JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **213,** 160–182 (1997)
ARTICLE NO. AY975412

Local Tomography for the Limited-Angle Problem

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Received July 24, 1996

We investigate local tomography in the case of limited-angle data. The main theoretical tool is analysis of the singularities of pseudodifferential operators (PDO) acting on piecewise-smooth functions. Amplitudes of the PDO we consider are allowed to be nonsmooth in the dual variable ξ across the boundary of a wedge. Results of numerical simulation of limited-angle local tomography confirm basic theoretical conclusions.

1. INTRODUCTION

Let $\Phi(x, \alpha, p)$, $x \in \mathbb{R}^n$, $\alpha \in S^{n-1}$, $p \in \mathbb{R}$, be a smooth, strictly positive function. Here S^{n-1} is the unit sphere in $\mathbb{R}^n$. The generalized Radon transform $R^{(\Phi)}$ is defined as follows [14],

$$(R^{(\Phi)} f)(\alpha, p) := \hat{f}^{(\Phi)}(\alpha, p) := \int_{\mathbb{R}^n} \Phi(x, \alpha, p) f(x) \delta(p - \alpha \cdot x)\, dx, \tag{1.1}$$

where δ is the delta-function. If the weight function Φ identically equals 1, we obtain the classical Radon transform $\hat{f} = Rf$:

$$\hat{f}(\alpha, p) = \int_{\mathbb{R}^n} f(x) \delta(p - \alpha \cdot x)\, dx. \tag{1.2}$$

Not many properties of the generalized Radon transform are known. In particular, no inversion formula is known for $R^{(\Phi)}$.

0022-247X/97 $25.00

Let n, the dimension of the space $\mathbb{R}^n$, be even. Define the local tomography function [9, 4]

$$f_\Lambda^{(\Phi)}(x) := \frac{(-1)^{n/2}\pi}{(2\pi)^n}\int_{S^{n-1}}\frac{1}{\Phi(x,\alpha,\alpha\cdot x)}\frac{\partial^n}{\partial p^n}\hat{f}^{(\Phi)}(\alpha,p)\bigg|_{p=\alpha\cdot x}d\alpha. \tag{1.3}$$

Using (1.1) and the oscillatory integral $\delta(t)=(1/2\pi)\int_{-\infty}^{\infty}e^{i\lambda t}\,d\lambda$, one shows that $f_\Lambda^{(\Phi)}=\mathcal{B}f$, where $\mathcal{B}$ is an elliptic pseudodifferential operator (PDO) with the principal symbol $|\xi|$ [9, 4]. In the case of the classical Radon transform, one has $f_\Lambda^{(\Phi)}=\mathcal{F}^{-1}(|\xi|\mathcal{F}f)$, $\Phi\equiv 1$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the direct and inverse Fourier transforms, respectively.

To compute $f_\Lambda^{(\Phi)}$ by formula (1.3) one has to know $\hat{f}^{(\Phi)}(\alpha,p)$ for all $\alpha\in S^{n-1}$. However, the full angle data are not always available, and one frequently has the limited-angle data $\hat{f}^{(\Phi)}(\alpha,p)$, $\alpha\in\Omega$, $p\in\mathbb{R}$. Here Ω is an open set, $\Omega\subsetneq S^{n-1}$. In [9] it was proposed to compute

$$f_{\Lambda\chi}^{(\Phi)}(x) := \frac{(-1)^{n/2}\pi}{(2\pi)^n}\int_{\Omega}\frac{\chi(\alpha)}{\Phi(x,\alpha,\alpha\cdot x)}\frac{\partial^n}{\partial p^n}\hat{f}^{(\Phi)}(\alpha,p)\bigg|_{p=\alpha\cdot x}d\alpha, \tag{1.4}$$

where $\chi\in C_0^\infty(\Omega)$ is a smooth cut-off function. It was shown in [9] that $f_{\Lambda\chi}^{(\Phi)}=\mathcal{B}_\chi f$, where $\mathcal{B}_\chi$ is a PDO of order one. Moreover, the "visible" singularities of f: $WF(f)\cap(\mathbb{R}^n\times\Omega)$ can be located using (1.4) in a relatively stable way (see [10] for the earlier work on the subject). Here and everywhere below, for convenience of notation, we consider wave fronts as subsets of the sphere bundle $\mathbb{R}^n\times S^{n-1}$.

Let us consider the practically important case $n=2$. In this paper we drop the assumption $\chi\in C_0^\infty(\Omega)$ and suppose only that $\chi\in C^\infty(\Omega)$ and $\chi(\Theta)=0$ if $\Theta\notin\Omega$. Therefore, χ can be nonsmooth across $\partial\Omega$, the boundary of Ω. The main reason for dropping the assumption $\chi\in C_0^\infty(\Omega)$ is as follows: if χ vanishes smoothly near $\partial\Omega$, the operator $\mathcal{B}_\chi$ suppresses the singularities of f located close to $\pi(WF(f)\cap(\mathbb{R}^2\times\partial\Omega))$ (see Section 4). Here $\pi:\mathbb{R}^2\times S^1\to\mathbb{R}^2$ is the natural projection.

If χ is not smooth, the relation $f_{\Lambda\chi}^{(\Phi)}=\mathcal{B}_\chi f$ still holds, but now $\mathcal{B}_\chi$ is not a PDO in the classical sense, because its amplitude is not smooth in the dual variable ξ. In the paper we study the singularities of $\mathcal{B}_\chi f$. We show that the singular support of $f_{\Lambda\chi}^{(\Phi)}$ consists of two parts: visible singularities $S_v:=\pi(WF(f)\cap(\mathbb{R}^2\times\Omega))$, and "extra" singularities S_e, which are caused by the nonsmoothness of the corresponding symbol. We study the behavior of $f_{\Lambda\chi}^{(\Phi)}$ in a neighborhood of S_v and S_e and show that

(1) Knowing $f_{\Lambda\chi}^{(\Phi)}$ in a neighborhood of S_v, one can recover values of jumps of f across S_v (recall that $S_v\subset$ singsupp f); and

(2) Extra singularities S_e, which cause artifacts in the tomographic reconstruction, are weaker than visible singularities S_v. This means that even if one uses a sharp cut-off function χ: $\chi(\Theta) = 0$ if $\Theta \notin \Omega$ and $\chi(\Theta) = 1$ if $\Theta \in \Omega$, the resulting artifacts will not be strong. Moreover, the faster χ decays to zero near the boundary of Ω, the weaker extra singularities S_e are.

Since we derive an asymptotics in smoothness of $f_{\Lambda\chi}^{(\Phi)}$ in a neighborhood of S_e, the results obtained in the paper can also be used for finding an optimal cut-off function χ, not necessarily $\chi \in C_0^\infty(\Omega)$, such that the largest possible part of the visible singularities is recovered with minimal distortions.

Note that the main point of the paper is theoretical investigation of the singularities of $f_{\Lambda\chi}^{(\Phi)}$ in the case when χ is not C^∞ smooth. Therefore, the numerical experiments presented in Section 4 are intended primarily for illustrating theoretical results obtained in the paper. The problem of finding an optimal cut-off function χ and testing the algorithm on complicated phantoms will be the subject of future investigations.

Local tomography for the classical Radon transform was proposed in [17, 16]. Investigation of some properties of the local tomography function and results of testing local tomography on real data were presented in [2]. Further investigation of local tomography using the classical theory of PDO was published in [11–13]. Local tomography for the generalized Radon transform was developed in [9, 4]. In [9] it was shown that locations of the visible singularities can be obtained using (1.4) in a relatively stable way. Alternative approaches to locating visible singularities were described in [10, 5]. A study of local tomography was the subject of the monograph [14]. First results describing the behavior of $\mathcal{B}f$ near visible singularities S_v and extra singularities S_e were obtained in [14, Chap. 5]. In this paper we investigate this subject in more detail. The present derivation is more simple, and the results we obtain are more general. New results include

(1) Theorem 1, which describes the wave front of $\mathcal{B}f$ in the case of an arbitrary compactly supported distribution $f \in \mathcal{E}'(\mathbb{R}^2)$;

(2) Consideration of the case when the radius of curvature of $S =$ singsupp f is infinite (see Remark 3 in Section 3);

(3) Consideration of cut-off functions of different degrees of smoothness (parameter m in Theorem 2);

(4) More detailed numerical experiments, which illustrate the need for choosing an optimal cut-off function $\chi \in C^\infty(\Omega)$, which is not necessarily in $C_0^\infty(\Omega)$; and

(5) A brief discussion of limited-angle local tomography for the generalized Radon transform.

