{Bmatrix} * & 1 & c_2 & c_3 & \cdot & \cdot & c_{s_1} & \frac{k}{2} & \frac{k}{2} & \cdot & \frac{k}{2} & c_{s_2} & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot \\ k & k-1 & b_2 & b_3 & \cdot & \cdot & b_{s_1} & \frac{k}{2} & \frac{k}{2} & \cdot & \frac{k}{2} & b_{s_2} & \cdot & * \end{Bmatrix} \begin{matrix} \left(c_{s_2} > \frac{k}{2}\right) \\ (k > 4) \end{matrix}$$

We show that $s_2 \leqslant 3s_1 + 2$ from which it follows that the diameter is bounded since $k_{s_2-1} > k_{s_2} > k_{s_2+1} > \cdots$ and the bound on $| G_\alpha |$ implies a bound on s_1. Assume $s_2 > 3s_1 + 2$.

LEMMA 1. *If $t = 2s_1 + 1$ and N is the number of vertices at distance $\leqslant t$ from any chosen vertex u,*

$$N > \frac{(2t+1)}{4} k_t .$$

Proof. Application of $k_i b_i = k_{i+1} c_{i+1}$ shows $k_{s_1+1} = k_{s_1+2} = \cdots = k_t$. Then

$$t - s_1 = s_1 + 1 = \frac{t+1}{2} > \frac{2t+1}{4}$$

and

$$k_0 + k_1 + k_2 + \cdots + k_t > k_{s_1+1} + k_{s_1+2} + \cdots + k_t$$
$$> k_t \cdot \frac{(2t+1)}{4}. \quad \blacksquare$$

LEMMA 2. *Let $2s_1 + 1 \leqslant q \leqslant s_2 - 1$ and choose x_1, x_2 such that $d(x_1, x_2) = q$. Suppose vertex y in $\{\Gamma_i(x_1) \cap \Gamma_{q-i}(x_2)\}$ is joined to $b_i'(y)$ vertices of $\{\Gamma_{i+1}(x_1) \cap \Gamma_{q-i-1}(x_2)\}$ $(i = 0, 1,..., q-1)$ and to $c_i'(y)$ vertices of $\{\Gamma_{i-1}(x_1) \cap \Gamma_{q-i+1}(x_2)\}$ $(i = 1, 2,..., q)$. Then*

(1) $$c_i'(y) = \begin{cases} \dfrac{k}{2} & (i = s_1 + 1,..., q) \\ c_i & (i = 1, 2,..., s_1) \end{cases}$$

(2) $$b_i'(y) = \begin{cases} \dfrac{k}{2} & (i = 0, 1,..., q - s_1 - 1) \\ c_{q-i} & (i = q - s_1,..., q-1) \end{cases}$$

(3) $$|\Gamma_i(x_1) \cap \Gamma_{q-i}(x_2)| = \frac{\left(\frac{k}{2}\right)^i}{c_1 c_2 \cdots c_i} \quad (1 \leqslant i \leqslant s_1)$$
$$= \frac{\left(\frac{k}{2}\right)^{s_1}}{c_1 c_2 \cdots c_{s_1}} \quad (s_1 + 1 \leqslant i \leqslant q)$$
$$= \frac{\left(\frac{k}{2}\right)^{q-i}}{c_1 c_2 \cdots c_{q-i}} \quad (q - s_1 \leqslant i \leqslant q)$$

Proof. (1) y is joined to c_i vertices z of $\Gamma_{i-1}(x_i)$ and $d(x_2, z) \leqslant d(x_2, y) + 1 \leqslant q - i + 1$. Since $x_2 \in \Gamma_q(x_1)$ and $z \in \Gamma_{i-1}(x_1)$, $d(x_2, z) \geqslant q - i + 1$. Hence $z \in \Gamma_{i-1}(x_1) \cap \Gamma_{q-i+1}(x_2)$ and so $c_i' = c_i$ $(i = 1, 2,..., q)$.

(2) y is joined to c_{q-i} vertices z' of $\Gamma_{q-i-1}(x_2)$ and

$$d(x_1, z') \leqslant d(x_1, y) + d(y, z') \leqslant i + 1.$$

Since $x_1 \in \Gamma_q(x_2)$, $z' \in \Gamma_{q-i-1}(x_2)$, $d(x_1, z') \geqslant i + 1$. Hence $z' \in \Gamma_{i+1}(x_1) \cap \Gamma_{q-i-1}(x_2)$ so $b_i'(y) = c_{q-i}$ $(i = 0, 1,..., q-1)$.