The paper is organized as follows. In Section 2 we describe the wave front of $\mathcal{B}f$ in the case when the amplitude of $\mathcal{B}$ is nonsmooth for finitely many directions $\xi/|\xi|$ and $f \in \mathcal{E}'(\mathbb{R}^2)$ is a compactly supported distribution. In Section 3 we obtain the asymptotics in smoothness of $\mathcal{B}f$ near visible singularities S_v and extra singularities S_e. Finally, application of the obtained results to local tomography is described in Section 4.

2. WAVE FRONT OF $\mathcal{B}f$

Consider the operator $\mathcal{B}$,

$$\mathcal{B}f(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} B(x,y,\xi) f(y) e^{-i\xi\cdot(x-y)}\,dy\,d\xi, \qquad f \in \mathcal{E}'(\mathbb{R}^2), \tag{2.1}$$

and suppose that for some $J > 1$ and a finite partition of the unit circle $S^1 = \bigcup_{j=1}^{J}[\theta_j, \theta_{j+1}]$, $\theta_{J+1} = \theta_1 + 2\pi$, the amplitude B can be represented as

$$B(x,y,\xi) = \sum_{j=1}^{J} \chi_j(\xi/|\xi|) B_j(x,y,\xi), \qquad B_j \in S^{\gamma}_{1,0}(\mathbb{R}^2),$$

$$\chi_j(\Theta) = \begin{cases} 1, & \theta \in [\theta_j, \theta_{j+1}], \\ 0, & \theta \notin [\theta_j, \theta_{j+1}]. \end{cases} \tag{2.2}$$

For convenience of the reader we recall that $B_j \in S^{\gamma}_{1,0}$ is equivalent to the two conditions

$$B_j(x,y,\xi) \in C^{\infty}(\mathbb{R}^6), \tag{2.3a}$$

and for any multi-indices $\alpha_u = (\alpha_{u1}, \alpha_{u2})$, $u = x, y, \xi$, and any compact set $G \in \mathbb{R}^4$, a constant $C_{\alpha_x\alpha_y\alpha_\xi G}$ exists for which

$$\left| \frac{\partial^{\alpha_x}}{\partial x^{\alpha_x}} \frac{\partial^{\alpha_y}}{\partial y^{\alpha_y}} \frac{\partial^{\alpha_\xi}}{\partial \xi^{\alpha_\xi}} B_j(x,y,\xi) \right| \le C_{\alpha_x\alpha_y\alpha_\xi G}(1+|\xi|)^{\gamma-|\alpha_\xi|},$$

$$\xi \in \mathbb{R}^2, \qquad (x,y) \in G, \qquad |\alpha_\xi| = \alpha_{\xi 1} + \alpha_{\xi 2}. \tag{2.3b}$$

In Eq. (2.2) and everywhere below, the variables Θ, $\Theta^{\perp}$, and θ are related as follows: $\Theta = (\cos\theta, \sin\theta)$ and $\Theta^{\perp} = (-\sin\theta, \cos\theta)$. Therefore, Θ_j, $j = 1, 2, \ldots, J$, is the set of directions across which the amplitude $B(x,y,t\Theta)$ is nonsmooth. For a set A, $U_\epsilon(A) \subset \mathbb{R}^2$ denotes an ϵ-neighborhood of A. In particular, $U_\epsilon(x_0)$ is a ball with radius $\epsilon > 0$ and center x_0. If A is a subset of S^1, then we assume that $U_\epsilon(A) \subset S^1$.

THEOREM 1. *Consider the operator* $\mathcal{B}$ *defined by* (2.1) *and* (2.2). *For a compactly supported distribution* $f \in \mathcal{E}'(\mathbb{R}^2)$, *define*

$$A_f := WF(f) \cap \left(\mathbb{R}^2 \times \cup_{j=1}^{J} \Theta_j\right). \tag{2.4}$$

Then $\mathcal{B}f \in \mathcal{D}'(\mathbb{R}^2)$ *and, moreover,* $(x,\Theta) \notin WF(\mathcal{B}f)$ *if*

(1) $(x,\Theta) \notin WF(f)$; *and*

(2) *either* $\Theta \notin \cup_{j=1}^{J} \Theta_j$ *or* $(x-y)\cdot\Theta \neq 0$ *for all* $(y,\Theta) \in A_f$.

In particular, $x \notin \operatorname{singsupp} \mathcal{B}f$ *if*

(1′) $x \notin \operatorname{singsupp} f$; *and*

(2′) $(x-y)\cdot\Theta \neq 0$ *for all* $(y,\Theta) \in A_f$.

Proof. Fix $\varphi \in C_0^\infty(\mathbb{R}^2)$. Clearly, the function $\Psi(y,\xi) := \int_{\mathbb{R}^2} B(x,y,\xi)\varphi(x)e^{-i\xi\cdot x}\,dx$ is C^∞ in y. Using (2.2), (2.3), and integrating by parts, we see that $\Psi(y,\xi)$ and its derivatives with respect to y decay rapidly (that is, faster than any power of $|\xi|$) as $|\xi| \to \infty$. Since the distribution f is of finite order, the function $\Psi_1(\xi) := \int_{\mathbb{R}^2} \Psi(y,\xi) f(y) e^{i\xi\cdot y}\,dy$ is well-defined and decays rapidly. In view of the relation $(\mathcal{B}f,\varphi) = (2\pi)^{-2}\int_{\mathbb{R}^2}\Psi_1(\xi)\,d\xi$, we conclude that $\mathcal{B}f$ is a continuous linear functional on $C_0^\infty(\mathbb{R}^2)$.

Denote $S = \operatorname{singsupp} f$. Fix two functions $\eta_\epsilon \in C_0^\infty(\mathbb{R}^2)$ and $\mu_\epsilon \in C^\infty(S^1)$ such that $\eta_\epsilon(x) = 1$ if $x \in U_\epsilon(S)$, $\eta_\epsilon(x) = 0$ if $x \notin U_{2\epsilon}(S)$, and $\mu_\epsilon(\Theta) = 1$ if $\Theta \in U_\epsilon(\cup_{j=1}^{J}\Theta_j)$, $\mu_\epsilon(\Theta) = 0$ if $\Theta \notin U_{2\epsilon}(\cup_{j=1}^{J}\Theta_j)$. We have

$$\begin{aligned}
(2\pi)^2\,\mathcal{B}f(x) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} B(x,y,\xi)(1-\eta_\epsilon(y))f(y)e^{-i\xi\cdot(x-y)}\,dy\,d\xi \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (1-\mu_\epsilon(\xi/|\xi|))B(x,y,\xi)\eta_\epsilon(y)f(y) \\
&\qquad \times e^{-i\xi\cdot(x-y)}\,dy\,d\xi \\
&\quad + \int_{\xi/|\xi| \in U_\epsilon(\cup_{j=1}^{J}\Theta_j)}\int_{U_\epsilon(S)} \mu_\epsilon(\xi/|\xi|)B(x,y,\xi)\eta_\epsilon(y)f(y) \\
&\qquad \times e^{-i\xi\cdot(x-y)}\,dy\,d\xi \\
&= I_1(x) + I_2(x) + I_3(x).
\end{aligned} \tag{2.5}$$

Since $(1-\eta_\epsilon)f \in C_0^\infty(\mathbb{R}^2)$, one easily verifies using properties (2.3) that

$$\int_{\mathbb{R}^2} B_j(x,y,\xi)(1-\eta_\epsilon(y))f(y)e^{i\xi\cdot y}\,dy = O(|\xi|^{-\infty}), \quad |\xi| \to \infty,\ 1 \le j \le J, \tag{2.6}$$

where $O(|\xi|^{-\infty})$ denotes a C^∞ function of x, which decays with all derivatives with respect to x faster than any negative power of $|\xi|$ as $|\xi| \to \infty$. Using (2.6), we immediately get the inclusion $I_1(x) \in C^\infty(\mathbb{R}^2)$.

The function $B(x, y, \xi)(1 - \mu_\epsilon(\xi/|\xi|))$ is a conventional amplitude. Using the pseudolocal property of PDO [15, pp. 15 and 224], we conclude

$$\operatorname{singsupp} I_2 \subset \operatorname{singsupp} \eta_\epsilon f = \operatorname{singsupp} f, \tag{2.7a}$$

$$WF(I_2) \subset WF(\eta_\epsilon f) = WF(f). \tag{2.7b}$$

Let us rewrite the integral I_3 in polar coordinates,

$$I_3(x) = \int_0^\infty \int_{U_\epsilon(\cup_{j=1}^J \Theta_j)} \int_{U_\epsilon(S)} \mu_\epsilon(\Theta) B(x, y, t\Theta) \eta_\epsilon(y) f(y) \times e^{-it\Theta\cdot(x-y)}\, dy\, d\theta\, t\, dt. \tag{2.8}$$

Using properties (2.3) and Theorem 1.1 in [15, p. 6], we see that the oscillatory integrals

$$\int_0^\infty B_j(x, y, t\Theta) e^{-itp} t\, dt = \Psi_j(x, y, \theta, p), \qquad 1 \le j \le J,$$

define C^∞ functions of x, y, θ, and p, provided that $p \neq 0$. This together with (2.2) and (2.8) implies that

$$\operatorname{singsupp} I_3 \subset \Big\{ x \in \mathbb{R}^2 : (x - y) \cdot \Theta = 0 \text{ for some } y \in U_\epsilon(S), \Theta \in U_\epsilon\big(\cup_{j=1}^J \Theta_j\big), (y, \Theta) \in WF(f) \Big\}. \tag{2.9}$$

Using that $\epsilon > 0$ can be taken arbitrarily small and taking into account (2.5), (2.7a), and (2.9), we have proved the assertion about the singular support of $\mathcal{B}f$.

Take an arbitrary $\varphi \in C_0^\infty(\mathbb{R}^2)$ and fix any $\Theta_0 \notin \cup_{j=1}^J \Theta_j$. Suppose $\epsilon > 0$ is such that $\Theta_0 \notin \overline{U_\epsilon\big(\cup_{j=1}^J \Theta_j\big)}$. Here the overbar denotes closure. Using (2.8), we have

$$\begin{aligned} \widetilde{I_3 \varphi}(s\Theta_0) &= \int_{\mathbb{R}^2} I_3(x) \varphi(x) e^{is\Theta_0 \cdot x}\, dx \\ &= \int_0^\infty \int_{U_\epsilon(\cup_{j=1}^J \Theta_j)} \int_{U_\epsilon(S)} \mu_\epsilon(\Theta) \Psi(y, t\Theta, s\Theta_0 - t\Theta) \eta_\epsilon(y) f(y) \\ &\quad \times e^{it\Theta\cdot y}\, dy\, d\theta\, t\, dt, \end{aligned} \tag{2.10}$$

where

$$\Psi(y, t\Theta, s\Theta_0 - t\Theta) := \int_{\mathbb{R}^2} B(x, y, t\Theta) \varphi(x) e^{i(s\Theta_0 - t\Theta)\cdot x}\, dx. \tag{2.11}$$

Integrating by parts with respect to x in (2.11) and using properties (2.3) and the fact that Θ_0 is bounded away from $U_\epsilon(\cup_{j=1}^J \Theta_j)$, we see that for every $N > 0$ there is a constant $c_N(y)$ such that

$$|\Psi(y, t\Theta, s\Theta_0 - t\Theta)| \le c_N(y) \frac{(1+t)^\gamma}{[\max(s,t)]^N}, \qquad \Theta \in S^1, s, t > 0.$$

Clearly, $\Psi(y, t\Theta, s\Theta_0 - t\Theta)$ is a C^∞ function of y. Therefore, the expression

$$\Psi_1(t\Theta, s\Theta_0 - t\Theta) := \int_{U_\epsilon(S)} \Psi(y, t\Theta, s\Theta_0 - t\Theta)\eta_\epsilon(y)e^{it\Theta\cdot y}f(y)\,dy$$

is well-defined and we have for some $c_1 > 0$ (which depends on N)

$$|\Psi_1(t\Theta, s\Theta_0 - t\Theta)| \le c_1 \frac{(1+t)^{\gamma+M}}{[\max(s,t)]^N}, \qquad \Theta \in S^1, s, t > 0,$$

where M is the order of the distribution f. Together with (2.10) this implies

$$\begin{aligned} |\widetilde{I_3\varphi}(s\Theta_0)| &\le c_1 \int_0^\infty \int_{U_\epsilon(\cup_{j=1}^J \Theta_j)} |\mu_\epsilon(\Theta)| \frac{(1+t)^{\gamma+M}}{[\max(s,t)]^N}\,d\theta\,t\,dt \\ &\le c_2 \int_0^\infty \frac{(1+t)^{\gamma+M}}{[\max(s,t)]^N}\,t\,dt. \end{aligned}$$

Since $N > 0$ can be taken arbitrarily large, this shows that $\widetilde{I_3\varphi}(s\Theta_0)$ decays rapidly as $s \to \infty$. Taking into account that $\epsilon > 0$ was arbitrarily small and using (2.7b), we have finished the proof. ∎

3. ANALYSIS OF THE BEHAVIOR OF $\mathcal{B}f$ IN A NEIGHBORHOOD OF singsupp $\mathcal{B}f$

Consider now the case when f can be represented in the form

$$f(x) = \sum_k \varphi_k(x)\chi_k(x), \qquad \varphi_k \in C^\infty(\mathbb{R}^2), \tag{3.1}$$

where the sum is finite and the χ_k are the characteristic functions of bounded domains D_k with piecewise smooth boundaries ∂D_k. Clearly, $S :=$ singsupp $f = \cup_k \partial D_k$. According to Theorem 1,

$$\text{singsupp } \mathcal{B}f \subset S \cup \left(\cup_j L_j\right),$$

where each L_j is tangent to S and perpendicular to some vector from the set $\bigcup_{j=1}^{J}\Theta_j$. Since $\mathcal{B}$ is linear, we can assume without loss of generality that each L_j is tangent to S at exactly one point.

Fix any $x_0 \in S$, $x_0 \notin \bigcup_j L_j$. Let $\epsilon > 0$ be sufficiently small. We have

$$\mathcal{B}f = \mathcal{B}[(1-\chi)f] + \mathcal{B}[\chi f], \tag{3.2}$$

where $\chi \in C_0^\infty(U_{2\epsilon}(x_0))$ is any function such that $\chi = 1$ on $U_\epsilon(x_0)$. According to Theorem 1, $\mathcal{B}[(1-\chi)f] \in C^\infty(U_\epsilon(x_0))$. According to our choice of $x_0 \in S$, $WF(\chi f) \cap (\mathbb{R}^2 \times \bigcup_{j=1}^{J}\Theta_j) = \varnothing$. Let $\delta > 0$ be so small that $WF(\chi f) \subset \mathbb{R}^2 \times (S^1 \setminus U_\delta(\bigcup_{j=1}^{J}\Theta_j))$. Take any $\eta \in C^\infty(S_1)$ so that $\eta(\Theta) = 0$ if $\Theta \notin U_\delta(\bigcup_{j=1}^{J}\Theta_j)$ and $\eta(\Theta) = 1$ if $\Theta \in U_{\delta/2}(\bigcup_{j=1}^{J}\Theta_j)$. As in the proof of Theorem 1,

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \eta(\xi/|\xi|) B(x,y,\xi)\chi(y)f(y)e^{-i\xi\cdot(x-y)}\,dy\,d\xi \in C^\infty(\mathbb{R}^2). \tag{3.3}$$

From (3.2) and (3.3) we conclude that

$$\mathcal{B}f(x) \overset{C^\infty(U_\epsilon(x_0))}{=} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(1-\eta(\xi/|\xi|))B(x,y,\xi)\chi(y)f(y) \times e^{-i\xi\cdot(x-y)}\,dy\,d\xi. \tag{3.4}$$

The notation $\overset{C^\infty(U_\epsilon(x_0))}{=}$ means that the equality holds up to a $C^\infty(U_\epsilon(x_0))$ function. According to our assumptions, $(1-\eta)B \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \times (\mathbb{R}^2 \setminus 0))$. Therefore the right-hand side of (3.4) defines a conventional PDO, and we can use the results obtained in [4] to find the asymptotics in smoothness of $\mathcal{B}f$ in a neighborhood of x_0.

Let U be an open set such that $S \cap U \neq \varnothing$. We say that S is smooth inside U if $S \cap U = \{x \in U : g(x) = 0\}$ for some $g \in C^\infty(\overline{U})$ such that $|\nabla g| \neq 0$ on $S \cap \overline{U}$. Here $\overline{U}$ denotes the closure of U. For a point $x_0 \in S$, $n(x_0)$ denotes a unit vector perpendicular to S at x_0, and $D(x_0)$ denotes a jump of f across S at x_0 in the direction $n(x_0)$: $D(x_0) = \lim_{s\to 0^+}[f(x_0 + sn(x_0)) - f(x_0 - sn(x_0))]$. The following proposition is a particular case of Theorem 2.1 in [4].