(3) This follows from repeated application of

$$|\Gamma_i(x_1) \cap \Gamma_{q-i}(x_2)|\, b_i' = |\Gamma_{i+1}(x_1) \cap \Gamma_{q-i-1}(x_2)|\, c_{i+1}' . \quad \blacksquare$$

Now let $q = 2s_1 + 1$, choose x_0 and choose $x_1 \in \Gamma_1(x_0)$, $x_2 \in \Gamma_{q+1}(x_0) \cap \Gamma_q(x_1)$, $x_3 \in \Gamma_{q+1}(x_1) \cap \Gamma_1(x_2) \cap \Gamma_{q+2}(x_0)$. Let $W_1 = \Gamma_{s_1}(x_1) \cap \Gamma_{q-s_1}(x_2)$ and $W_2 = \Gamma_{s_1+1}(x_1) \cap \Gamma_{q-s_1-1}(x_2)$. Then since $b_{s_1}' = k/2 = c_{s_1+1}'$, the vertices of $W_1 \cup W_2$ together with the edges joining them form a (not necessarily connected) bipartite regular graph B of valency $k/2$.

LEMMA 3. $\Gamma_{s_1}(x_1) \cap \Gamma_{q-s_1}(x_2) = \Gamma_{s_1+1}(x_0) \cap \Gamma_{q-s_1}(x_2)$.

Proof. From Lemma 2 $|\Gamma_{s_1}(x_1) \cap \Gamma_{q-s_1}(x_2)| = |\Gamma_{s_1+1}(x_0) \cap \Gamma_{q-s_1}(x_2)|$. Since $\Gamma_{s_1}(x_1) \cap \Gamma_{q-s_1}(x_2) \subset \Gamma_{s_1+1}(x_0) \cap \Gamma_{q-s_1}(x_2)$ the result follows. $\blacksquare$

LEMMA 4. $\Gamma_{s_1+1}(x_1) \cap \Gamma_{q-s_1}(x_3) = \Gamma_{s_1+1}(x_1) \cap \Gamma_{q-s_1-1}(x_2)$.

Proof. From Lemma 2

$$|\Gamma_{s_1+1}(x_1) \cap \Gamma_{q-s_1}(x_3)| = |\Gamma_{s_1+1}(x_1) \cap \Gamma_{q-s_1-1}(x_2)| .$$

Since $\Gamma_{s_1+1}(x_1) \cap \Gamma_{q-s_1}(x_3) \supset \Gamma_{s_1+1}(x_1) \cap \Gamma_{q-s_1-1}(x_2)$ the result follows. $\blacksquare$

Now choose an automorphism h such that $x_0h = x_1$, $x_2h = x_3$. Then

$$\begin{aligned} W_1h &= (\Gamma_{s_1}(x_1) \cap \Gamma_{q-s_1}(x_2))\, h \\ &= (\Gamma_{s_1+1}(x_0) \cap \Gamma_{q-s_1}(x_2))\, h \quad \text{(Lemma 3)} \\ &= \Gamma_{s_1+1}(x_1) \cap \Gamma_{q-s_1}(x_3) \\ &= \Gamma_{s_1+1}(x_1) \cap \Gamma_{q-s_1-1}(x_2) \quad \text{(Lemma 4)} \\ &= W_2 . \end{aligned}$$

Also, since $W_2h \subset \Gamma_{q-s_1-1}(x_3)$, $W_1 \subset \Gamma_{s_1}(x_1)$ and $d(x_1, x_3) = q + 1$, $W_1 \cap W_2h = \phi$. Hence W_2 consists of half of the vertices of the graph B, and also it consists of half of the vertices of a graph Bh, isomorphic to B, with vertices $W_2 \cup W_2h$.

We now have vertices x_1, $x_1h \in \Gamma_1(x_1)$, $x_3 \in \Gamma_{q+1}(x_1) \cap \Gamma_q(x_1h)$, $x_3h \in \Gamma_{q+1}(x_1h) \cap \Gamma_1(x_3) \cap \Gamma_{q+2}(x_1)$ with $W_2 = \Gamma_{s_1}(x_1h) \cap \Gamma_{q-s_1}(x_3)$, $W_2h = \Gamma_{s_1+1}(x_1h) \cap \Gamma_{q-s_1-1}(x_3)$. Repeating the process we find that W_2h, W_2h^2 are the vertices of a graph Bh^2 isomorphic to B. Hence we obtain in this way a sequence of graphs B, Bh, Bh^2, Bh^3,... with Bh^i having vertices $\{W_2h^{i-1} \cup W_2h^i\}$. Since the graph is finite and since we have already found all edges adjacent to W_2, W_2h,... the only possibility is that there exists vertex $v \in W_2h^j \cap W_1$ for some $j > 0$. Suppose v is chosen so that j is as small as possible. We show that in this case $W_2h^j = W_1$.