PROPOSITION 1. *Suppose that f satisfies* (3.1). *Consider a classical PDO* $\mathcal{B} \in CL^\gamma_{1,0}(\mathbb{R}^n)$ *with amplitude* $B(x,y,\xi)$:

$$B(x,y,t\Theta) \sim \sum_{k\ge 0} b_k(x,y,\Theta)t^{\gamma-k},\ t\to\infty,$$
$$b_k(x,y,\Theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times S^{n-1}). \tag{3.5}$$

Suppose that $B(x, y, \xi)$ *is even in* ξ: $B(x, y, \xi) = B(x, y, -\xi)$. *Fix a sufficiently small open set* $U, S \cap U \neq \emptyset$. *Suppose that* S *is smooth inside* U *and* $b_0(x_0, x_0, n(x_0)) \neq 0$ *for* $x_0 \in S \cap U$. *Then one has*

$$\mathcal{B}f(x) = \frac{b_0(x_0, x_0, n(x_0))}{\pi} \operatorname{Im}\left\{ \int_0^\infty \Psi(x, t) e^{ith}\, dt \right\},$$
$$x = x_0 + hn(x_0) \in U,\ x_0 \in S \cap U, \quad (3.6)$$

where $\Psi(x, t) \in C^\infty(U \times [0, \infty))$. *Moreover,* Ψ *admits the asymptotic expansion*

$$\Psi(x, t) \sim t^{\gamma - 1}\left(D(x_0) + \sum_{k \geq 1} \frac{d_k(x)}{t^k} \right), \qquad t \to \infty,\ d_k \in C^\infty(U), \quad (3.7)$$

which can be differentiated with respect to $x \in U$ *and* t.

Remark 1. The coefficients d_k can, in principle, be expressed in terms of f and B. However, the resulting formulas are rather cumbersome and, therefore, are not given here.

The following result is an immediate corollary to Theorem 3.1 in [4].

PROPOSITION 2. *Put* $\gamma = 1$ *in* (3.5). *Then one has*

$$\mathcal{B}f(x) = \frac{b_0(x_0, x_0, n(x_0))}{\pi} \frac{D(x_0)}{h} + O(\ln|h|),$$
$$x = x_0 + hn(x_0) \in U,\ h \to 0.$$

Now let L be any of the lines L_j which are in singsupp $\mathcal{B}f$. Let y_0 be the point of contact of L and S. Fix a sufficiently small neighborhood U of y_0. Since the operator $\mathcal{B}$ is linear, we may assume without loss of generality that $\operatorname{supp} f \subset U$ and, according to Theorem 1, singsupp $\mathcal{B}f \subset \operatorname{singsupp} f \cup L$. The main reason for truncating $\operatorname{supp} f$ is that this allows us to get rid of all the lines L_j that are perpendicular to Θ_0 and tangent to S at other points $y_j \neq y_0$. In view of Theorem 1, the behavior of $\mathcal{B}f(x)$ as $x \to S \setminus y_0$ is given by Proposition 1. Therefore, it remains to find the behavior of $\mathcal{B}f(x)$ as $x \to x_0$ for all $x_0 \in L$. Thus, in what follows, we always assume that x is in a sufficiently small neighborhood U of x_0.

Suppose that $S \cap U$ is smooth and strictly convex. Consider the integral

$$\mathcal{B}_+ f(x) = \frac{1}{(2\pi)^2} \int_0^\infty \int_{\theta_0}^{\theta_0 + \epsilon} \int_{\mathbb{R}^2} B(x, y, t\Theta) f(y) e^{-it\Theta \cdot (x-y)}\, dy\, d\theta\, t\, dt, \quad (3.8)$$

where B admits the following asymptotic expansion,

$$B(x, y, t\Theta) \sim (\theta - \theta_0)^m \sum_{k \geq 0} b_k(x, y, \Theta) t^{\gamma - k}, \qquad t \to \infty; \quad (3.9a)$$

$$b_k \in C^\infty\big(\mathbb{R}^2 \times \mathbb{R}^2 \times [\theta_0, \theta_0 + \epsilon]\big), \qquad k = 0, 1, 2, \ldots; \ b_0(x, y, \Theta_0) \not\equiv 0; \quad (3.9b)$$

$$\frac{\partial^j}{\partial \theta^j} b_k(x, y, \Theta) \,|_{\theta = \theta_0 + \epsilon} = 0, \qquad k, j = 0, 1, 2, \ldots, \quad (3.9c)$$

and expansion (3.9a) can be differentiated with respect to x, y, θ, and t. The integer parameter m used in (3.9a) regulates the degree of smoothness of the amplitude $B(x, y, t\Theta)$ across $\theta = \theta_0$ (cf. the discussion of the degree of smoothness of the cut-off function χ given in the Introduction).

Introduce a local coordinate system with the origin at y_0, the x_1-axis of which points in the direction Θ_0. Let $y_1 = g(y_2)$ be the local equation of S in a neighborhood of $y = y_0$. Clearly, $\theta_0 = 0$ in the new coordinate system. First, consider the integral

$$J_1(x, \theta, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(x, (y_1, y_2), t\Theta) f(y_1, y_2) e^{it(y_1 \cos\theta + y_2 \sin\theta)} \, dy_1 \, dy_2 .$$

Since $f(y_1, y_2)$ is discontinuous at $y_1 = g(y_2)$, substituting (3.9a) into the last equation and integrating by parts with respect to y_1, we find

$$\Psi_1(x, y_2, \theta, t) = \int_{-\infty}^{\infty} B(x, (y_1, y_2), t\Theta) f(y_1, y_2) e^{it(y_1 - g(y_2)) \cos\theta} \, dy_1, \quad (3.10a)$$

$$J_1(x, \theta, t) = \int_{-\infty}^{\infty} \Psi_1(x, y_2, \theta, t) e^{it(g(y_2)\cos\theta + y_2 \sin\theta)} \, dy_2, \quad (3.10b)$$

where Ψ_1 admits the asymptotic expansion

$$\Psi_1(x, y_2, \theta, t) \sim \sum_{k \geq 0} \psi_{1,k}(x, y_2, \theta) t^{\gamma - 1 - k}, \ t \to \infty,$$

$$\psi_{1,k} \in C^\infty(U \times \mathbb{R} \times [\theta_0, \theta_0 + \epsilon]), \quad (3.11a)$$

$$\psi_{1,0}(x, y_2, \theta) = i(\theta - \theta_0)^m b_0(x, (g(y_2), y_2), \theta) D(g(y_2), y_2)/\cos\theta. \quad (3.11b)$$

Here $D(g(y_2), y_2)$ is the jump of f across S at the point $y = (g(y_2), y_2)$.

Since f is compactly supported, Eq. (3.10a) implies that the integration in (3.10b) is over a compact set. Substituting (3.10b) into (3.8), we find

$$\mathcal{B}_+ f(x) = \frac{1}{(2\pi)^2} \int_0^\infty \int_0^\epsilon \int_{-\infty}^\infty \Psi_1(x, y_2, \theta, t) \times e^{it((g(y_2)-x_1)\cos\theta + (y_2-x_2)\sin\theta)}\, dy_2\, d\theta\, t\, dt. \tag{3.12}$$

Now consider the integral

$$J_2(x, y_2, t) = \int_0^\epsilon \Psi_1(x, y_2, \theta, t) e^{ita(x, y_2, \theta)}\, d\theta,$$
$$a(x, y_2, \theta) = (g(y_2) - x_1)\cos\theta + (y_2 - x_2)\sin\theta. \tag{3.13}$$

Let $x_2 \neq 0$ be fixed. Without loss of generality we may assume that $\epsilon > 0$ is so small that $\partial a/\partial\theta \neq 0$ if $0 \leq \theta \leq \epsilon$. Indeed, if $\epsilon > 0$ is not sufficiently small, we can represent $\mathcal{B}_+$ as

$$\begin{aligned}\mathcal{B}_+ f(x) &= \frac{1}{(2\pi)^2} \int_0^\infty \int_0^{2\delta} \int_{\mathbb{R}^2} \eta_\delta(\Theta) B(x, y, t\Theta) f(y) e^{-it\Theta\cdot(x-y)}\, dy\, d\theta\, t\, dt \\ &\quad + \frac{1}{(2\pi)^2} \int_0^\infty \int_\delta^\epsilon \int_{\mathbb{R}^2} (1 - \eta_\delta(\Theta)) B(x, y, t\Theta) f(y) \\ &\quad \times e^{-it\Theta\cdot(x-y)}\, dy\, d\theta\, t\, dt \\ &=: \mathcal{B}_1 f(x) + \mathcal{B}_2 f(x),\end{aligned}$$

$$\eta_\delta \in C^\infty([0, \epsilon]), \qquad \eta_\delta(\Theta) = \begin{cases} 1, & 0 \leq \theta \leq \delta, \\ 0, & \theta \geq 2\delta, \end{cases}$$

where $\delta > 0$ is sufficiently small. Since $\mathcal{B}_2$ is a conventional PDO, $\mathcal{B}_2 f$ is C^∞ in a neighborhood of x_0. The operators $\mathcal{B}_1$ and $\mathcal{B}_+$ are of the same form. This shows that we can take $\epsilon > 0$ in (3.8) as small as we like. Integrating by parts in (3.13) and using (3.11b), we get

$$J_2(x, y_2, t) = \Psi_2(x, y_2, t) e^{it(g(y_2)-x_1)}, \tag{3.14}$$

where Ψ_2 admits the asymptotic expansion

$$\Psi_2(x, y_2, t) \sim \sum_{k \geq 0} \psi_{2,k}(x, y_2) t^{\gamma-2-m-k}, \qquad t \to \infty,\ \psi_{2,k} \in C^\infty(U \times \mathbb{R}), \tag{3.15}$$

and

$$\psi_{2,0}(x,y_2) = \frac{i^{m+1}m!\psi_{1,k}(x,y_2,0)}{(y_2-x_2)^{m+1}}$$
$$= -i^m m!\frac{b_0(x,(g(y_2),y_2),0)D(g(y_2),y_2)}{(y_2-x_2)^{m+1}}. \quad (3.16)$$

From (3.12)–(3.14) it follows that we have to study the integral

$$J_3(x,t) = \int_{-\infty}^{\infty} \Psi_2(x,y_2,t)e^{it(g(y_2)-x_1)}\,dy_2. \quad (3.17)$$

Suppose $g''(0) \neq 0$, that is, the radius of curvature of S at y_0 is finite. The case $g''(0) = 0$ is briefly discussed in Remark 3 below. Then the stationary phase method yields (see [18, pp. 76–81] and Theorem 14.5.2 in [14, p. 421])

$$J_3(x,t) = \Psi_3(x,t)e^{-itx_1}, \quad (3.18)$$

where Ψ_3 admits the asymptotic expansion

$$\Psi_3(x,t) \sim \sum_{k\geq 0} \psi_{3,k}(x)t^{\gamma-2.5-m-k}, \qquad t\to\infty,\ \psi_{3,k}\in C^\infty(U), \quad (3.19)$$

and

$$\psi_{3,0}(x) = \left(\frac{2\pi}{|g''(0)|}\right)^{0.5} e^{i(\pi/4)\operatorname{sgn} g''(0)}\psi_{2,0}(x,0). \quad (3.20)$$

Suppose now that $x_2 = 0$. In this case, the stationary point of the phase $a(x,y_2,\theta)$ (cf. (3.13)) is given by $(y_2,\theta) = (0,0)$. Consider the following double integral (cf. (3.12))

$$J_4(x,t) = \int_0^{\epsilon}\int_{-\infty}^{\infty} \Psi_1(x,y_1,\theta,t)e^{ita(x,y_2,\theta)}\,dy_2\,d\theta. \quad (3.21)$$

The stationary point is located on the boundary of the domain of integration. Using expansion (3.11a) and applying the stationary phase method term by term (see Eq. (8.4.46) in [1, p. 348] and [18, pp. 440–442, 470, 471]), we get

$$J_4(x,t) = \Psi_4(x_1,t)e^{-itx_1}, \qquad x = (x_1,0), \quad (3.22)$$

where Ψ_4 admits the asymptotic expansion

$$\Psi_4(x_1,t) \sim \sum_{k\geq 0} \psi_{4,k}(x_1) t^{\gamma-2-(k/2)}, \qquad t\to\infty,\ \psi_{4,k}\in C^\infty((-\delta,\delta)), \tag{3.23}$$

where $\delta > 0$ is sufficiently small. In particular, if $m = 0$, we have

$$\psi_{4,0}(x_1) = \frac{\pi\psi_{1,0}((x_1,0),0,0)}{\sqrt{1-g''(0)x_1}}. \tag{3.24}$$

If $m \geq 1$, the formula for $\psi_{4,0}$ is very cumbersome and we do not give it here.

Let $R(y_0)$ be the radius of curvature of S at y_0. Returning to the original coordinate system and using (3.11b), (3.16), we can rewrite Eqs. (3.18)–(3.20) and (3.22)–(3.24) as

$$\begin{aligned} J_3(x,t) &= \Psi_3(x,t)e^{-ith}, \qquad x = x_0 + h\Theta_0,\ x_0\in L,\ x_0\neq y_0, \\ \Psi_3(x,t) &\sim \sum_{k\geq 0}\psi_{3,k}(x)t^{\gamma-2.5-m-k}, \qquad t\to\infty, \\ \psi_{3,0}(x) &= (-i)^m m!\sqrt{2\pi R(y_0)}\, e^{\mp i(\pi/4)}\frac{b_0(x,y_0,\theta_1)D(y_0)}{\left[(x_0-y_0)\cdot\Theta_0^\perp\right]^{m+1}}; \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} J_4(x,t) &= \Psi_4(h,t)e^{-ith}, \qquad x = y_0 + h\Theta_0, \\ \Psi_4(h,t) &\sim \sum_{k\geq 0}\psi_{4,k}(h)t^{\gamma-2-(k/2)}, \qquad t\to\infty, \\ \psi_{4,0}(h) &= \frac{i\pi b_0(x,y_0,\theta_1)D(y_0)}{\sqrt{1\pm(h/R(y_0))}}, \qquad m = 0. \end{aligned} \tag{3.26}$$

In (3.25) and (3.26), the signs in $\pm$ and $\mp$ are chosen according to where the center of curvature of S at $y_0 \in S$ is located. More precisely, if O is the center of curvature, then the top signs are chosen if $(y_0 - O)\cdot\Theta_0 > 0$, and the bottom signs are chosen if $(y_0 - O)\cdot\Theta_0 < 0$.

Therefore, we have proved the following result.

THEOREM 2. *Suppose that f satisfies* (3.1). *Consider the distribution* $\mathcal{B}_+ f$ *defined by* (3.8) *and* (3.9). *Fix the line L perpendicular to Θ_0 and tangent to S at y_0. Let $R(y_0)$ be the radius of curvature of S at y_0,* $0 < R(y_0) < \infty$, *and* $D(y_0) \neq 0$ *be the value of the jump of f across S at y_0 in the direction Θ_0:*

$D(y_0) = \lim_{s \to 0^+}[f(y_0 + s\Theta_0) - f(y_0 - s\Theta_0)]$. Fix any $x_0 \in L$ and let U be a sufficiently small neighborhood of x_0. If $x_0 \neq y_0$, we have

$$\mathcal{B}_+ f(x) = (-i)^m m! e^{\mp i(\pi/4)} \frac{\sqrt{R(y_0)}\, b_0(x, y_0, \Theta_0) D(y_0)}{(2\pi)^{1.5}[(x_0 - y_0)\cdot\Theta_0^\perp]^{m+1}}$$

$$\times \int_0^\infty \Psi_1(x,t) e^{-ith}\, dt,$$

$$x = x_0 + \Theta_0 h \in U, \tag{3.27}$$

where $\Psi_1 \in C^\infty(U \times [0,\infty))$ admits the asymptotic expansion

$$\Psi_1(x,t) \sim t^{\gamma - 1.5 - m}\left(1 + \sum_{k \geq 1} \psi_{1,k}(x) t^{-k}\right), \qquad t \to \infty,\ \psi_{1,k} \in C^\infty(U), \tag{3.28}$$

which can be differentiated with respect to $x \in U$ and t.

If $x_0 = y_0$ and $m = 0$ in (3.9a), *we have*

$$\mathcal{B}_+ f(x) = \frac{b_0(x, y_0, \Theta_0) D(y_0)}{4\pi} i \int_0^\infty \Psi_2(h,t) e^{-ith}\, dt,$$

$$x = y_0 + \Theta_0 h \in U, \tag{3.29}$$

where $\Psi_2 \in C^\infty((-\delta,\delta) \times [0,\infty))$ and $\delta > 0$ is sufficiently small. Moreover, Ψ_2 admits the asymptotic expansion

$$\Psi_2(h,t) \sim t^{\gamma-1}\left(1 + \sum_{k\geq 1} \psi_{2,k}(h) t^{-(k/2)}\right), t \to \infty,$$

$$\psi_{2,k} \in C^\infty((-\delta,\delta)), \tag{3.30}$$

which can be differentiated with respect to $h \in (-\delta,\delta)$ and t.