Since $W_1 \subset \Gamma_{s_1}(x_1)$, $W_2 \subset \Gamma_{s_1+1}(x_1)$ it can be seen from the intersection array that $W_2h \subset \Gamma_{s_1+2}(x_1)$, $W_2h^2 \subset \Gamma_{s_1+3}(x_1), \ldots, W_2h^{s_2-s_1-1} \subset \Gamma_{s_2}(x_1)$. Thus if $w_1 \in W_2h^{s_2-s_1-1}$, $w_2 \in W_1$ $d(w_1, w_2) \geqslant s_2 - s_1$. Hence $j + 1 \geqslant 2(s_2 - s_1) > 2(2s_1 + 2) = 2q + 2$ so $j > 2q + 1$. Then since $(j - s_1) - (q - s_1) > q$ we can choose $Z_1 \in W_2h^{j-s_1}$, $Z_2 \in W_1h^{q-s_1}$ such that $d(Z_1, v) = s$, $d(Z_2, v) = q - s_1$, $d(Z_1, Z_2) = q$. Then $W_1 \subset \Gamma_{q-s_1}(Z_2)$, $W_2h^j \subset \Gamma_{s_1}(Z_1)$. Let $v' \in \Gamma_{s_1}(Z_1) \cap \Gamma_{q-s_1}(Z_2)$ so $v' \in W_2h^j \cup W_2h^{j-1} \cup \cdots \cup W_2h^{j-2s_1}$ and $v' \in W_1 \cup W_2 \cup W_2h \cup \cdots \cup W_2h^q$. Since $j > 2q + 1$ these sets are disjoint except for vertices of $W_1 \cap W_2h^j$. It then follows from

$$|\Gamma_{s_1}(Z_1) \cap \Gamma_{q-s_1}(Z_2)| = |W_1| = |W_2h^j| \text{ that } \Gamma_{s_1}(Z_1) \cap \Gamma_{q-s_1}(Z_2)$$

$$= W_1 = W_2h^j .$$

Hence we have seen that the graph consists of a "ring" of copies of the graph B. We are now ready to obtain a contradiction. Suppose

(1) $b_2 > k/2$ so $b_2 \geqslant (k/2) + 1$. Let $p = 2s_1 + 1$. The number M of vertices at distance $\leqslant p$ from any given vertex is given by

$$M \leqslant (2p + 1) |W_1| = \frac{(2p+1)\left(\frac{k}{2}\right)^{s_1}}{c_1c_2 \cdots c_{s_1}} .$$

Repeated application of $k_ib_i = k_{i+1}c_{i+1}$ gives

$$k_p = \frac{k(k-1)\, b_2 \cdots b_{s_1}}{c_1c_2c_3 \cdots c_{s_1} \dfrac{k}{2}}$$

so

$$\begin{aligned} M &\leqslant \frac{(2p+1)\left(\frac{k}{2}\right)^{s_1+1} k_p}{k(k-1)\, b_2b_3 \cdots b_{s_1}} \\ &\leqslant \frac{(2p+1)\left(\frac{k}{2}\right)^{s_1-2} k_p \left(\frac{k}{2}\right)\left(\frac{k}{2}\right)}{2b_3b_4 \cdots b_{s_1}(k-1)\left(\frac{k}{2}+1\right)} \\ &\leqslant \frac{(2p+1)\left(\frac{k}{2}\right)^{s_1-2} k_p \left(\frac{k}{2}\right)\left(\frac{k}{2}\right)}{4b_3b_4 \cdots b_{s_1}\left(\left(\frac{k}{2}\right)^2 + \frac{k}{4} - \frac{1}{2}\right)} \\ &\leqslant \frac{(2p+1)}{4} k_p . \end{aligned}$$

Combining this with Lemma 1 gives a contradiction so $s_2 \leqslant 3s_1 + 2$.

(2) Suppose $b_2 = k/2$. In this case the intersection array is

$$\left\{\begin{matrix} * & 1 & \frac{k}{2} & \frac{k}{2} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ k & k-1 & \frac{k}{2} & \frac{k}{2} & \cdots \end{matrix}\right\} \quad (k > 4)$$

Then $| W_1 | = k/2$ and so every vertex of W_2 is joined to every vertex of W_1 and of W_2h. Similarly every vertex of W_2h and of W_1 is joined to every vertex of W_2. Hence if $y_1, y_2 \in W_2$, y_1 is joined to k vertices of $\Gamma_1(y_2)$ contradicting $c_2 = k/2$. Hence $b_2 \neq k/2$.

(3) If $b_2 < k/2$ the diameter is at most 3. If $d > 3$ choose $w \in \Gamma_4(u)$, $u' \in \Gamma_2(u)$ such that $d(u', w) = 2$. u, u' are joined by c_2 arcs of length 2. Hence u', w are joined by c_2 arcs of length 2 by distance-transitivity. Since $u' \in \Gamma_2(u)$, $w \in \Gamma_4(u)$, u', w are joined by at most b_2 arcs of length 2, so $b_2 \geqslant c_2$, which is a contradiction.

In case of diameter 2 the graph is a complete bipartite graph and in case of diameter 3 the graph is the incidence graph of a symmetric block design.

We have now proved the following theorem:

THEOREM. *Let Γ be a bipartite distance-transitive graph of valency >4 and suppose a bound for $| G_\alpha |$ exists. Then a bound for the diameter of Γ can be found.*

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