Let O be the center of curvature of S at y_0. Top signs $\pm$ and $\mp$ are chosen if $(y_0 - O)\cdot\Theta_0 > 0$, and bottom signs are chosen if $(y_0 - O)\cdot\Theta_0 < 0$.

Since the function $(1 \pm (h/R(y_0)))^{-1/2}$ is smooth for small h and equals 1 when $h = 0$, we absorbed this function by the integral on the right-hand side of (3.29), and this did not change the leading term in the expansion of Ψ_2.

Remark 2. Similarly to the proof of Theorem 2, one can show that in the case of the operator given by the formula

$$\mathsf{B}_{-}f(x) = \frac{1}{(2\pi)^2}\int_0^\infty \int_{\theta_0-\epsilon}^{\theta_0}\int_{\mathbb{R}^2} B(x,y,t\Theta)f(y)e^{-it\Theta\cdot(x-y)}\,dy\,d\theta\,t\,dt,$$

where $B \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2)$ satisfies the conditions

$$B(x,y,t\Theta) \sim (\theta-\theta_0)^m \sum_{k\geq 0} b_k(x,y,\theta)t^{\gamma-k}, \qquad t\to\infty, \quad (3.31a)$$

$$b_k \in C^\infty\big(\mathbb{R}^2 \times \mathbb{R}^2 \times [\theta_0-\epsilon,\theta_0]\big), \qquad k=0,1,2,\ldots, b_0(x,y,\theta_0)\not\equiv 0, \quad (3.31b)$$

$$\frac{\partial^j}{\partial\theta^j}b_k(x,y,\theta)\,|_{\theta=\theta_0-\epsilon} = 0, \qquad k,j=0,1,2,\ldots, \quad (3.31c)$$

the analog of Eq. (3.27) becomes

$$\mathsf{B}_{-}f(x) = -(-i)^m m!e^{\mp i(\pi/4)}\frac{\sqrt{R(y_0)}\,b_0(x,y_0,\Theta_0)D(y_0)}{(2\pi)^{1.5}\big[(x_0-y_0)\cdot\Theta_0^\perp\big]^{m+1}} \times\int_0^\infty \Psi_1(x,t)e^{-ith}\,dt,$$
$$x = x_0+\Theta_0 h\in U, \quad (3.32)$$

where Ψ_1 admits asymptotic expansion (3.28), and Eqs. (3.29), (3.30) remain unchanged.

Remark 3. Using Eq. (3.17), we see that the stationary phase method allows one to find the behavior of $\mathsf{B}_{+}f$ and $\mathsf{B}_{-}f$ in a neighborhood of L in the case when the function $y_1 = g(y_2)$ has a degenerate critical point at $y_2 = 0$:

$$g(0) = g'(0) = \cdots = g^{(l-1)} = 0, \qquad g^{(l)} \neq 0,\ l>2.$$

Suppose, for example, that l is even. Then we get

$$\mathsf{B}_{+}f(x) = \frac{(-i)^m m!}{(2\pi)^2}\,\frac{b_0(x,y_0,\Theta_0)D(y_0)}{\big[(x_0-y_0)\cdot\Theta_0^\perp\big]^{m+1}}\,\frac{2\Gamma(1/l)}{l}\left(\frac{l!}{|g^{(l)}(0)|}\right)^{1/l} \times e^{i(\pi/2l)\operatorname{sgn} g^{(l)}(0)}\int_0^\infty \Psi_1(x,t)e^{-ith}\,dt,$$
$$x = x_0+\Theta_0 h \in U, \qquad l = 2k,$$

where Γ is the gamma-function, and the leading term in the expansion of Ψ_1 is given by

$$\Psi_1(x,t) \sim \text{const}\, t^{\gamma-1-(1/l)-m}, \qquad t \to \infty.$$

Remark 4. From Eq. (3.24) it follows that (3.29) holds even if $g''(0) = 0$, that is, $R(y_0) = \infty$.

4. APPLICATION TO LOCAL TOMOGRAPHY

Let $\chi(\Theta)$ be a piecewise-smooth even function: $\chi(\Theta) = \chi(-\Theta)$. Define the family of local tomography functions $f_{\Lambda\chi}$ [11, 12],

$$f_{\Lambda\chi}(x) := -\frac{1}{4\pi}\int_0^{2\pi} \chi(\Theta)\hat{f}_{,pp}(\theta, \Theta\cdot x)\, d\theta. \tag{4.1}$$

Here $\hat{f}_{,pp} := \partial^2/\partial p^2 \hat{f}$. Suppose the Radon transform $\hat{f}(\theta, p)$ is given for $\theta \in [\theta_1, \theta_2]$ and $p \in \mathbb{R}$. Since $\hat{f}$ is even, $\hat{f}(\theta+\pi, p) = \hat{f}(\theta, -p)$, we may assume that $\hat{f}$ is known for $\theta \in [\theta_1, \theta_2]$, $\theta \in [\theta_1 + \pi, \theta_2 + \pi]$, and $p \in \mathbb{R}$. Denote $\Omega := \{\Theta \in S^1 : \theta \in [\theta_1, \theta_2]$ or $\theta \in [\theta_1 + \pi, \theta_2 + \pi]\}$. Putting $\chi(\Theta) = 0$, $\Theta \notin \Omega$, in (4.1), we obtain the local tomography function which uses only the known data. From (4.1) one easily gets using the Fourier slice theorem,

$$f_{\Lambda\chi} = F^{-1}\big(\chi(\xi/|\xi|)|\xi|\tilde{f}(\xi)\big), \qquad \hat{f} = Ff, \tag{4.2}$$

where F and F^{-1} denote the direct and inverse Fourier transforms, respectively. From Eq. (4.2) we see that the theory developed in Sections 2 and 3 is directly applicable to the analysis of the singularities of $f_{\Lambda\chi}$. Let us suppose for simplicity that $\chi(\Theta) = 1$, $\Theta \in \Omega$. Note that in this case $\chi(\Theta)$ is discontinuous. As usual, $n(x_0)$ denotes a unit vector perpendicular to S at $x_0 \in S$. Theorem 1 implies that the singular support of $f_{\Lambda\chi}$ consists of

(1) *Visible singularities*: corner points of S and points $x_0 \in S$, where S is smooth and $n(x_0) \in \Omega$; and

(2) *Extra singularities*: the lines which are tangent to S and which are perpendicular to vectors Θ_1 or Θ_2, $\Theta_k = (\cos\theta_k, \sin\theta_k)$, $k = 1, 2$.

Pick any x_0 such that $n(x_0)$ is strictly inside Ω. Using Proposition 2 (see Section 2), we get

$$f_{\Lambda\chi}(x_0 + hn(x_0)) \sim \frac{D(x_0)}{\pi} h^{-1}, \qquad h \to 0,\ x_0 \in S. \tag{4.3}$$

The last equation shows that knowing $f_{\Lambda\chi}$ in a neighborhood of the visible singularities S_v, one can recover values of jumps of f across S_v. This can be done using, for example, the algorithm in [4].

Now pick any line $L := \{x : \Theta \cdot x = p\}$, where $\Theta = \Theta_1$ or $\Theta = \Theta_2$, which is tangent to S. Take, for example, $\Theta = \Theta_1$ and let y_0 be the point of contact. Fix any $x_0 \in L, x_0 \neq y_0$. Clearly, we may always assume that $\Theta_1 = n(y_0)$. Equations (3.27), (3.28) yield with $b_0(x, y, \Theta) = \chi(\Theta) \equiv 1$, $\Theta \in \Omega$, $m = 0$, and $\gamma = 1$,

$$\begin{aligned} f_{\Lambda\chi}(x_0 + hn(y_0)) \sim e^{-i(\pi/4)} & \frac{\sqrt{R(y_0)}\, D(y_0)}{(2\pi)^{1.5}(x_0 - y_0) \cdot n^{\perp}(y_0)} \\ & \times \int_0^{\infty} \Psi_1(x, t) e^{-ith}\, dt \\ & + e^{i(\pi/4)} \frac{\sqrt{R(y_0)}\,(-D(y_0))}{(2\pi)^{1.5}(x_0 - y_0) \cdot \left(-n^{\perp}(y_0)\right)} \\ & \times \int_0^{\infty} \Psi_1(x, t) e^{ith}\, dt, \end{aligned} \tag{4.4}$$

where $n^{\perp}(y_0)$ is the unit vector perpendicular to $n(y_0)$ such that $n^{\perp}(y_0)$ is obtained by rotating $n(y_0)$ 90 degrees counterclockwise. The first and the second terms on the right-hand side of (4.4) correspond to the contributions from the discontinuities of $\chi(\Theta)$ at $\Theta = n(y_0)$ and $\Theta = -n(y_0)$, respectively. We made the following changes in the second term:

(1) h was replaced by $-h$, so that the point under consideration $x_0 + hn(y_0)$ does not change when we replace $n(y_0)$ by $-n(y_0)$;

(2) We took into account that $D(y_0)$ and $n^{\perp}(y_0)$ change signs when we replace $n(y_0)$ by $-n(y_0)$.

After simple transformations, we get using Eq. (21) in [3, p. 360]

$$\begin{aligned} f_{\Lambda\chi}(x_0 + hn(y_0)) &\sim \frac{2\sqrt{R(y_0)}\, D(y_0)}{(2\pi)^{1.5}(x_0 - y_0) \cdot n^{\perp}(y_0)} \operatorname{Re}\left[e^{i(\pi/4)} \int_0^{\infty} t^{-0.5} e^{ith}\, dt\right] \\ &= \frac{2\sqrt{R(y_0)}\, D(y_0)}{(2\pi)^{1.5}(x_0 - y_0) \cdot n^{\perp}(y_0)} \\ &\quad \times \operatorname{Re}\left[e^{i(\pi/4)} i\Gamma(1/2)\left(e^{-i(\pi/4)} h_+^{-1/2} - e^{i(\pi/4)} h_-^{-1/2}\right)\right] \\ &= \frac{\sqrt{2R(y_0)}\, D(y_0)}{2\pi(x_0 - y_0) \cdot n^{\perp}(y_0)} h_-^{-1/2}, \qquad h \to 0. \end{aligned} \tag{4.5}$$

Here $h_+ = 0$ if $h < 0$, $h_+ = h$ if $h > 0$, and $h_- = h - h_+$. From (4.5) we see that the leading singular term of $f_{\Lambda\chi}(x)$ as $x \to x_0 \in L$, $x_0 \neq y_0$, is on the same side of L as S in a neighborhood of y_0. In (4.5) we took into account the contribution of the leading term of Ψ_1 as $t \to \infty$. The second term of the expansion of Ψ_1 is $O(t^{-1.5})$ as $t \to \infty$ (see (3.28)). Since the function $\int_0^\infty O(t^{-1.5})e^{-ith}\,dt$ is continuous at $h = 0$, together with (4.5) this implies that there exists the limit of $f_{\Lambda\chi}(x)$ as x approaches $x_0 \in L$, $x_0 \neq y_0$, from the side of L opposite to the location of S in a neighborhood of y_0.

Equations (4.3) and (4.5) are illustrated by Fig. 1, where the behavior of $f_{\Lambda\chi}$ in a neighborhood of singsupp $f_{\Lambda\chi}$ is sketched. The shaded disc represents the phantom, which is more dense than the surrounding

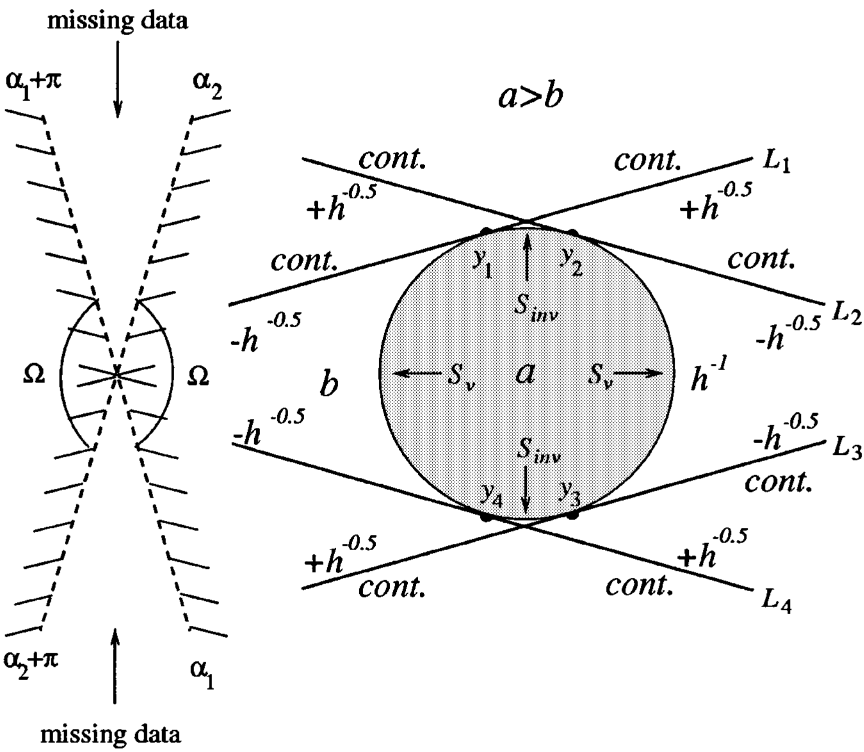

FIG. 1. Schematic behavior of the local tomography function $f_{\Lambda\chi}$ in a neighborhood of singsupp $f_{\Lambda\chi}$ in the case of the limited-angle data. Ω, angular interval of available data; S_v, pieces of the boundary of the phantom which are in singsupp $f_{\Lambda\chi}$; S_{inv}, pieces of the boundary of the phantom which are not in singsupp $f_{\Lambda\chi}$. One has singsupp $f_{\Lambda\chi} = S_1 \cup L_1 \cup L_2 \cup L_3 \cup L_4$.

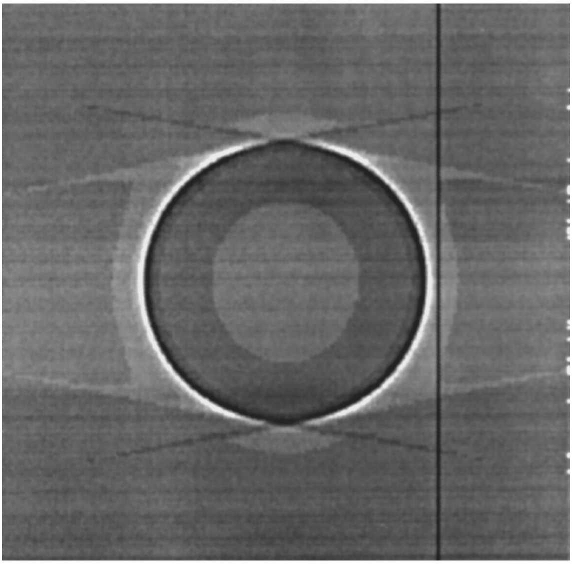

FIG. 2. Density plot of the local tomography function computed from the limited angle data. The phantom consists of one disk.

medium. According to (4.3), $f_{\Lambda\chi}(x) \sim \text{const}\, h^{-1}$ as $h = \text{dist}(x, S_v) \to 0$, where S_v denotes the visible singularities. Now let us consider, for example, the line L_1 (see Fig. 1). The function $f_{\Lambda\chi}$ is continuous as x approaches L_1 from the side opposite to S. In Fig. 1 this is denoted by cont. Equation (4.5) implies that $f_{\Lambda\chi}(x)$ is proportional to $h^{-0.5}$ as $h = \text{dist}(x, L_1) \to 0$ if x approaches L_1 from the side of S. Moreover, since the disc is more dense than the surrounding medium, the coefficient of proportionality is positive to the right of the point of contact y_1, and it is negative to the left of y_1. In Fig. 1 this is denoted by $+h^{-0.5}$ and $-h^{-0.5}$, respectively.

In Fig. 2 we see the density plot of $f_{\Lambda\chi}(x)$ computed for the same phantom as in Fig. 1. The intervals of missing data are [80°, 100°] and [260°, 280°]. The vertical cross-section of Fig. 2 along the black line is shown in Fig. 3. Let us note that Figs. 1 and 3 are in complete agreement.

Figure 4 illustrates the influence of the degree of smoothness of the cut-off function χ on the limited-angle local tomographic reconstructions. We took the function χ of the form,

$$\chi(\theta) = \begin{cases} \left(1 - (\theta/80^\circ)^{10}\right)^m, & |\theta| \le 80^\circ, \\ 0, & 80^\circ \le |\theta| \le 90^\circ, \end{cases} \qquad \chi(\theta + 180^\circ) = \chi(\theta),$$

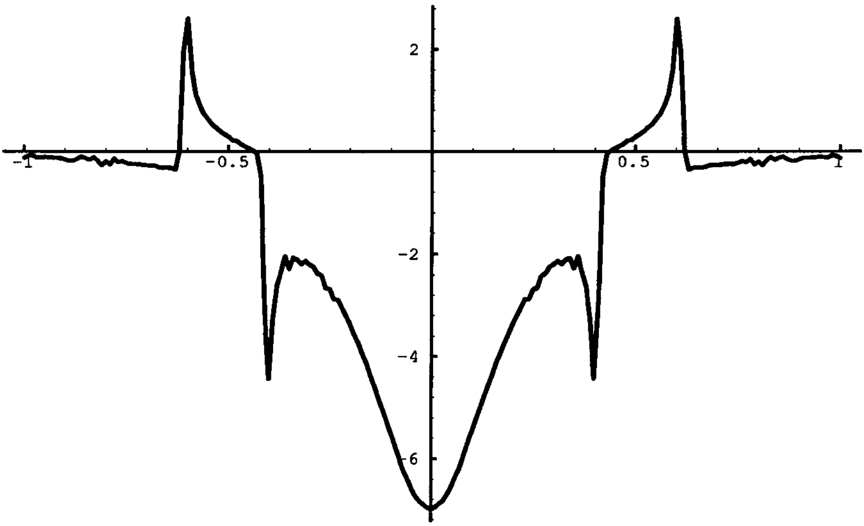

FIG. 3. Vertical cross-section of the local tomography function.

and computed $f_{\Lambda\chi}$ for different values of m. The top panel in Fig. 4 corresponds to $m = 1$, the center panel corresponds to $m = 5$, and the bottom panel to $m = 10$. As we can see, the extra singularities L_j are much less visible in the top panel of Fig. 4 than those in Fig. 2. However, a part of the visible singularities of $f_{\Lambda\chi}$ located close to the points of contact of L_j and S are suppressed a little. As m increases, we do not see a significant improvement in suppressing the artifacts caused by the nonsmoothness of χ at $\theta = \pm 80°$. On the other hand, the visible singularities became more strongly distorted. This shows that when choosing an optimal χ, there should be a trade-off between suppressing artifacts caused by the nonsmoothness of the cut-off function χ and preserving visible singularities.

Consider now the case of the generalized Radon transform. Choosing the function $\chi(\theta)$ as above, one can easily show that

$$f_{\Lambda\chi}^{(\Phi)}(x) = \int_{\xi/|\xi|\in\Omega}\int_{\mathbb{R}^2}\chi(\xi/|\xi|)B(x,y,\xi)f(y)e^{-i\xi\cdot(x-y)}\,dy\,d\xi, \quad (4.6)$$

where $f_{\Lambda\chi}^{(\Phi)}$ is defined by (1.4) (with $n = 2$) and B can be represented as a

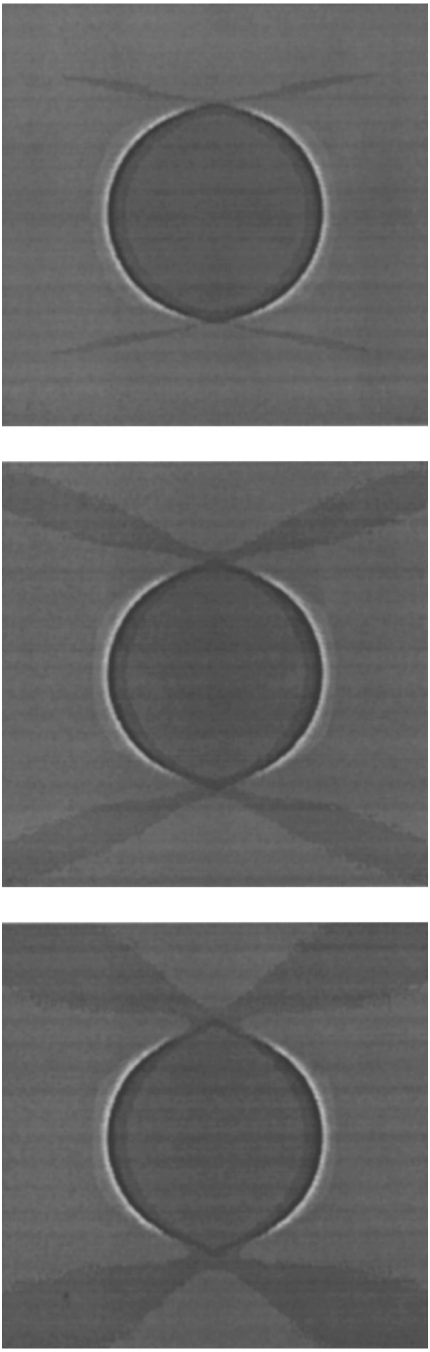

FIG. 4. Density plots of $f_{\Lambda\chi}$ in the case of the cut-off function $\chi(\theta) = (1 - (\theta/80^\circ)^{10})^m$. From top to bottom, $m = 1, 5, 10$.

sum

$$B(x, y, \xi) = |\xi| + b_1(x, y, \xi/|\xi|) + b_2(x, y, \xi/|\xi|)|\xi|^{-1},$$
$$b_1, b_2 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \times S^1). \quad (4.7)$$

Formulas (4.6) and (4.7) imply that Eqs. (4.3)–(4.5) remain valid if we replace $f_{\Lambda\chi}$ by $f_{\Lambda\chi}^{(\Phi)}$. Therefore, the behavior of $f_{\Lambda\chi}^{(\Phi)}$ in a neighborhood of singsupp $f_{\Lambda\chi}^{(\Phi)}$ is the same as the one depicted in Fig. 1.

REFERENCES

1. N. Bleistein and R. Handelsman, "Asymptotic Expansions of Integrals," Dover, New York, 1986.
2. A. Faridani, E. Ritman, and K. Smith, Local tomography, *SIAM J. Appl. Math.* **52** (1992), 459–484 and 1193–1198.
3. I. M. Gelfand and G. E. Shilov, "Generalized Functions," Vol. I, Academic Press, New York, 1964.
4. A. I. Katsevich, Local tomography for the generalized Radon transform, *SIAM J. Appl. Math.*, **57**, No. 4 (1997), 1128–1162.
5. A. I. Katsevich and A. G. Ramm, A method for finding discontinuities of functions from the tomographic data, *in* "Tomography, Impedance Imaging, and Integral Geometry" (E. T. Quinto, M. Cheney, and P. Kuchment, Eds.), pp. 115–123, Lectures in Applied Mathematics, Vol. 30, Amer. Math. Soc., New York, 1994.
6. A. I. Katsevich and A. G. Ramm, Asymptotics of PDO on discontinuous functions near singular support, *Appl. Anal.* **58**, Nos. 3–4 (1995), 383–390.
7. A. I. Katsevich and A. G. Ramm, Finding jumps of a function using local tomography, *PanAmer. Math. J.* **6**, No. 2, (1996), 1–21.
8. A. I. Katsevich and A. G. Ramm, New methods for finding values of the jumps of a function from its local tomographic data, *Inverse Problems* **11**, No. 5 (1995), 1005–1023.
9. P. Kuchment, K. Lancaster, and L. Mogilevskaya, On local tomography, *Inverse Problems* **11** (1995), 571–589.
10. E. T. Quinto, Singularities of the x-ray transform and limited data tomography in $\mathbb{R}^2$ and $\mathbb{R}^3$, *SIAM J. Math. Anal.* **24**, No. 5 (1993), 1215–1225.
11. A. G. Ramm, Optimal local tomography formulas, *PanAmer. Math. J.* **4**, No. 4 (1994), 125–127.
12. A. G. Ramm, Finding discontinuities from tomographic data, *Proc. Amer. Math. Soc.* **123**, No. 8 (1995), 2499–2505.
13. A. G. Ramm, Necessary and sufficient conditions for a PDO to be a local tomography operator, *C. R. Acad. Sci. Paris* **332**, No. 7 (1996), 613–618.
14. A. G. Ramm and A. I. Katsevich, "The Radon Transform and Local Tomography," CRC Press, Boca Raton, FL, 1996.

15. M. A. Shubin, ''Pseudodifferential Operators and Spectral Theory,'' Springer-Verlag, Berlin, 1980.
16. K. T. Smith and F. Keinert, Mathematical foundations of computed tomography, *Appl. Optics* (1985), 3950–3957.
17. E. Vainberg, E. Kazak, and V. Kurczaev, Reconstruction of the internal three-dimensional structure of objects based on real-time integral projections, *Soviet J. Nondestr. Testing* **17** (1981), 415–423.
18. R. Wong, ''Asymptotic Approximations of Integrals,'' Academic Press, Boston, 1989.